Tutorial 1, Statistical Mechanics: Concepts and applications
2018/19 ICFP Master (first year)

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Tutorial
(Dated: September 9, 2018)

1. **General properties of the characteristic functions.**

   (a) [EASY] Prove the following properties:

   - $\Phi_\xi(0) = 1$.
   - $\Phi_\xi(-t) = \Phi_\xi^*(t)$.
   - $|\Phi_\xi(t)| \leq 1$.
   - $\Phi_{a\xi+b}(t) = e^{ibt}\Phi_\xi(at)$.

   (b) [EASY] Let ξ_1 and ξ_2 two independent random variables, what is the characteristic function of their sum? What about the sum of n independent random variables?

   (c) [EASY] Name the first two cumulants. What is the variance of the sum of two independent random variables?

2. **Sum of random variables with uniform distribution.**

   (a) [EASY] Compute the characteristic function of the sum of n random variables ξ_j with uniform distribution $\pi_{\xi_j}(x) = \frac{1}{2a}\theta_H(x+a)\theta_H(a-x)$, where $\theta_H(x+a)$ is the Heaviside theta function.

(b) [EASY-MEDIUM] Show that the characteristic function of $\xi^{(n)} = \sum_{j=1}^{n} \xi_j$ can be written in the following form:

$$\Phi_{\xi^{(n)}} = \frac{1}{(2ia)^n} t^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{i(n-2k)at} \tag{1}$$

Hint 1: Express the sine functions using complex exponentials ($\sin x = \frac{e^{ix}-e^{-ix}}{2i}$) and use the binomial theorem $(a+b)^j = \sum_{k=0}^{j} \binom{j}{k} a^j b^{j-k}$.

(c) [HARD] Compute the inverse Fourier transform of the characteristic function and show that the distribution of $\xi^{(n)}$ can be written as[2]

$$\pi_{\xi^{(n)}} = \frac{1}{(n-1)!(2a)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \max\Big((n-2k)a - x, 0\Big)^{n-1}. \tag{2}$$

Hint 1: Move the sum outside of the integral of the inverse Fourier transform. *Warning*: the resulting integrals are divergent, but the divergencies have to simplify, so don't worry too much! The finite part of the integrals can be extracted using the *Cauchy principal value*, usually denoted by P.V., which, in the case of a singularity at zero, reads as

$$\text{P.V.} \int_{-\infty}^{\infty} f(t) = \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} f(t) + \int_{\epsilon}^{\infty} f(t) \right]. \tag{3}$$

Hint 2: Compute the (finite part of the) integrals by integrating by parts $n-1$ times (note that the original product of sin functions has a zero of order n at $t=0$).

Hint 3: P.V. $\int_{-\infty}^{\infty} \mathrm{d}t t^{-1} e^{itb} = i\pi \mathrm{sgn}(b)$.

Hint 4: $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (x+k)^j = 0$ for any x and integer $j = 1, \ldots, n-1$.

(d) [MEDIUM] Verify the validity of the central limit theorem for the sum of variables with uniform distribution (you can work with the characteristic function).

Hint 1: $\log \frac{\sin t}{t} = \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{2n(2n)!} (2t)^{2n}$, where the coefficients B_n are known as "Bernoulli numbers", $B_0 = 1$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, et cetera.

3. **Stable distributions. Definition**: A non-degenerate distribution π_ξ is a stable distribution if it satisfies: let ξ_1 and ξ_2 be independent copies of a random variable ξ (they have the same distribution π_ξ). Then π_ξ is said to be stable if for any constants $a > 0$ and $b > 0$ the random variable $a\xi_1 + b\xi_2$ has the distribution $\pi_{c\xi+d}$ for some constants $c > 0$ and d.

   (a) [MEDIUM] Prove that the Gaussian $\pi_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is a stable distribution.

   (b) [EASY] Consider a characteristic function of the form

$$\Phi_\xi(t) = \exp\bigl(it\mu - (c_0 + ic_1 f_\alpha(t))|t|^\alpha)\bigr) \,, \tag{4}$$

   with $1 \leq \alpha < 2$. Show that $f_\alpha(t) = \mathrm{sgn}(t)$, for $\alpha \neq 1$, and $f_1(t) = \mathrm{sgn}(t) \log |t|$ produce stable distributions. These are also known as *Lévy distributions*, after Paul Lévy, the first mathematician to study them.

   (c) [EASY] Find a distinctive feature of the Lévy distributions.

   (d) [EASY] Assume $\alpha \neq 1$ and show that, in order for $\Phi_\xi(t)$ to be the Fourier transform of a probability distribution, the coefficient c_1 cannot be arbitrarily large. Determine its maximal value.

   Hint 1: One can show (MEDIUM-HARD) that the inverse Fourier transform of (4) has the tails

$$\pi_\xi(x) \xrightarrow{|x| \gg 1} \frac{\Gamma(1+\alpha)}{2\pi|x|^{1+\alpha}} \Bigl(c_0 \sin\frac{\pi\alpha}{2} - c_1 \mathrm{sgn}(x) \cos\frac{\pi\alpha}{2} \Bigr) \,. \tag{5}$$

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[1] A. Rényi, *Probability Theory*, North-Holland Publishing, Amsterdam (1970).

[2] D. M. Bradley and R. C. Gupta, *On the Distribution of the Sum of n Non-Identically Distributed Uniform Random Variables*, Annals of the Institute of Statistical Mathematics (2002) 54: 689 [arXiv:0411298]

[3] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, John Wiley & Sons, New York (1966).