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Effect of Helium Elasticity on Torsional Oscillator Measurements

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Abstract In 2004 Kim and Chan performed a torsional oscillator measurement of the rotational inertia of solid helium-4. They found frequency changes which were interpreted in terms of a non-classical rotational inertia, that is a partial superfluidity or "supersolidity" of solid helium-4. Since then there have been many further studies using various versions of this technique. One important question that arises in these experiments is the possible effect on the oscillator frequency of changes in the elasticity of the solid helium; this can produce a change in frequency that adds to any effect due to superfluidity. In this paper we give a general discussion of the effect of changes in elasticity on the oscillator frequency and consider how the magnitude of the effect is influenced by the oscillator design. Our results should help make it possible to discriminate between frequency changes due to changes in elasticity and changes due to supersolidity.

Keywords Supersolids · Solid helium

1 Introduction

During the past few years there have been many experiments performed to investigate the possible supersolidity of solid helium-4. Kim and Chan [1–3] performed torsional oscillator experiments with solid helium and found that below about 200 mK the period of the oscillator began to decrease. This indicated that some fraction of the mass of the helium was not rotating with the oscillator and suggested that a transition

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to a supersolid state had occurred. This uncoupled mass is referred to as non-classical rotational inertia (NCRI). This effect has since been found in similar experiments performed in many other laboratories [4–8].[1] The size of the effect and the precise dependence on temperature of the frequency change vary significantly between the different experiments; this is presumed to be a result of variations in the way the solid helium samples were produced [11], dependence on the amount of helium-3 impurity present [12], and differences in the sample geometry and the oscillator frequency.

There are also large changes in the shear modulus that occur in approximately the same temperature range. It was found by Paalanen et al. [13] that the shear modulus at around 1 K was less than the value at 50 mK by between 20 and 40%. They suggested that dislocations can move freely at high temperature, but are pinned by helium-3 impurities at sufficiently low temperature [13]. This large variation in the shear modulus has also been studied by Day and Beamish [14–16], by Mukharsky et al. [17] and by Rojas et al. [18, 19]. Remarkably, the general form of the temperature-dependence of the shear modulus is very similar to the temperature-dependence of the frequency shift found in the torsional oscillator experiments [14], thus suggesting that the two effects could have a common origin. This "common origin" could in fact arise in two very distinct ways. There may be some physical mechanism that gives rise to both the superfluidity and the change in shear modulus. However, one can also consider the possibility that the change in the shear modulus gives a shift in the frequency of the torsional oscillator that makes it appear as if some of the mass of the helium is becoming uncoupled. Experimental evidence that this is *not* what is happening has been presented by West et al. [20]. In addition, for one particular oscillator, the magnitude of this effect has been estimated by means of a finite element calculation that considers the detailed structure of the torsional oscillator as well as the elasticity of the helium [21]. However, calculations of this type are difficult and require the use of a large number of finite elements because (1) the oscillator has both large and small geometrical features (e.g., the main body and the fill line, respectively), (2) the elastic constants and density of solid helium are much smaller than the elastic constants and density of the materials used to construct the oscillator, and (3) it is necessary to allow for the elastic anisotropy of the helium if the sample is a single crystal. In this note, we describe some simpler methods for the calculation of the effect of the elasticity of the helium, and give some numerical examples in representative cases.

2 Perturbation Approach

At first sight it is tempting to consider that if the helium becomes softer, the frequency of the oscillator will increase. The argument is that if the helium is soft it will not perfectly follow the rotation of the cell and so the contribution to the inertia from the helium will be reduced. Here, we use the term "cell" to refer to the main body of the torsional oscillator, i.e., excluding the helium. In fact, as we now show by a formal calculation this is incorrect. For generality, we will present the calculation

[1] For a review see [9] or [10].

using a lattice dynamics approach; one can also derive the same result using continuum elasticity. Consider any linear mechanical system composed of a number N of masses $m(l)$ with displacement in the α direction $u_\alpha(l)$. Let the potential energy be

$$U = \frac{1}{2}\sum_{ll'} \Phi_{\alpha\beta}(ll')u_\alpha(l)u_\beta(l'), \tag{1}$$

where $\Phi_{\alpha\beta}(ll')$ are coupling parameters specifying the forces between the masses. These coefficients have the symmetry property [22]

$$\Phi_{\alpha\beta}(ll') = \Phi_{\beta\alpha}(l'l). \tag{2}$$

Then the equation of motion is

$$m(l)\frac{\partial^2 u_\alpha(l)}{\partial t^2} = -\sum \Phi_{\alpha\beta}(ll')u_\beta(l'). \tag{3}$$

We look for normal mode solutions of this equation of the form

$$u_\alpha(l) = E_\alpha(l)\exp(-i\Omega t), \tag{4}$$

where the coefficients $\{E_\alpha(l)\}$ can be considered as the components of the polarization vector. By making the change of variables to $\{e_\alpha(l)\}$ defined by

$$e_\alpha(l) \equiv E_\alpha(l)\sqrt{m(l)}, \tag{5}$$

we find that the frequencies are given by the eigenvalue problem

$$\Omega^2 e_\alpha(l) = \sum_{l'} D_{\alpha\beta}(ll')e_\beta(l'), \tag{6}$$

where the dynamical matrix is

$$D_{\alpha\beta}(ll') = \frac{1}{\sqrt{m(l)m(l')}}\Phi_{\alpha\beta}(ll'). \tag{7}$$

This matrix is real and symmetric in the sense that

$$D_{\alpha\beta}(ll') = D_{\beta\alpha}(l'l). \tag{8}$$

We suppose that a normal mode frequency has been found from this equation and that there is then a small change $\delta\Phi_{\alpha\beta}(ll')$ in the coupling parameters. Then the change in the mode frequency is given by

$$\frac{\delta\Omega}{\Omega} = \frac{1}{2\Omega^2}\frac{\sum_{ll'} e_\alpha(l)D_{\alpha\beta}(ll')e_\beta(l')}{\sum_l e_\alpha(l)e_\alpha(l)} = \frac{1}{2\Omega^2}\frac{\sum_{ll'} E_\alpha(l)\delta\Phi_{\alpha\beta}(ll')E_\beta(l')}{\sum_l E_\alpha^2(l)m(l)}. \tag{9}$$

We could write a corresponding expression for the frequency shift due to a change in mass in some part of the system.

We now consider the application of this formula to a macroscopic system in which the displacements of the normal modes vary slowly with position. By equating macroscopic and microscopic expressions for the strain energy it follows that

$$\sum_{ll'} \Phi_{\alpha\beta}(ll')u_\alpha(l)u_\beta(l') = \int \eta_{\alpha\beta}(\vec{r})c_{\alpha\beta\gamma\delta}(\vec{r})\eta_{\gamma\delta}(\vec{r})dV, \tag{10}$$

and so

$$\sum_{ll'} \delta\Phi_{\alpha\beta}(ll')u_\alpha(l)u_\beta(l') = \int \eta_{\alpha\beta}(\vec{r})\delta c_{\alpha\beta\gamma\delta}(\vec{r})\eta_{\gamma\delta}(\vec{r})dV. \tag{11}$$

Then we obtain the result

$$\frac{\delta\Omega}{\Omega} = \frac{1}{2\Omega^2}\frac{\int \eta_{\alpha\beta}(\vec{r})\delta c_{\alpha\beta\gamma\delta}(\vec{r})\eta_{\gamma\delta}(\vec{r})dV}{\int \rho(\vec{r})u^2(\vec{r})dV}, \tag{12}$$

where $\vec{r}$ is the position, $\delta c_{\alpha\beta\gamma\delta}(\vec{r})$ are the changes of the elastic constant tensor, $\{\eta_{\alpha\beta}(\vec{r})\}$ and $u(\vec{r})$ are the strain tensor and magnitude of the displacement associated with the normal mode vibration, respectively, and $\rho(\vec{r})$ is the mass density. An analogous expression holds for the frequency shift if there is a change in density.

Consider first the application of this formula to polycrystalline solid helium. At the macroscopic level this must be described by the elastic constants which in Voigt notation are c_{11}, c_{12}, and c_{44} satisfying the isotropy condition $c_{11} = c_{12} + 2c_{44}$. It is believed that the changes in the shear modulus arise from the pinning of dislocations. As a result, the bulk modulus B should remain constant as the shear modulus changes while $c_{11} = B + 4c_{44}/3$ and $c_{12} = B - 2c_{44}/3$. Then for a change in c_{44} it is straightforward to show that

$$\begin{aligned}\eta_{\alpha\beta}\delta c_{\alpha\beta\gamma\delta}\eta_{\gamma\delta} = \frac{\delta c_{44}}{3}\big[&(\eta_2-\eta_3)^2 + (\eta_3-\eta_1)^2 + (\eta_1-\eta_2)^2 \\ &+ 2(\eta_1^2+\eta_2^2+\eta_3^2) + 6(\eta_4^2+\eta_5^2+\eta_6^2)\big],\end{aligned} \tag{13}$$

where the right hand side of the equation is expressed in Voigt notation. One can see that if δc_{44} is positive this expression is positive definite and so the frequency of the normal mode will increase. If the helium stiffens, the TO period necessarily decreases.

For a single crystal one can consider the possibility that c_{44} is the only elastic coefficient that changes. In this case

$$\eta_{\alpha\beta}\delta c_{\alpha\beta\gamma\delta}\eta_{\gamma\delta} = 4\delta c_{44}\big(\eta_5^2 + \eta_6^2\big) \tag{14}$$

which again is positive definite.

The results just given are general in the sense that they apply regardless of the way in which the cell is constructed. Now restrict attention to the oscillation of a cell which is sufficiently rigid that all parts of the cell move with the same angular velocity. What will be the effect of changing the stiffness of the torsion rod so that the frequency of the oscillator changes? We note that the strain in the helium has to

be sufficient to provide the angular acceleration of the helium. Thus this strain has to be proportional to the displacement times Ω^2. It then follows from (12) that if the stiffness of the torsion rod is changed, the fractional change $\delta\Omega/\Omega$ in frequency that results from a small change in the elastic constants of the helium is proportional to Ω^2. Thus, to minimize the effect of the elasticity of helium it is advantageous to use a low frequency oscillator.

3 General Method for a Rigid Cell

The perturbation method just described is restricted to small changes in the elastic constants of the helium because it is assumed that the pattern of strain in the helium remains constant. For a rigid cell one can construct a convenient method that is not restricted to small changes in the elastic constants. Let K be the torsion constant and let the moment of inertia of the cell be I_c. We imagine that we start with the cell and the helium at rest and then at time zero make a sudden rotation of the cell by an angle $\Delta\theta$ and then hold it fixed. At later time the angular momentum of the helium is

$$L_{hel} = G(t)\Delta\theta, \tag{15}$$

where $G(t)$ is a response function. More generally, if we rotate the cell so that at time t its angular position is $\theta(t)$, the angular momentum of the helium will be

$$L_{hel}(t) = \int^t G(t-t')\frac{d\theta(t')}{dt'}dt'. \tag{16}$$

If $\theta(t) = A\cos(\Omega t)$ then

$$L_{hel}(t) = -2\pi A\Omega G(\Omega)\sin(\Omega t), \tag{17}$$

where

$$G(\Omega) = \frac{1}{2\pi}\int_0^\infty G(t)\cos(\Omega t)dt. \tag{18}$$

Then since the torque τ from the torsion rod must be equal to the total rate of change of the angular momentum, it follows that

$$\tau = -KA\cos(\Omega t) = \frac{d}{dt}[-I_c\Omega A\sin(\Omega t)] + \frac{dL_{hel}}{dt}, \tag{19}$$

which gives the result

$$\Omega^2 = \frac{K}{I_c + 2\pi G(\Omega)}. \tag{20}$$

If the helium were rigid, the frequency of the cell plus helium would be

$$\Omega_0^2 = \frac{K}{I_c + 2\pi G(0)}. \tag{21}$$

To a very good approximation we can use Ω_0 in place of Ω on the right hand side of (20). It then follows that the lack of complete rigidity of the helium makes it appear that there is an apparent supersolid density fraction of

$$\left.\frac{\Delta\rho}{\rho}\right|_{app} = \frac{G(\Omega_0) - G(0)}{G(0)}. \tag{22}$$

The great advantage of this approach is that one only has to calculate the response function of the helium; if we assume that the cell is rigid the details of the construction of the cell enter only through the frequency Ω_0 and the geometry of the volume containing the helium sample.

We now give some examples of the application of this method. We present here a calculation in which the solid helium is treated as an elastic continuum. At the walls of the cell a "no-slip" boundary condition is applied, i.e., the displacement of the helium is equal to the displacement of the cell wall. There is no viscous damping in the helium. A discussion of the possible effects of dissipative processes in the helium, including glassy phenomena, has been given by Nussinov et al. [23, 24] and by Yoo and Dorsey [25].

To calculate $G(t)$ we use a simple finite-difference simulation to calculate the displacement as a function of time. This simulation uses a cubic mesh of grid points with coordinates x, y, z. To perform the simulation for a single crystal in which the c-axis lies in direction $\hat{n}$, we make a transformation of the elastic constants from the frame of the crystal axes to the coordinate frame x, y, z. Initially, the displacements and velocities are set to be zero for all mesh points. Then at time $t = 0$ all mesh points that lie *outside* the volume of the helium (i.e., mesh points that lie in the walls of the cell) are given a displacement that corresponds to unit rotation around the axis of oscillation. At later times the displacement positions of these mesh points is held fixed, while the change in the displacement of the mesh points within the helium is calculated. At each step in the time development the angular momentum of the helium is calculated. The integral in (18) is then found. This simple method has the advantage that, unlike in a finite element method, it is not necessary to adapt the mesh to the geometry of the helium volume. Also, it is easy to perform the simulation for an anisotropic solid.

As an example, we consider the geometry of a cell that has a helium space which is a cylinder of radius 0.5 cm. We calculate the apparent supersolid fraction for a single crystal as a function of the value of c_{44}. All other elastic constants are held constant at the values $c_{11} = 4.05, c_{12} = 2.13, c_{13} = 1.05$ and $c_{33} = 5.54$ in units of 10^8 g cm^{-1} s^{-2} as measured by Greywall [26]. The frequency of the oscillator is taken to be 1000 Hz. Results for solid helium with the c-axis parallel and perpendicular to the rotation axis are shown in Figs. 1 and 2. The spacing of mesh points was 0.05 cm and the uncertainty due to the finite mesh spacing is estimated to be $\pm 2\%$.

The results indicate that once the height of the cell becomes greater than twice the diameter, the value of $(\Delta\rho/\rho)_{app}$ becomes almost independent of the height. In addition, one can see from Fig. 1 that if the c-axis is vertical, then in the limit $c_{44} \to 0$ the value of $(\Delta\rho/\rho)_{app}$ is independent of height regardless of the height. This is because in this limit each layer of the solid that lies in a plane normal to the c-axis is uncoupled to adjacent layers and therefore moves independently.

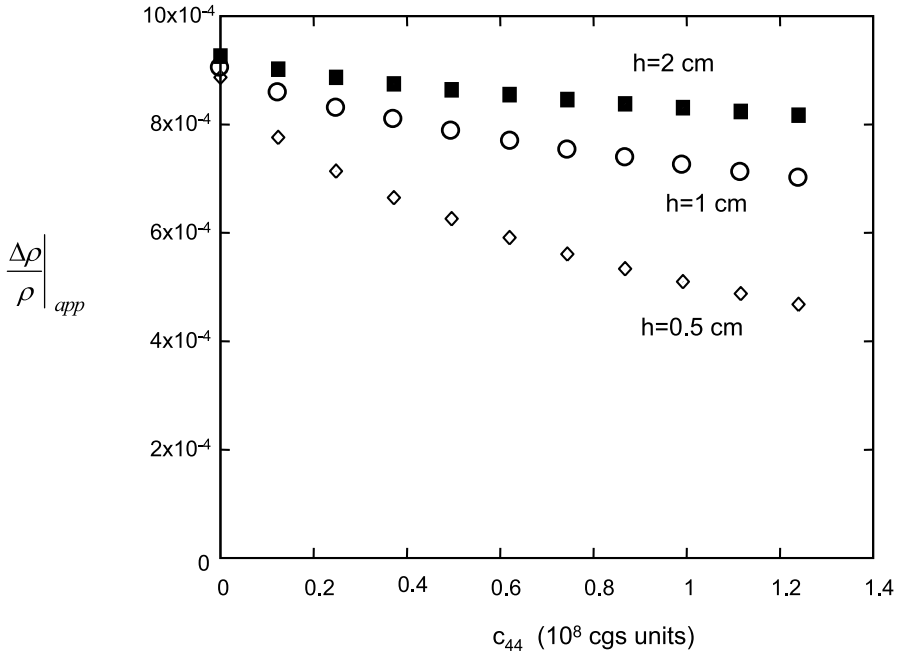

Fig. 1 Apparent supersolid fraction $(\delta\rho/\rho)_{app}$ for a single crystal of solid helium as a function of the value of c_{44}. The cell has a radius of 0.5 cm and a height h as indicated in the figure. Results are for a crystal with c-axis parallel to the rotation axis of the oscillator

In a recent paper [27] the elastic constants of polycrystalline helium have been estimated based on the single crystal values. Using the constants measured by Greywall the estimated constants for the polycrystal are $c_{11} = 4.1280, c_{12} = 1.6210$ and $c_{44} = 1.2535$ in units of 10^8 g cm^{-1} s^{-2}. If c_{44} is reduced because dislocations are able to move in the basal plane, we can expect that the bulk modulus $B(B = (c_{11} + 2c_{12})/3)$ will remain constant. Also for a polycrystal the effective elastic constants must satisfy the isotropy condition $c_{11} = c_{12} + 2c_{44}$. It follows that if c_{44} is reduced to the value fc_{44} then c_{11} and c_{12} must take on the values $2.4567 + 1.6713f$ and $2.4567 - 0.8357f$, respectively, again in units 10^8 g cm^{-1} s^{-2}. In Fig. 3 we show results for polycrystalline helium based on these elastic constants.

The apparent supersolid fraction that we have calculated for these geometries is of the order of a few parts in 10^4. For example, for a polycrystal a decrease of c_{44} by 20% with a cell of height 1 cm gives $(\Delta\rho/\rho)_{app} = 1.35 \times 10^{-4}$. This is considerably less than the values of $(\Delta\rho/\rho)_{app}$ measured in many torsional oscillator experiments.

It follows from the equations given above that if the *shape* (e.g., ratio of height to radius) of the helium volume remains constant, the value of $\delta\rho/\rho$ increases as the square of the linear dimensions of the volume. Thus, the results given for cells of different ratios of height to radius can be used to provide an estimate of the effect of elasticity for any cylindrical cell. In addition, provided that the oscillator frequency is small compared to the frequency of the torsional modes of the helium confined in the cell, the value of $\Delta\rho/\rho$ is proportional to Ω^2. In an interesting experiment, Aoki et al.

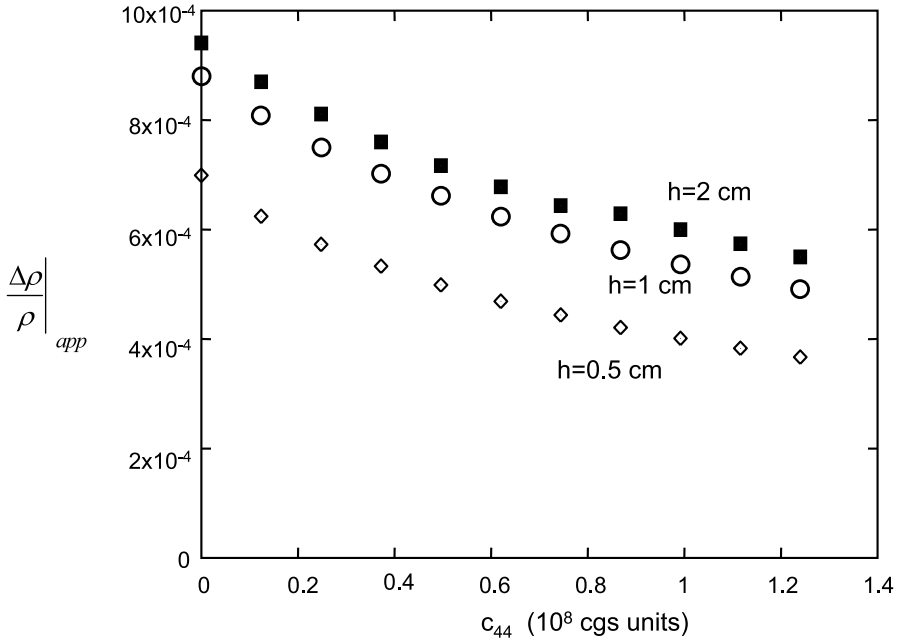

Fig. 2 Apparent supersolid fraction $(\delta\rho/\rho)_{app}$ for a single crystal of solid helium as a function of the value of c_{44}. The cell has a radius of 0.5 cm and a height h as indicated in the figure. Result are for a crystal with c-axis perpendicular to the rotation axis of the oscillator

[5–10], made measurements at two different frequencies with the same oscillator cell and found the same apparent supersolid fraction. Thus, in this particular experiment it appears that elastic effects cannot explain the supersolid fraction.

4 Results for a Non-rigid Cell

In the case of a rigid cell there exists strain in the helium only because there has to be a stress to accelerate the helium. However, if the cell is *not* rigid then there will be relative motion of the different parts of the cell and there can be a much larger strain in the helium and, consequently, a much larger shift of the frequency due to a change in the shear modulus. For example, in a non-rigid cell in which the helium is contained in a narrow annulus there will be some relative motion of the cell material on the inside and outside of the annulus; this may result in a large strain in the helium and a large frequency shift. To derive general results for this is difficult because more parameters are required to describe the structure of the cell. However, the following simplified discussion may be of value.

We imagine that we can divide the cell into two parts "inner" and "outer", with moments of inertia I_{in} and I_{out}, respectively. For simplicity we take each of these parts to be rigid, i.e., for each of the parts the motion is completely described by a rigid body rotation around the oscillator axis. The solid helium is in a narrow annulus

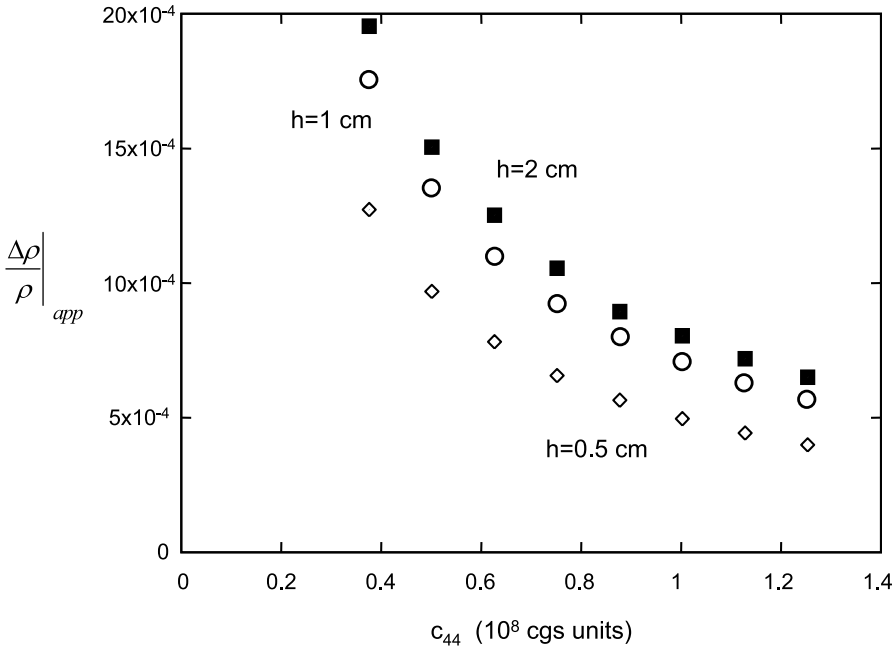

Fig. 3 Apparent supersolid fraction $(\delta\rho/\rho)_{app}$ for polycrystalline solid helium as a function of the value of c_{44}. The radius of the cell is 0.5 cm and a height h as indicated in the figure

of width w, mean radius R, and height h between these two parts of the cell.[2] We suppose that these two parts of the cell are coupled together by a torsion element of stiffness K' which to a first approximation is sufficiently stiff that the inner and outer part always have nearly the same angular displacement. Normally the main contribution to K' will come from the material of the cell itself, but the solid helium will also provide some coupling (see below). We consider the outer part of the cell to be connected to the outside world by the torsion rod K. Then the angular positions of the two parts vary with time as

$$I_{in}\frac{d^2\theta_{in}}{dt^2} = -K'(\theta_{in} - \theta_{out}), \tag{23}$$

$$I_{out}\frac{d^2\theta_{out}}{dt^2} = -K\theta_{out} - K'(\theta_{out} - \theta_{in}). \tag{24}$$

The equation for the normal mode frequencies can be written in the form

$$\Omega^2 = \frac{K}{I_{in} + I_{out}} + \frac{I_{in}I_{out}\Omega^4 - K\,I_{in}\Omega^2}{(I_{in} + I_{out})K'}. \tag{25}$$

[2]The effect of changes in the elasticity of helium in a rigid cell and with the helium in an annulus has been considered by J.D. Reppy in [28].

In a normal torsional oscillator $K' \gg K$ and the mode that is measured is the low frequency mode of the system. It follows that we can find an approximate solution for this low frequency mode as

$$\Omega^2 \approx \frac{K}{I_{in} + I_{out}} - \frac{I_{in}^2 K^2}{(I_{in} + I_{out})^3 K'}. \tag{26}$$

Note that we have not included the moment of inertia of the helium in this expression. If there is a small change $\delta K'$ in K' the change in frequency will be given by

$$\frac{\delta\Omega}{\Omega_0} = \frac{1}{2} \frac{I_{in}^2 K \delta K'}{(I_{in} + I_{out})^2 K'^2}. \tag{27}$$

Now we include the effect of the shear stiffness of the helium. If the inner part rotates by $\Delta\theta$ relative to the outer part the shear strain in the solid helium will be $\Delta\theta R/w$ and so the torque acting between these parts is $2\pi R^3 h c_{44} \Delta\theta / w$. Thus there is a contribution to K' which is equal to $2\pi R^3 h c_{44}/w$. Hence if there is a change in the shear modulus of the helium there will be a frequency shift

$$\frac{\delta\Omega}{\Omega_0} = \frac{\pi R^3 h I_{in}^2 K \delta c_{44}}{(I_{in} + I_{out})^2 K'^2 w}. \tag{28}$$

We can compare this with the frequency shift due to the introduction of the solid helium which is

$$\frac{\delta\Omega_0}{\Omega_0} = \frac{\pi R^3 h w \rho_{\mathrm{He}}}{I_{in} + I_{out}}, \tag{29}$$

where ρ_{He} is the density of the helium. Comparison of (28) and (29) gives the result that the change in stiffness of the helium leads to an apparent superfluid density of

$$\left(\frac{\delta\rho}{\rho}\right)_{app} = \frac{\Omega_0^2 \delta c_{44}}{\rho_{\mathrm{He}} w^2 \omega_0^4}, \tag{30}$$

where $\omega_0 = \sqrt{K'/I_{in}}$. This is the frequency at which the inside part would vibrate if the outside were to be held fixed.

One can see from this formula that the effect of a change in the elastic properties of the helium depends critically on the cell construction and geometry, i.e., through the values of w and ω_0. Suppose for example that the oscillator frequency is 1 kHz and that $\omega_0/2\pi$ is 100 kHz which may be a reasonable value. Then if $w = 100$ μm, a 20% change in c_{44} gives an apparent superfluid fraction of only 3.33×10^{-4}. However, it is clear that if the cell is *not* rigidly constructed the effect could be much larger since it varies as ω_0^4. To get the superfluid fraction to be 1% (as has been observed in some experiments) would require $\omega_0/2\pi = 43$ kHz.

5 Summary

We have given a general discussion of the effect of changes in elastic properties of solid helium in torsional oscillator measurements. From our results, we can see that

even large changes in c_{44} give changes in oscillator frequency that mimic a superfluid fraction that is of the order of a few parts in 10^4 for typical cylindrical cells. Significantly larger effects could occur in cells in which the helium is contained in a narrow annulus if the cell is constructed in a way such that the inside and outside of the annulus are not strongly coupled together. It is not possible to discuss this further without considering the details of the construction of each torsional oscillator.

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