# Tutorial 1, Advanced MCMC
# 2022/23 ICFP Master (second year)

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1. ***A priori* probabilities and filters**

   (a) In the lecture, we discussed *a priori* probabilities $\mathcal{A}$ and filters $\mathcal{P}$. Discuss those for the $3 \times 3$ pebble game.

   (b) We discussed that a finite Markov chain takes place on a graph. Draw this graph for the $3 \times 3$ pebble game.

2. **Irreducibility and uniqueness of stationary distribution**

   In the lecture, we discussed that an irreducible transition matrix has a unique stationary distribution (but that convergence is not guaranteed). We now consider the matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

   (a) Show that $P$ is a transition matrix of a Markov chain.

   (b) Draw the graph of $P$.

   (c) Is $P$ irreducible?

   (d) Does $P$ have a unique stationary distribution?

   (e) Is there a contradition with what was said in the lecture?? (look up "Unique essential communicating class" in Levin & Peres).

3. **Different categories of transition matrices**

   In the lecture, we discussed the different categories of transition matrices. We now consider the matrix:

$$P = \begin{pmatrix} 5/12 & 5/12 & 1/6 \\ 1/4 & 1/4 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \tag{2}$$

(a) Is $P$ the transition matrix of a Markov chain?

(b) If yes, is $P$ an irreducible transition matrix?

(c) If yes, is $P$ an aperiodic transition matrix?

(d) Is $P$ reversible or non-reversible?

(e) What is its category?

4. **Total variation distance and mixing time**

In the lecture, we discussed the total variation distance (TVD)

(a) There was a "tiny" theorem to be proven... Prove it!

(b) Compute the TVD for the $3 \times 3$ pebble game, starting from the upper right corner, for the first few steps (if you have no computer) or for all $t$ (if you have a computer).

(c) Compute the quantity $d(t)$ (with the maximum taken over initial *configurations*), and the mixing time as a function of the parameter $\epsilon$ (with a computer).

5. **Conductance on the path graph**

The path graph $P_n = (\Omega, V)$ is the one-dimensional lattice without periodic boundary conditions, with $\Omega = \{1, \dots, n\}$, and $V = \{(1,2), \dots, (n-1, n)\}$. In week 2, we will consider a number of different MCMC algorithms, variants of the Metropolis algorithm, where one proposes a move from $i$ to $j = i \pm 1$ with probability $\frac{1}{2}$, and accepts it with probability $\min(1, \pi_j/\pi_i)$. It will be interesting to know the conductance of the Metropolis algorithm for different choices of $\pi$.

(a) What is the conductance for the Metropolis algorithm for the constant distribution $\pi_i = \text{const}$? (NB: the distribution must be normalized)

(b) What is the conductance for the Metropolis algorithm for the square-wave distribution $\pi_{2k-1} = \frac{2}{3n}$, $\pi_{2k} = \frac{4}{3n}$ $\forall k \in \{1, \dots, n/2\}$ (for even $n$)?

(c) What is the conductance for the Metropolis algorithm for the V-shaped distribution: $\pi_i = \text{const}|\frac{n+1}{2} - i|$ $\forall i \in \{1, \dots, n\}$?

NB: Next week, we will connect these conductances to the mixing and correlation times. If you have time, you can compute the TVD (by simulation) as a function of time (start from configuration $i = 1$).