Tutorial 2, Advanced Topics in Markov-chain Monte Carlo
2021/22 ICFP Master (second year)

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1. **Lifting for the V-shaped stationary distribution**

   In this exercise for the V-shaped stationary distribution on sites $i = \{1, \dots, n = 2k\}$, namely $\pi_i = \frac{4}{n^2}|\frac{n+1}{2} - i|)$, you can either run the MCMC algorithm many times or else do the repeated matrix multiplication as $\pi^{\{t\}} = \pi^{\{t-1\}}P$ (in numpy: `pi = pi @ P`).

   (a) Program the time evolution of π, for different n, and track the TVD starting from $i = 1$ (suppose that this is the most unfavorable initial condition, that is, suppose that TVD $= d(t)$). Do this for the collapsed Markov chain (Metropolis algorithm), and for the lifted Markov chain (with transport and resampling).

   (b) For the collapsed MCMC, plot $d(t)$ vs t/n^2 and also vs $t/(n^2 \log n)$. The latter gives a better data collapse.

   (c) For the lifted MCMC, plot $d(t)$ vs t/n for small times (for a few values of n) and interpret the results.

   (d) For the lifted MCMC, plot $d(t)$ vs t/n^2 for all times (for a few values of n and interpret the results.

2. **Lifting and global balance**

   In lecture 2, we discussed the lifting on a path graph P_n for $\pi_i = 1/n$ (NB: the one with transport + resampling)

   (a) Prove that it satisfies the global-balance condition for any lifted configuration.

   (b) Prove that it is really a lifting.

3. **Conductance and correlation times (projects for next week)**

   (a) In lecture 2, we discussed that a lifting cannot increase the conductance of a Markov chain. Prove this (and present it in week 3, if you want).

(b) In lecture 2, we discussed an important achievement by Sinclair and Jerrum (1989), telling us that the conductance gives an upper bound for the correlation time. Look up the proof (for a reversible chain) in the original paper (Lemma 3.3, page 15-17), which consists in just two pages of basic linear algebra. Present this in week 3, if you want.