Advanced topics in Markov-chain Monte Carlo

Lecture 8:
Meta algorithms, consensus sampling
Part 2/2: Consensus sampling

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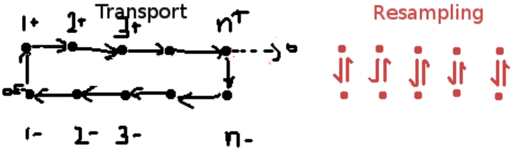
References

W. Krauth Frontiers in Physics (2022)

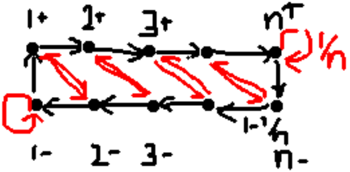
Lifted MCMC in one dimension

Probability distribution $\pi = (1/n, \dots, 1/n)$ (Diaconis et al. 2000)

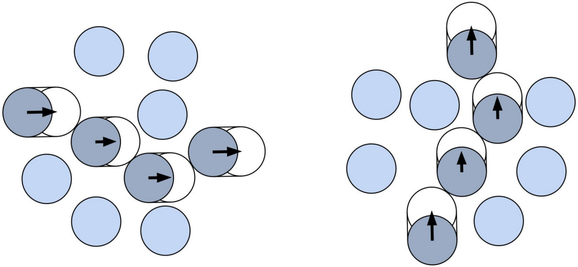
- Transport + resampling



- "Lifted" Markov chain $\hat{\Omega} = \Omega \times \{-, +\}$:

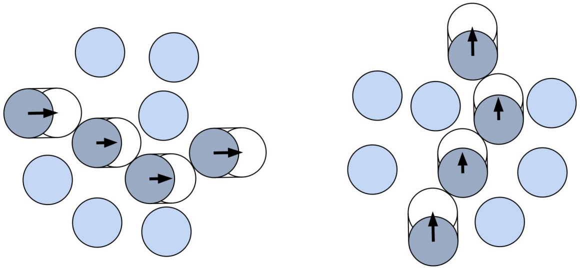


Lifted MCMC in higher dimensions (1/5)



- Infinitesimal moves (avoid overlaps).
- This algorithm is correct without moving “down” nor “up”.
- See algorithms presented by Gabriele Tartero (later today)
- There are weirder versions, even for hard spheres.

Lifted MCMC in higher dimensions (2/5)



- Consensus sampling: Active sphere moves until one of the other spheres becomes “unhappy”.

Lifted MCMC in higher dimensions (3/5)

- Metropolis filter:

$$p^{\text{Met}}(a \to b) = \min\left[1, \prod_{i<j} \exp\left(-\beta \Delta U_{i,j}\right)\right]$$

- Factorized Metropolis filter:

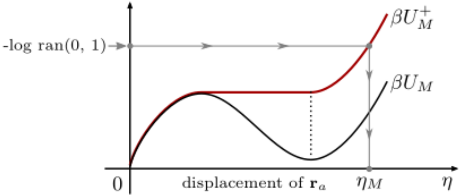
$$p^{\text{Fact.}}(a \to b) = \prod_{i<j} \min\left[1, \exp\left(-\beta \Delta U_{i,j}\right)\right].$$

both satisfy the detailed-balance condition...

- Interpretation in terms of Boolean random variables.

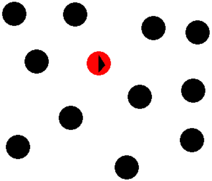
$$X^{\text{Fact.}}(a \to b) = X_{1,2} \wedge X_{1,3} \wedge \cdots \wedge X_{N-1,N}$$

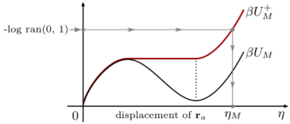
Lifted MCMC in higher dimensions (4/5)



- $$p_M(m) = \underbrace{\prod_{l=1}^{m-1} \mathrm{e}^{-\beta\,\Delta U_M^+(l)}}_{\text{accepted}} \overbrace{\left[1 - \mathrm{e}^{-\beta\,\Delta U_M^+(m)}\right]}^{\text{move } m \text{ rejected}}, \qquad (1)$$
- Peters, de With (2012)

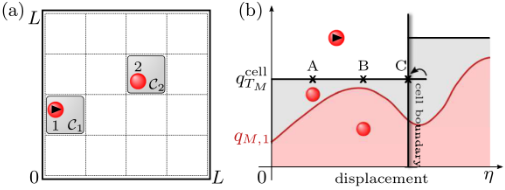
Lifted MCMC in higher dimensions (5/5)





- Compute event-times for all factors.
- Select smallest one.
- Complexity $\mathcal{O}(N)$.

Thinning and sampling (1/1)



- Time-dependent Poisson process $q(x)\mathrm{d}x$

$$\underbrace{q(x)\mathrm{d}x}_{\text{variable}} = \underbrace{q^{\max}\mathrm{d}x}_{\text{constant}} \underbrace{\frac{q(x)}{q^{\max}}}_{\text{rejection}}$$

- This is called “Thinning”, many generalizations.

- Rejection sampling (see SMAC book)

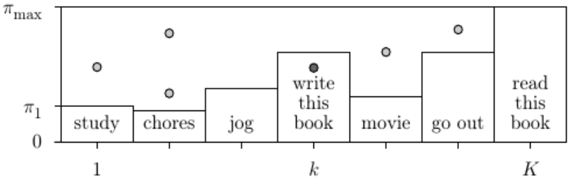


Fig. 1.28 Saturday night problem solved by Alg. 1.13 (`reject-finite`).

Sampling from a discrete distribution (2/3)

- Tower sampling (see SMAC book)

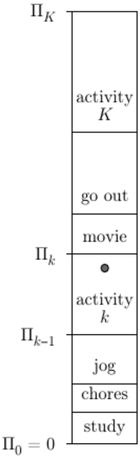
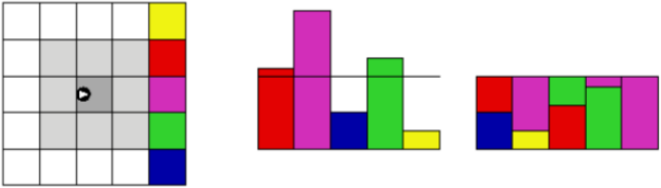


Fig. 1.29 Saturday night problem solved by tower sampling.

Sampling from a discrete distribution (3/3)

- Walker's method of aliases (see SMAC book 2nd edition)



.. Cell-veto sampling using Walker's method.

- Complexity $\mathcal{O}(1)$
- ... gives $\mathcal{O}(1)$/event algorithm for any potential