Tutorial 2, Advanced Topics in Markov-chain Monte Carlo
2021/22 ICFP Master (second year)

Werner Krauth
ENS Paris
(Dated: 26 January 2022)

1. **Conductance and correlation times**

   (a) (Easy) In lecture 2, we discussed that a lifting cannot increase the conductance of a Markov chain. Prove this (and present it in week 3, if you want).

   (b) (Fun) In lecture 2, we discussed an important achievement by Sinclair and Jerrum, telling us that the conductance gives an upper bound for the mixing time. Look up the proof (for a reversible chain) in the original paper (Lemma 3.3, page 15-17), which consists in just two pages of basic linear algebra.

2. **Lifting and global balance**

   In lecture 2, we discussed the (original) lifting on a path graph P_n for $\pi_i = 1/n$ (NB: original: the one with the diagonal transition probabilities).

   (a) (Easy) Prove that the original variant is really a lifting.

   (b) (Easy) Prove that the two-step variant is really a lifting, but of a slightly modified collapsed chain.

   (c) (Super-easy) Prove that the (original) lifted Markov chain satisfies the global-balance condition (for any lifted configuration).

   (d) (Super-easy) Prove that the two-step lifting discussed in the lecture satisfies the global-balance condition for any lifted configuration for $\pi_i = 1/n$,

   (e) (Super-easy to get confused) Prove that the two-step lifting discussed in the lecture satisfies the global-balance condition for any lifted configuration, supposing that $\pi_i^+ = \pi_i^- = \pi_i/2$.

3. **Lifting for the V-shaped stationary distribution**

   In this problem, we follow Hildebrand (2004), but with the two-step lifting.

(a) Program the time evolution of π, as in the 3×3 pebble game, for different n, and track the TVD starting from $i = 1$ (suppose that this is the most unfavorable initial condition, that is, suppose that TVD $= d(t)$). Do this for the collapsed Markov chain (that is, for the Metropolis algorithm), and for the two-step-lifted Markov chain

(b) For the collapsed MCMC, plot $d(t)$ vs t/n^2 and also vs $t/(n^2 \log n)$. The latter should give a better data collapse.

(c) For the lifted MCMC, plot $d(t)$ vs t/n for small times (for a few values of n) and interpret the results.

(d) For the lifted MCMC, plot $d(t)$ vs t/n^2 for all times (for a few values of n and interpret the results.