Advanced topics in Markov-chain Monte Carlo

Lecture 5:
Perfect sampling in Markov-chain Monte Carlo
Part 2/3: Coupling from the past in the Ising model

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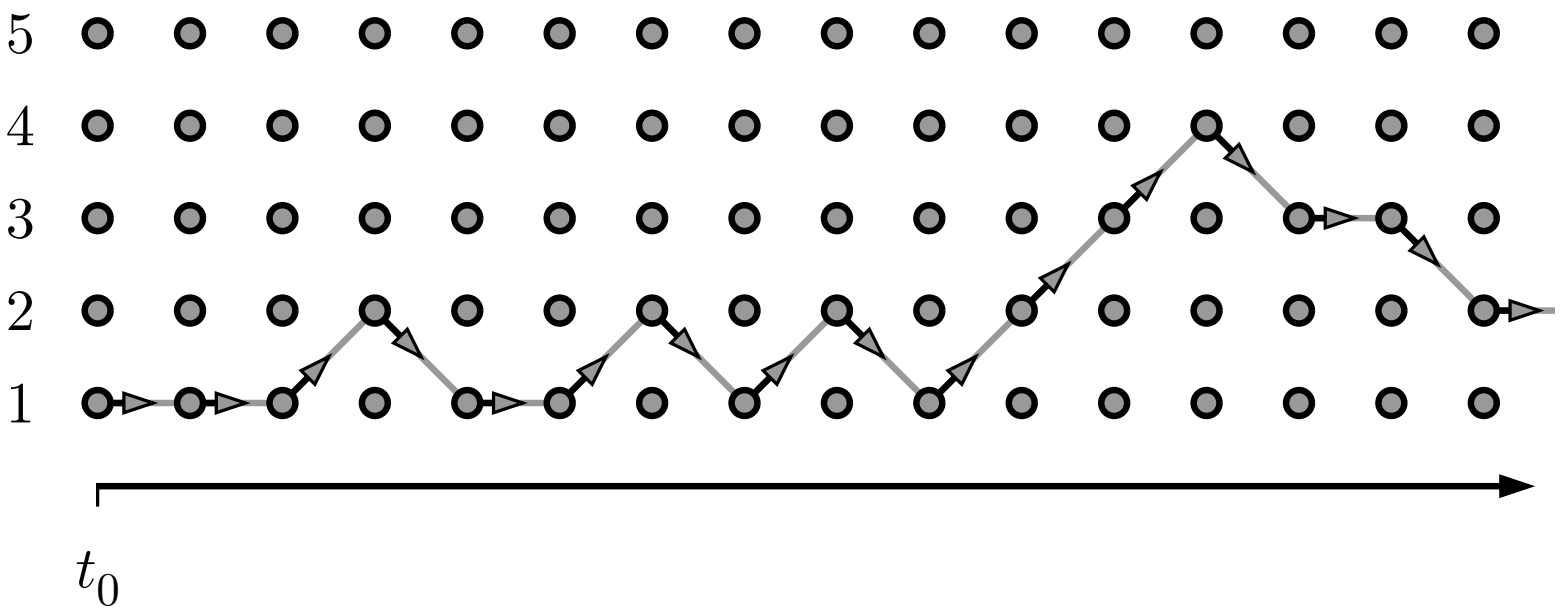
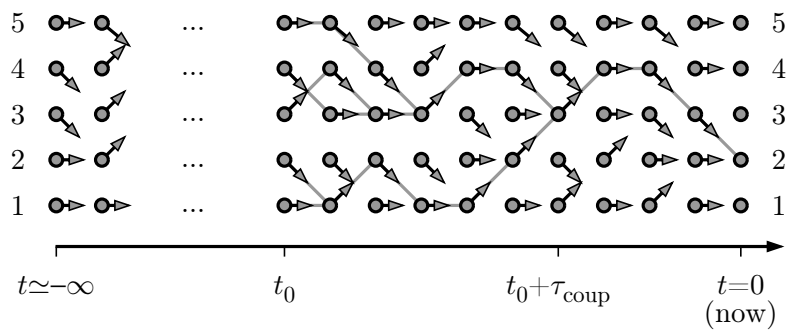
Markov chain (traditional view)



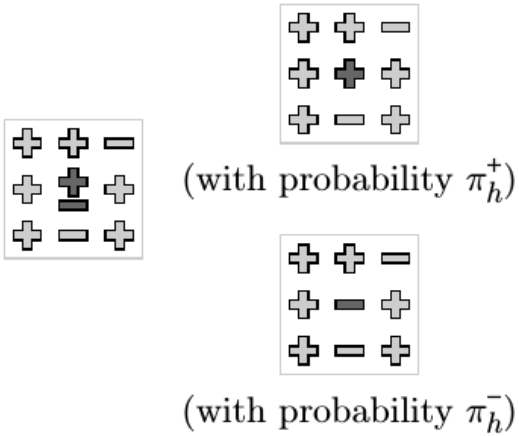
- Configuration c_t, move δ_t.
- Set $t_0 = 0$.
- Transition matrix easy to write down (TD)

Markov chain coupling



- Coupling (Doeblin, 1930s).
- Random maps, coupling from the past
- In the following: Perfect-sampling MCMC algorithm for the Ising model

Ising model - heat bath

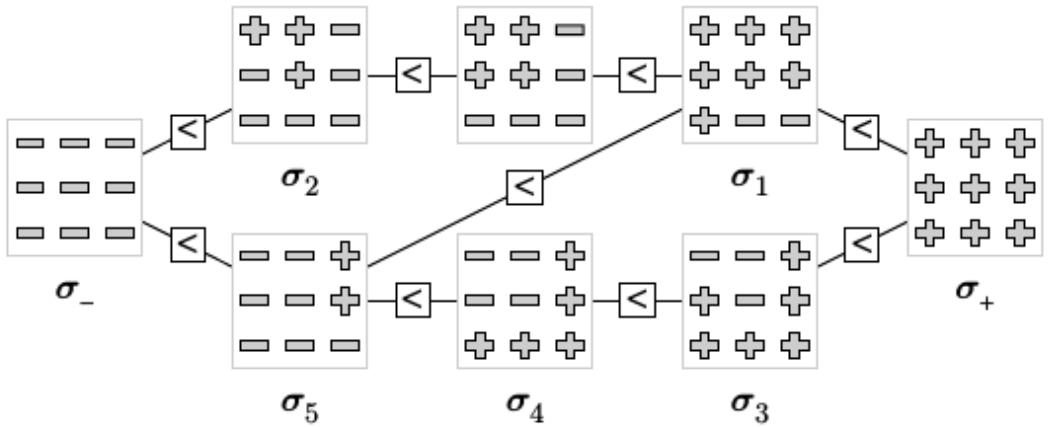


$$\pi_h^+ = \frac{e^{-\beta E^+}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{-2\beta h}},$$

$$\pi_h^- = \frac{e^{-\beta E^-}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{+2\beta h}}.$$

- Roughly equivalent to Metropolis algorithm.

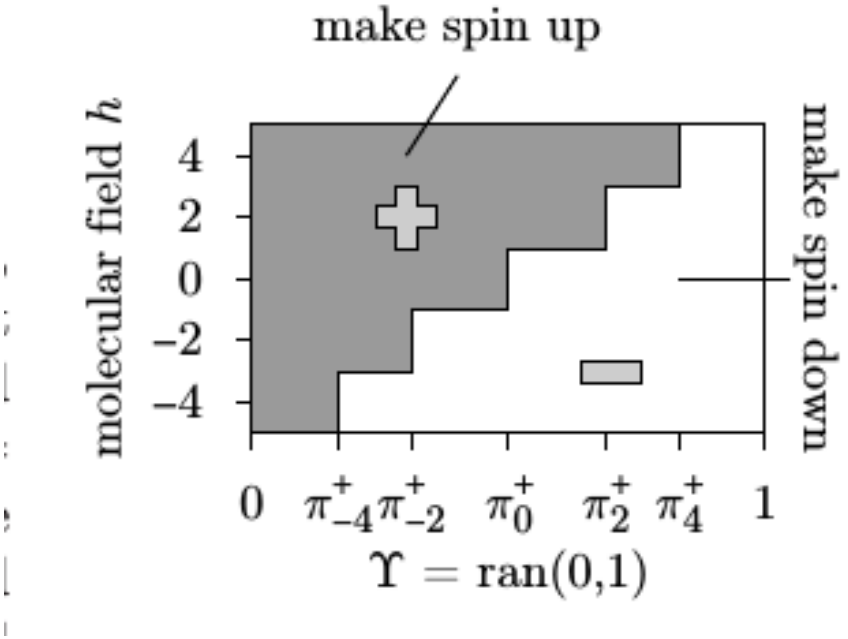
Ising model - half order



$$\pi_h^+ = \frac{e^{-\beta E^+}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{-2\beta h}},$$

$$\pi_h^- = \frac{e^{-\beta E^-}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{+2\beta h}}.$$

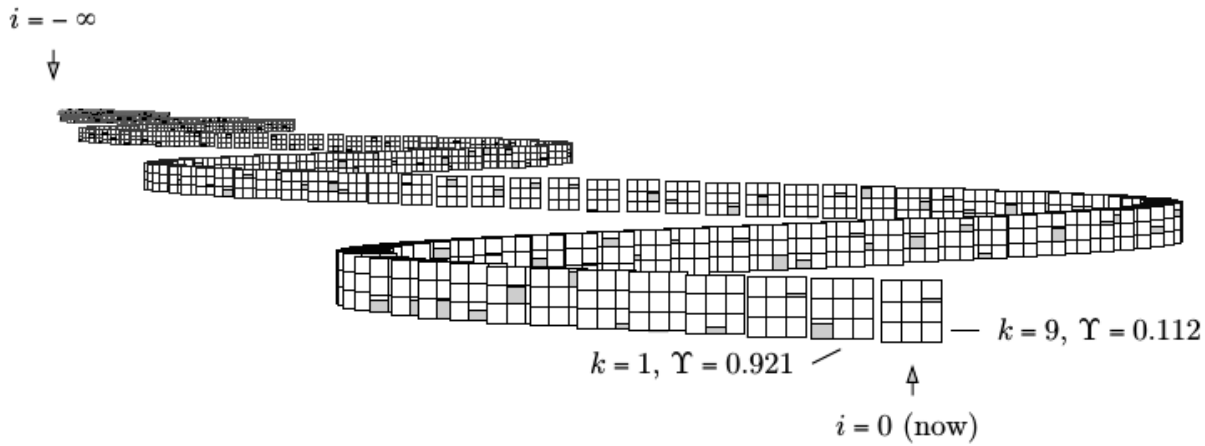
Ising model - half order



$$\pi_h^+ = \frac{\mathrm{e}^{-\beta E^+}}{\mathrm{e}^{-\beta E^+} + \mathrm{e}^{-\beta E^-}} = \frac{1}{1 + \mathrm{e}^{-2\beta h}},$$

$$\pi_h^- = \frac{\mathrm{e}^{-\beta E^-}}{\mathrm{e}^{-\beta E^+} + \mathrm{e}^{-\beta E^-}} = \frac{1}{1 + \mathrm{e}^{+2\beta h}}.$$

Ising model - coupling from the past



- Ising-model simulation that has run since time $i = -\infty$.
- Produces perfect samples in any dimension.