Modelling the individual and collective dynamics of the propensity to offend

J.-P. NADAL[1,2], M. B. GORDON[3], J. R. IGLESIAS[4] and V. SEMESHENKO[5]

[1] *Centre d'Analyse et de Mathématique Sociales (CAMS, UMR 8557 CNRS - EHESS), Ecole des Hautes Etudes en Sciences Sociales, 54 Bd. Raspail - 75270 Paris Cedex 06, France,*
email: Jean-Pierre.Nadal@ehess.fr
[2] *Laboratoire de Physique Statistique (LPS, UMR 8550 CNRS - ENS - UPMC Univ. Paris 6 - Paris Diderot Paris 7), Ecole Normale Supérieure, 24 rue Lhomond - 75231 Paris Cedex 05, France*
[3] *Laboratoire TIMC-IMAG (UMR 5525), Université de Grenoble I, Domaine de La Merci - Jean Roget 38706 La Tronche, France*
[4] *Instituto de Física, Universidade Federal do Rio Grande do Sul - 91501-970 Porto Alegre, Brazil*
[5] *Departamento de Economía, Facultad de Ciencias Económicas, Universidad de Buenos Aires, Av. Córdoba 2122 - 1120 AAQ Buenos Aires, Argentina*

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We introduce a general framework for modelling the dynamics of the propensity to offend in a population of (possibly interacting) agents. We consider that each agent has a "honesty index" which parametrizes his probability to abide by the law. This probability also depends on a composite parameter associated to the attractiveness of the crime outcome and of the crime setting (the context which makes a crime more or less likely to occur, such as the presence or not of a guardian). Within this framework we explore some consequences of the working hypothesis that punishment has a deterrent effect, assuming that, after a criminal act, an agent's honesty index may increase if he is caught and decrease otherwise. We provide both analytical and numerical results. We show that in the space of parameters characterizing the probability of punishment, there are two "phases", one corresponding to a population with a low crime rate, and the other to one with large crime rate. We speculate on the possible existence of a self-organized state in which, due to the society reaction against crime activities, the population dynamics would be stabilized on the critical line, leading to a wide distribution of propensities to offend in the population. In view of empirical works on the causes of the recent evolution of crime rates in developed countries, we discuss how change of socio-economic conditions may affect the model parameters - hence the crime rate in the population. We suggest possible extensions of the model that will allow to take into account more realistic features.

Contents

1 Introduction

As put forward by the UCL Jill Dando Institute of Crime Science, "*most criminal acts are not undertaken by deviant psychopathic individuals, but are more likely to be carried out by ordinary people reacting to a particular situation with a unique economic, social, environmental, cultural, spatial and temporal context. It is these reactionary responses to the opportunities for crime which attract more and more people to become involved in criminal activities rather than entrenched delinquency*" [Jill Dando Institute of Crime Science (2001)]. The goal of this paper is to discuss a simple multi-agent model of a society where the agents' law-abidingness is represented by an "honesty index": the higher the honesty index the lower the propensity to offend, as detailed below. We will focus on the effect of the punishment policy on the global crime rate, assuming that the propensity to offend evolves according to the actual risk associated to criminal behavior (either of his own criminal activity, if any, and/or that of other agents). That is, we will consider a learning dynamics whereupon each agent's honesty index may increase or decrease depending on whether criminal offenses are punished or not.

We are not going to enter in a detailed discussion on the origins of crime, nor the effectiveness of different kinds of punishments. There is an extensive and detailed literature on crime from the fields of Law, Economics, Politics, Sociology, Psychology, Religion, and even Mathematics and Physics. A brief review can be found in Gordon et al. (2009), where we discuss common characteristics of criminal activity and some attempts to explain, prevent and deter criminality. Here we will only present the main aspects needed to motivate the model and relate it to the literature.

Societies try to cope with crime by both prevention and punishment. These issues are not independent: prevention also partly relies on the fear of a penalty. Common sense justice requires punishment to be commensurate with the gravity of the offense, and most countries, and not only democratic ones, enforce in a way or another this basic principle. One goal of the model is to discuss the link between the global level of criminal activity and the probability distribution of being caught and punished conditioned on the gravity of the offense. This issue is clearly related to the "Zero Tolerance" [Kelling et al. (1994)] debate (can crime be significantly reduced by imposing a policy of very low tolerance?), which originated from the "broken windows" theory of crime developped by Kelling & Coles (1996), and the controversial issue of whether or not such policy has had any role in the decrease of criminality in some US cities during the 1990s. It has also been argued that there is a link between the crime rate and the probability of conviction (rather than with the severity of the punishment, see Langan & Farrington (1998), Eide (1999)). However the causality relationship behind this statistical correlation has been challenged (see e.g. Smith (1999)), since some statistics show a decrease of criminality whereas the probability of punishment decreases as well. Yet drawing definite conclusions is difficult since crime rates obviously do not depend on the risk of punishment alone - e.g. socio-economics conditions play a major role, as discussed in Rosenfeld & Messner (2009). In any case we will certainly not settle these issues here, yet we think that our approach can bring new arguments in these debates, and be developed within more elaborated and realistic models. In the present work we propose a simple model in order to explore the consequences of a punishment policy, *under the working hypothesis* of a deterrent effect of punishment. In the Discussion we will come back to this hypothesis and on how the model can be adapted to include other assumptions.

The type of crimes we have in mind here are non organized crimes, with gravity ranking from petty crimes to violent crimes whenever they are of an economic nature (hence excluding specific violent offenses such as sexual ones). The issues evoked above have been addressed within an economic approach, based on the assumption that the decision to commit a crime results from a trade off between the expected profit and the risk of punishment. In a now classical article Becker (1968) presents for the first time an economic analysis of costs and benefits of crime, with the aim of developing optimal policies to combat illegal behavior. Considering the social losses from offenses, which depend on their number and on the produced harm, the cost of apprehension and conviction, and the probability of punishment per offense, the model tries to determine how many offenses should be permitted and how many offenders should go unpunished, through minimization of the social loss function. Using a similar point of view, Ehrlich (1975, 1996) develops an economic theory to explain participation in illegitimate activities. He assumes that a person's decision to participate in an illegal activity is motivated by the relation between cost and gain, or risks and benefits, arising from such activity. This model seems to provide strong empirical evidence of the deterrent effectiveness of sanctions.

Recently in Gordon et al. (2009) and Semeshenko et al. (2009) we presented a simple economic model with crime and punishment that stands on the assumption that punishment has a deterrent effect on criminality. In the model each agent's law-abidingness level is quantified by an honesty index, his inclination to abide by the law, that may be psychological, ethical, or a reflection of his educational level and/or socio-economical environment, in a way similar to Bourguignon et al. (2003), and to Fajnzylber et al. (2002) (where a "moral stance" parameter analogous to the honesty index is introduced). However, contrary to the latter works, we assume that this index is not a fixed idiosyncratic characteristic of the agents: it evolves in time according to the risk of apprehension upon performing a crime. We have studied the model (actually several variants of it) through extensive numerical simulations. The main results are: (i) the existence of a sharp phase transition, from a society with high crime-rates to one with low crime-rates as the probability of punishment is increased; this transition exhibits a clear hysteresis effect, suggesting that the cost of reversing a deteriorate situation might be much higher than that of maintaining a relatively low level of criminal activity; (ii) tolerance with respect to small felonies has a global negative consequence because it requires bigger efforts to cope with important crimes in order to keep a given level of honesty. Or the other way round, being relatively tolerant with important crimes (for example white-collar crimes) requires a harsh policy towards minor crimes in order to keep a given honesty level. It is also observed an avalanche effect since a small change in the probability of punishment may reduce or increase the average criminality significatively. In these works the consequences of the criminal activity on the distribution of wealth in the society are also explored, an aspect which will not be discussed here. In the present paper we focus on the analysis of the honesty levels dynamics. Combining analytical and numerical results, this will allow us in particular to discuss in a simpler setting the phase transition observed in Gordon et al. (2009) and Semeshenko et al. (2009).

The paper is organized as follows. In section 2 we describe the model, in sections 3 and 4 we provide some analytical results, and in section 5 we present some numerical simulations. In section 6 we discuss the results and present perspectives.

2 Model: Dynamics of the honesty index

We consider a population of $N >> 1$ agents, where each agent's law-abidingness level is characterized by an honesty index. The higher the honesty index the lower the propensity to offend of the agent. Here (contrary to Gordon et al. (2009), Semeshenko et al. (2009)), we consider that the honesty index takes a number L of possible discrete values, $\{H_0, ..., H_{L-1}\}$ (with $H_k < H_{k+1}$ for every $k \in \{0, .., L-2\}$). In the following we will indifferently denote the kth honesty level by H_k or simply by the level number k ($k \in \{0, ..., L-1\}$).

Starting from initial values drawn at random, the agents' honesty levels are modified in the course of life according to the following dynamics. At each time step an agent is picked at random. With some probability α he commits a crime, and with some probability π he is caught and punished. Then, depending on whether he was or not punished, his honesty index changes. Let us now specify our choices for the probabilities α and π, and the detailed rules for the honesty dynamics.

2.1 Crime and punishment

The probabilities to offend (α) and to be punished (π) may depend on two important ingredients. The first one is the attractiveness of the target for the potential offender: a crime opportunity presents some expected payoff that we represent by an attractiveness variable S_a. Essentially one may think of S_a as quantifying the gravity of the crime (petty crimes corresponding to small values of S_a, crimes with large potential loots to large values of S_a). The other essential ingredient is the "setting". The popular statement "opportunity makes the thief" evokes key ingredients for the analysis and modelling of criminal activities. Some criminologists have stressed that to prevent crime one should focus on the *setting* preluding to the crime, rather than on the criminal himself. This is particularly developed in the Routine Activity Theory of L. Cohen and M. Felson, which states that for a (possibly predatory) crime to occur, three elements must be present:

a target (a possible victim), a place (a given location at a given time, with/without a guardian), a possible offender (see Cohen & Felson (1979) and Clarke & Felson (1993)). Prevention of crime can be enhanced by limiting the settings which are more likely to see the occurrence of a crime. In the present model, one may introduce the prevention effort against crime by a scalar $S_s \geqslant 0$: the larger S_s, the better the prevention (hence the smaller the likeliness of a crime). One may also interpret S_s as a measure of the perceived risk by the potential criminal.

We have now to specify how the probabilities to offend and to be caught depend on S_a and S_s. To simplify further the analysis, we replace this pair of variables by a single variable $S \geqslant 0$, which we will (partially improperly) call the "setting" value. This value S thus aggregates here the two features, the intrinsic target attractiveness on one side and the actual setting on the other side. A more elaborated model should consider separately these two variables, as suggested below. The justification for our "amalgam" is that the behavior of the quantities of interest, with respect to variations in S_a and S_s, are qualitatively the same. In particular, situations with large values of S_s are less frequent than those with small values, and likewise it is natural to assume that the opportunities with a large attractiveness S_a are less frequent than those with a small one (petty crimes). The probability to commit a crime should clearly decrease with increasing S_s, as well as for increasing S_a, at least for large values of S_a. One may note, however, that if the probability to offend results from a balance between expected profit and risk (see also below), one may expect this probability to increase with S_a at small S_a, up to some value S_a^*, and then decrease only for $S_a > S_a^*$, with S_a^* depending on the honesty index of the agent (a larger S_a^* for a smaller honesty index). Again for simplicity we will restrict to the case of a decreasing probability for any S_a value, and in the following we thus consider that probabilities of commiting a crime and of punishment depend on the single composite variable S.

Thus, we assume that the likeliness $\rho(S)$ of the occurrence of a setting (of an opportunity) of a given level S decreases with S, e.g.:

$$\rho(S) = \frac{1}{S_0} \exp -S/S_0, \quad S \geqslant 0 \tag{2.1}$$

This setting distribution, which plays an important role in the following analysis, will be assumed fixed, independent of the criminal activity (in Gordon et al. (2009), Semeshenko et al. (2009), the variable playing a role analogous to S is related to the potential loot of the randomly encountered victim, and evolves with the economic characteristics of the society).

We assume that the probability to commit an offense $\alpha(H, S)$ depends on both the honesty index and the setting. As discussed above, one should have $\partial\alpha(H, S)/\partial H \leqslant 0$, and we restrict the analysis to the case where $\partial\alpha(H, S)/\partial S \leqslant 0$ for all S.

Similarly, we can expect the probability π to be caught to be larger for a serious crime, and even larger if the crime is done in the presence of a guardian. This pdf $\pi(S)$ may be put in correspondence with the level of elucidation of crimes (in France, about 14% for petty crimes, 40% for violent crimes). As a specific example, we will consider:

$$\pi(S) = p_1 - (p_1 - p_0) \exp -\beta S \tag{2.2}$$

for some $\beta > 0$, $0 \leqslant p_0 \leqslant p_1 \leqslant 1$, which satisfies $d\pi/dS \geqslant 0$. In the rest of this paper we will mainly consider how the global criminal activity depends on the parameters p_0 and p_1.

Hereafter we denote by $\alpha_{\pi,k}$ and $\alpha_{1-\pi,k}$ the conditional probabilities, given the honesty index level H_k, to offend and, respectively, to be punished and to not be punished:

$$\alpha_{\pi,k} = \int \pi(S)\ \alpha(H_k, S)\ \rho(S)dS, \tag{2.3}$$
$$\alpha_{1-\pi,k} = \int (1-\pi(S))\ \alpha(H_k, S)\ \rho(S)dS.$$

In the following we explore successively the following hypothesis of increasing complexity:

(1) Neither honesty nor setting dependency (hereafter considered as the reference case). In this case

$$\alpha_{\pi,k} = \alpha_0\ \pi,\ \ \alpha_{1-\pi,k} = \alpha_0\ (1-\pi). \tag{2.4}$$

for some $0 < \alpha_0 < 1$ and $0 < \pi < 1$.

(2) Probability to commit an offense function of the honesty index but not of the setting. We consider, for some $0 < \alpha_0 < 1$,

$$\alpha(H_k) = \alpha_0^{k+1}. \tag{2.5}$$

The mean probability to be punished is $\pi = \int \pi(S)\rho(S)dS$, which gives

$$\alpha_{\pi,k} = \alpha_0^{k+1}\ \pi,\ \ \alpha_{1-\pi,k} = \alpha_0^{k+1}\ (1-\pi). \tag{2.6}$$

Since $0 < \alpha_0 < 1$, the offending probability is a decreasing function of the honesty index.

(3) Probability to offend and to be punished function of the attractiveness of the target and of the "setting". One may envisage two different types of dependency: a separable case, $\alpha(H, S) = f_h(H)\, f_s(S)$ and a non separable one. The preceding cases are trivial examples of a separable function, with no S-dependency. Note that for an arbitrary separable case, $\alpha_{\pi,k} = f_h(H_k) \int \pi(S) f_s(S)\rho(S)dS$ and $\alpha_{1-\pi,k} = f_h(H_k) \int (1-\pi(S)) f_s(S)\rho(S)dS$. In what follows we use (2.2) for $\pi(S)$, and for illustrative purposes we consider the following non separable case:

$$\alpha(H_k, S) = \alpha_0^{(k+1)S} \tag{2.7}$$

for some $0 < \alpha_0 < 1$. This leads to

$$\alpha_{\pi,k} = \frac{p_1}{1 - S_0(k+1)\ln\alpha_0} - \frac{p_1 - p_0}{1 + S_0(\beta - (k+1)\ln\alpha_0)}$$
$$\alpha_{1-\pi,k} = \frac{1-p_1}{1 - S_0(k+1)\ln\alpha_0} + \frac{p_1 - p_0}{1 + S_0(\beta - (k+1)\ln\alpha_0)}. \tag{2.8}$$

(4) Finally we mention a possible formulation which is more in the spirit of the economic approaches of Fajnzylber et al. (2002), Bourguignon et al. (2003), and Gordon et al. (2009), and which gives an example of a modelling where the attractivity S_a and the setting S_s have distinct roles. Both the probabilities to offend and to

be punished are functions of the attractivity S_a and of the perceived risk, parameterized by S_s. Given a probability to be punished $\pi(\mathbf{S})$, with $\mathbf{S} = \{S_a, S_s\}$, there is an expected cost of punishment, $c(H,\mathbf{S})\pi(\mathbf{S})$, where $c(H,\mathbf{S})$ is typically an increasing function of the honesty index H for each given value of $\mathbf{S}$, and an increasing function of S_a and S_s for each H value. If one assumes for the criminal an utility $U(S_a)$ (with U an increasing function of S_a, e.g. $U(S_a) = S_a$), then we can consider that the offending probability α is function of $U(S_a) - c(H,\mathbf{S})\pi(\mathbf{S})$. A reasonable hypothesis could be:

$$\alpha(H,\mathbf{S}) = \Theta(U(S_a) - c(H,\mathbf{S})\pi(\mathbf{S}))\, \alpha_0 \exp -\gamma(H,S_s)S_a \tag{2.9}$$

where Θ is the Heavyside distribution ($\Theta(u) = 1$ if $u > 0$, and 0 otherwise), and $\gamma(H,S_s)$ is larger for larger H values as well as for larger S_s values. Another reasonable choice for α would be:

$$\alpha(H,\mathbf{S}) = \Theta(U(S_a) - c(H,\mathbf{S})\pi(\mathbf{S}))\, \frac{\alpha_1}{1 + \frac{\alpha_1-\alpha_0}{\alpha_0}\exp -\gamma(H,S_s)(U(S_a) - c(H,\mathbf{S})\pi(\mathbf{S}))} \tag{2.10}$$

To keep the analysis simple enough, and to focus more specifically on the dynamics of the honesty levels, we will not explore here the consequences of such a probability to be punished that is a function of the risk. The analysis of this more involved case will be the subject of a future work.

2.2 Honesty dynamics

Let us now specify how the honesty index of an agent evolves according to his experience. We will consider the following extreme cases:

- **A.** No externalities (i.e. no social influence): the agent's honesty index depends on his own past experience alone. Whenever the agent commits a crime and is punished, with probability ϵ_+ his honesty index increases from its current level to the one just above (no change if it is at the maximal value, H_{L-1}). If he is not punished, with probability ϵ_- his honesty index decreases to the level just below (no change if it is at the minimal value, H_0).
- **B1.** Global social influence (instantaneous): at each time step, for each committed crime, *every* agent updates his honesty level, depending on whether the crime has been punished or not. For each agent the updating rule is the same as in case **A**, so that for every punished (resp. non punished) crime there is a fraction ϵ_+ (resp. ϵ_-) of agents who increase (resp. decrease) their honesty index.
- **B2.** Global social influence (variant: delayed, with threshold): each agent is influenced by the knowledge of the fraction of crimes which are punished in a period of N elementary time steps, and updating of honesty levels occur only at the end of each period. After one such period, if the fraction of punished crimes (which in this model is equivalent to the elucidation rate) is larger than some threshold value, say 30%, then every agent increases his honesty index with probability ϵ_+; otherwise, every agent decreases his honesty index with probability ϵ_-.

3 Mathematical analysis

We are mainly interested in the stationary states that result from these dynamics. In the case without externalities, **A**, and in the case with externalities **B1**, a simple but crude argument, identical to the one given in Gordon et al. (2009), allows to predict the existence of a phase transition from a society with high crime-rates to one with low crime-rates. Indeed, consider the simplest case of a uniform probability α_0 and a probability π of being punished (our reference case above). For each agent, the mean variation Δh of its honesty index is proportional to: {[the number of crimes with arrest] *times* [the probability of increasing the honesty level]} *minus* {[the number of crimes without arrest]*times* [the probability of decreasing the honesty level]}, that is:

$$\Delta h = (\epsilon_+ \pi - \epsilon_-(1-\pi))\; C\; \delta h,$$

where δh is the difference between two successive honesty levels, and C the typical number of crimes between two updates of the honesty index ($C = \alpha_0$ for case **A** and $C = \alpha_0 N$ for case **B1**). There is thus a critical value

$$\pi_c = \frac{\epsilon_-}{\epsilon_- + \epsilon_+}. \tag{3.1}$$

For $\pi < \pi_c$, every agent's honesty index will decrease, whereas if $\pi > \pi_c$, every agent's honesty index will increase. We will see below a more rigorous analysis.

In Case **A**, for finite L there is no sharp transition, but a crossover where the honesty index stationary distribution changes continuously from a distribution biased towards the small index values to a distribution biased towards the large index values.

Let us now turn to a more rigorous analysis of the dynamics.

3.1 Single agent dynamics

We first consider case **A**: since agents' behaviours are not coupled, we can consider the dynamics for a single agent.

Let $P_t(k)$ be the probability to have the honesty level k at time t. Starting from some initial probability $P_0(k)$, one can easily write the Master equations

$$\begin{aligned}
P_{t+1}(0) - P_t(0) &= \epsilon_- \alpha_{1-\pi,1} P_t(1) - \epsilon_+ \alpha_{\pi,0} P_t(0) \\
P_{t+1}(L-1) - P_t(L-1) &= \epsilon_+ \alpha_{\pi,L-2} P_t(L-2) - \epsilon_- \alpha_{1-\pi,L-1} P_t(L-1) \\
\text{for } \; k = 1, ..., L-2, & \\
P_{t+1}(k) - P_t(k) &= \epsilon_+ \left[\alpha_{\pi,k-1} P_t(k-1) - \alpha_{\pi,k} P_t(k)\right] \\
&\quad + \epsilon_- \left[\alpha_{1-\pi,k+1} P_t(k+1) - \alpha_{1-\pi,k} P_t(k)\right]
\end{aligned} \tag{3.2}$$

where we recall that $\alpha_{\pi,k}$ (respectively $\alpha_{1-\pi,k}$) is the probability to commit an offense and to be punished (resp. not punished), given the honesty level k.

3.2 Stationary distribution

Let us compute the stationary distribution $P_\infty(k)$. From (3.2) one gets, for $k = 0, ..., L-2$,

$$P_\infty(k) = \left[\prod_{j=k+1}^{L-1} \lambda_j \right] P_\infty(L-1) \tag{3.3}$$

with, for $k = 1, ..., L-1$,

$$\lambda_k \equiv \frac{\epsilon_- \, \alpha_{1-\pi,k}}{\epsilon_+ \, \alpha_{\pi,k-1}} \tag{3.4}$$

and the normalization constraint $\sum_k P_\infty(k) = 1$ gives

$$P_\infty(L-1) = 1 \Big/ \left[1 + \sum_{k=0}^{L-2} \prod_{j=k+1}^{L-1} \lambda_j \right]. \tag{3.5}$$

Specific cases

(1) Reference case, $\alpha_{\pi,k} = \alpha_0 \, \pi$ (equation (2.4)). Then

$$\lambda_k = \frac{\epsilon_- \, (1-\pi)}{\epsilon_+ \, \pi} \equiv \lambda, \tag{3.6}$$

and for $k = 0, ..., L-1$,

$$P_\infty(k) = \lambda^{L-k-1} \, \frac{1-\lambda}{1-\lambda^L} \tag{3.7}$$

The occupation of honesty levels is thus either concentrated near the highest or near the lowest level, depending on whether λ is smaller or larger than 1 respectively. The critical value $\lambda_c = 1$ is obtained for

$$\pi = \pi_c \equiv \frac{\epsilon_-}{\epsilon_- + \epsilon_+}. \tag{3.8}$$

(which gives the transition at $\pi = 1/2$ for $\epsilon_- = \epsilon_+$). This already gives the spirit of the transition observed in the more involved models.

(2) Honesty-dependent probability to commit an offense, uniform probability to be punished. With $\alpha_{\pi,k} = \alpha_0^{k+1} \, \pi$ (equation (2.6)) one gets exactly the same result as in the previous case.

(3) Probability to offend and to be punished both function of the setting S. If $\alpha(H,S)$ depends only on S but not H, the result is again quite similar to the previous cases. Here

$$\lambda = \frac{\epsilon_- \, \alpha_{1-\pi}}{\epsilon_+ \, \alpha_\pi}, \tag{3.9}$$

with $\alpha_\pi = \int \pi(S) \, \alpha(S) \, \rho(S) dS$ and $\alpha_{1-\pi} = \int (1-\pi(S)) \, \alpha(S) \, \rho(S) dS$ (see eq. 2.4).

For $\alpha(H,S)$ depending on both H and S, under the specific hypotheses listed above, with α given by equation (2.7), the probabilities $\alpha_{\pi,k}$ and $\alpha_{1-\pi,k}$ are given by eq. (2.8). This is the most interesting case which we analyze in what follows.

3.3 Alternative approach: population dynamics

Instead of considering the single agent dynamics, one may consider the dynamics of the population of honesty indices, that is of the fractions x_k of honesty index of level k (with $\sum_{k=0}^{L-1} x_k = 1$). This approach can be easily adapted to cases with social interactions, as discussed in the next section. Assume that at each time t one agent is picked at random and the dynamics described in the previous section is applied. Then there is a probability $x_k(t)$ to pick an honesty index of level k, and this level may loose one element, and contribute to populate the level $k+1$ or the level $k-1$, depending on whether an offense is committed and whether the criminal is punished or not. One can thus write the dynamics for the $\mathbf{x}(t) = \{x_k(t), k = 0, ..., L-1\}$:

$$
\begin{array}{lll}
\text{with probability} & x_k(t)\alpha_{\pi,k}\epsilon_+ \ (k \leqslant L-2), & x_k(t+1) = x_k(t) - 1/N \\
 & & x_{k+1}(t+1) = x_{k+1}(t) + 1/N \\
 & & (\text{and for } j \neq k, k+1, x_j(t+1) = x_j(t)) \\
\text{with probability} & x_k(t)\alpha_{1-\pi,k}\epsilon_- \ (k \geqslant 1), & x_k(t+1) = x_k(t) - 1/N \\
 & & x_{k-1}(t+1) = x_{k-1}(t) + 1/N \\
 & & (\text{and for } j \neq k, k-1, x_j(t+1) = x_j(t)) \\
\text{and otherwise} & \text{for all } j, x_j(t+1) = x_j(t). &
\end{array}
\tag{3.10}
$$

One can write the expected value of $\mathbf{x}(t+1)$ given $\mathbf{x}(t)$:

$$
\begin{aligned}
&\text{for} \quad k = 1, ..., L-2: \\
E[x_k(t+1)|\mathbf{x}(t)] - x_k(t) &= -\frac{1}{N} x_k(t)(\alpha_{\pi,k}\epsilon_+ + \alpha_{1-\pi,k}\epsilon_-) \\
&\quad + \frac{1}{N} x_{k+1}(t)\alpha_{1-\pi,k+1}\epsilon_- + \frac{1}{N} x_{k-1}(t)\alpha_{\pi,k-1}\epsilon_+ \\
&\text{and:} \\
E[x_0(t+1)|\mathbf{x}(t)] - x_0(t) &= -\frac{1}{N} x_0(t)\alpha_{\pi,0}\epsilon_+ + \frac{1}{N} x_1(t)\alpha_{1-\pi,1}\epsilon_- \\
E[x_{L-1}(t+1)|\mathbf{x}(t)] - x_{L-1}(t) &= -\frac{1}{N}\alpha_{1-\pi,L-1}\epsilon_- + \frac{1}{N} x_{L-1}(t)\alpha_{\pi,L-2}\epsilon_+
\end{aligned}
\tag{3.11}
$$

One gets the fixed point equation for the mean value of $\mathbf{x}$ (average over all histories) by setting to zero the left hand side of the above equation. The solution is identical to (3.3)-(3.5): for $k = 0, ..., L-2$,

$$
x_k = \left[\prod_{j=k+1}^{L-1} \lambda_j \right] x_{L-1} \tag{3.12}
$$

with the λ_k given by (3.4), and

$$
x_{L-1} = 1 \Big/ \left[1 + \sum_{k=0}^{L-2} \prod_{j=k+1}^{L-1} \lambda_j \right]. \tag{3.13}
$$

Figure 1 shows a simulation with 5 levels of honesty, and the $\alpha_{\pi,k}$ randomly generated (in such a way that the constraints $0 < \alpha_{\pi,k+1} < \alpha_{\pi,k} < 1$ are satisfied).

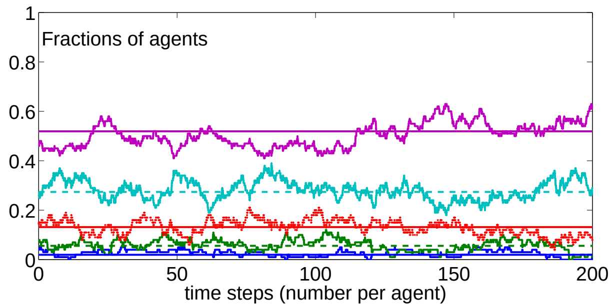

FIGURE 1. Simulation with $L = 5$ levels of honesty, 100 agents. The plot shows the time evolution of the fractions of agents having a given honesty index value. Horizontal lines: theoretical means obtained from the numerical computation of (3.12), (3.13).

3.4 Critical line

At the begining of this section we gave a simple argument predicting the existence of a sharp transition from a society with high crime-rates to one with low crime-rates. In this dynamics with no interaction between agents, for any finite L there is actually no sharp transition but a continuous crossover from one regime to the other. It is only in the limit of L going to infinity that one gets a sharp transition, with only the first or the last honesty level populated outside the transition line.

There is however, for any finite L, a critical line where the stationary distribution of honesty levels changes from a decreasing to an increasing function of the honesty level, as already mentioned section 3.2. The line is obtained by writing $\lambda = 1$ in the simplest cases where the value of λ_k is independent of k (see (3.6) and (3.9)), and in the more general case by the condition $\left[\prod_{j=1}^{L-1} \lambda_j\right] = 1$.

The critical line is shown on Figure 2 for the model characterized by p_0 and p_1, Eq. (2.2).

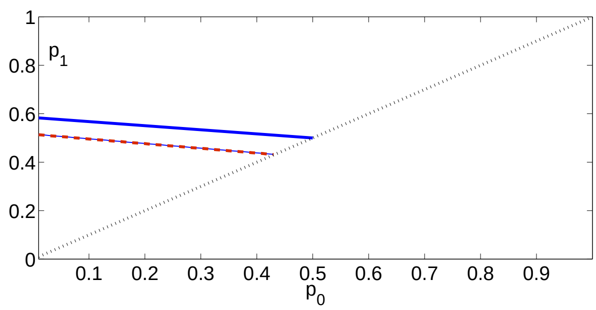

FIGURE 2. Dashed line: Transition line for the single agent dynamics. Solid line: Transition line in the case of interacting agents, model **B1**, for the same parameters: reaching the non criminal state, above the critical line, is harder with social influence.

4 Global social influence

4.1 Model B1: instantaneous global interactions

4.1.1 *Stationary distribution*

In the case **B1** of a global social influence, assuming that every agent is influenced in the same way by every criminal event in the society, and not only by his own past criminal history, one can easily adapt the previous calculation. Here, whenever a criminal is punished (resp. not punished), there is a randomly chosen fraction ϵ_+ (resp. ϵ_-), of the population which increases (resp. decreases) his honesty index (except at the boundaries $k = 0$ and $k = L - 1$). More explicitly, with

$$\langle \alpha_\pi \rangle(t) \equiv \int \pi(S) \sum_k x_k(t)\ \alpha(H_k, S)\ \rho(S) dS = \sum_k x_k(t)\ \alpha_{\pi,k} \tag{4.1}$$

(and similarly for $\langle \alpha_{1-\pi} \rangle(t)$), one has,

$$\begin{aligned} \text{with proba. } \epsilon_+ \langle \alpha_\pi \rangle, \quad & x_k(t+1) = x_{k-1}(t), k = 1, .., L-2 \\ & x_0(t+1) = 0 \\ & x_{L-1}(t+1) = x_{L-1}(t) + x_{L-2}(t) \end{aligned}$$

and

$$\begin{aligned} \text{with proba. } \epsilon_- \langle \alpha_{1-\pi} \rangle, \quad & x_k(t+1) = x_{k+1}(t), k = 1, .., L-2 \\ & x_0(t+1) = x_0(t) + x_1(t) \\ & x_{L-1}(t+1) = 0 \end{aligned}$$

and otherwise (that is with probability $1 - \epsilon_+ \langle \alpha_\pi \rangle - \epsilon_- \langle \alpha_{1-\pi} \rangle$) there is no change.

As in the previous case, one gets the fixed point considering the average over all possible histories. Here the solution is:

$$x_k = \lambda^{L-k-1}\ \frac{1-\lambda}{1-\lambda^L} \tag{4.2}$$

with λ obtained as solution of

$$\lambda = \frac{\epsilon_-}{\epsilon_+}\ \frac{\sum_{j=0}^{L-1} \alpha_{1-\pi,j}\ \lambda^{L-1-j}}{\sum_{j=0}^{L-1} \alpha_{\pi,j}\ \lambda^{L-1-j}} \tag{4.3}$$

Note that whenever $\alpha_{\pi,j} = \pi \alpha_j$ and $\alpha_{1-\pi,j} = (1-\pi)\alpha_j$ for some $\pi < 1$, one recovers the expression (3.6) for λ. Interestingly, since here λ is obtained as the solution of a polynomial of degree L, there might exist several equilibria. However. the simulations always converged to a single one, and we leave for further work the more detailed analysis of this equation.

4.1.2 *Critical line*

In the space of parameters, which is the half-plane $\{p_0 \leqslant p_1\}$, there is the critical curve which separates the domain where the honesty distribution is a decreasing function of the honesty level, from the domain where the honesty distribution is an increasing function of the honesty level. On the critical line the honesty levels are equally populated. This curve is obtained by setting $\lambda = 1$ in (4.3). This critical condition can be written as:

$$\frac{\bar{\alpha}_\pi}{\bar{\alpha}} = p_c \tag{4.4}$$

where

$$p_c = \frac{\epsilon_-}{\epsilon_- + \epsilon_+} \tag{4.5}$$

and with $\bar{\alpha}_\pi = \sum_k \alpha_{\pi,k}$, $\bar{\alpha} = \sum_k (\alpha_{\pi,k} + \alpha_{1-\pi,k})$. This equation can be understood as defining a critical fraction of punished crimes when all honesty levels are equally populated. For the specific choice of $\pi(S)$ given by (2.2), the critical curve is a straight line,

$$a\, p_0 + (1-a)\, p_1 = p_c \tag{4.6}$$

where $0 \leqslant a \leqslant 1$ is given by

$$a = \frac{\bar{\alpha}[\beta]}{\bar{\alpha}[0]} \tag{4.7}$$

with

$$\bar{\alpha}[\beta] \equiv \int e^{-\beta S} \sum_{k=0}^{L-1} \alpha(H_k, S)\, \rho(S) dS \tag{4.8}$$

Note that p_c is the critical value when the probability to be punished is independent of the setting, that is for $p_0 = p_1$. An example of critical line is shown on Fig. 2 together with the one for the model without social interaction.

The critical line depends through a on the probability distribution of the setting value S. Analysis of this dependency gives reasonnable results. Indeed, if there is almost no situations benefiting from a good prevention, then the setting distribution is peaked near the null value. As a result, the parameter a is close to 1, so that to enforce a non criminal state one has to increase the punishment probability p_0 for the less serious crimes. On the contrary, for a broad distribution of S, a will be smaller and thus one can afford a smaller value for p_0.

For an arbitrary $\pi(S)$ (with $\partial\pi/\partial S \geqslant 0$), the critical curve is no more a straight line. However (4.6) gives the spirit of what to expect in the general case: the smaller p_0, the larger must be p_1 to reach the critical line. This is indeed also what is found in the simulations in Gordon et al. (2009), Semeshenko et al. (2009) .

On the critical line, the stationary distribution is the uniform distribution (all the honesty levels are equally populated). Hence, for large L, far from the critical line, essentially only the smallest or the largest level of honesty is present in the population, whereas near the critical line there is a more or less flat distribution of the honesty index.

4.2 Model B2: delayed global interactions (with threshold)

In the variant **B2**, one assumes that the updating of the honesty index depends on a threshold on the fraction of punished crimes: if the elucidation rate is larger than some threshold θ, the honesty index increases with probability ϵ_+, otherwise it decreases with probability ϵ_-. In the reference case, (2.4), the mean fraction of punished crimes is simply

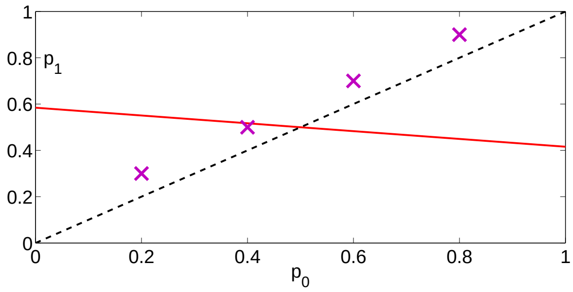

FIGURE 3. Critical line (solid line) in the half plane ($p_0 \leqslant p_1$), and the values (crosses) at which the simulations corresponding to Fig. (4) have been done.

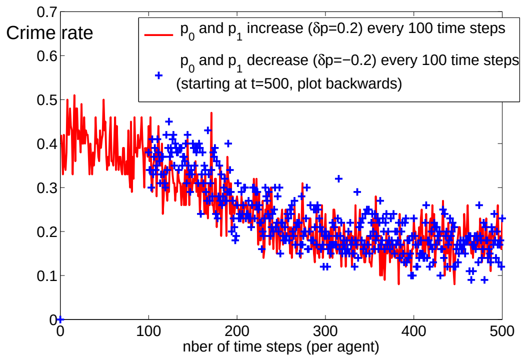

FIGURE 4. Time evolution of the crime rate with probabilities p_0 and p_1 changing every 100 time steps per agent. Simulation with 5 levels of honesty, 100 agents.

equal to π. Hence the critical value is obviously $\pi_c = \theta$: the punishment rate has to match the psychological level which is necessary to provoke an increase in honesty levels. In the general case, at each time t the condition for having (in average) an increase of the honesty indices is

$$\frac{\langle \alpha_\pi \rangle}{\langle \alpha \rangle} > \theta \tag{4.9}$$

with, as in the preceding section, $\langle \alpha_\pi \rangle = \sum_k x_k \; \alpha_{\pi,k}$, and $\langle \alpha \rangle = \sum_k x_k \; (\alpha_{\pi,k} + \alpha_{1-\pi,k})$. The critical line, where the change from a decreasing to an increasing distribution of honesty levels occurs, is given by the equality $\frac{\langle \alpha_\pi \rangle}{\langle \alpha \rangle} = \theta$ with all honesty levels equally populated, that is for

$$\frac{\bar{\alpha}_\pi}{\bar{\alpha}} = \theta \tag{4.10}$$

with $\bar{\alpha}_\pi = \sum_k \; \alpha_{\pi,k}$, $\bar{\alpha} = \sum_k \; (\alpha_{\pi,k} + \alpha_{1-\pi,k})$. This is the same condition as (4.4) obtained in the case **B1**, but with θ in place of p_c – that is here the critical condition is independent of ϵ_+ and ϵ_-.

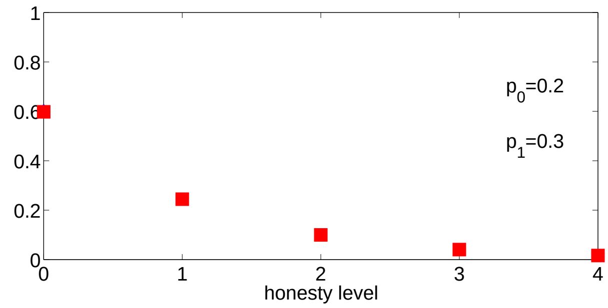

FIGURE 5. Theoretical distribution of the honesty levels for particular values of p_0 and p_1 (in parameter space, the lower left point shown in Fig 3).

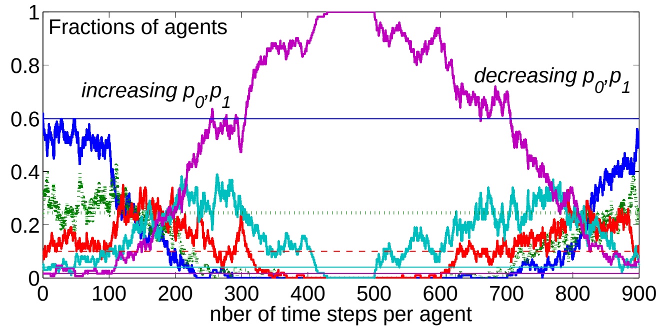

FIGURE 6. Time evolution of the fractions of agents with a given honesty level, with the same protocol as in Fig. 4. Simulation with 5 levels of honesty, 100 (non interacting) agents. Horizontal lines: theoretical means for the initial values of (p_0, p_1), as shown on Fig. 5.

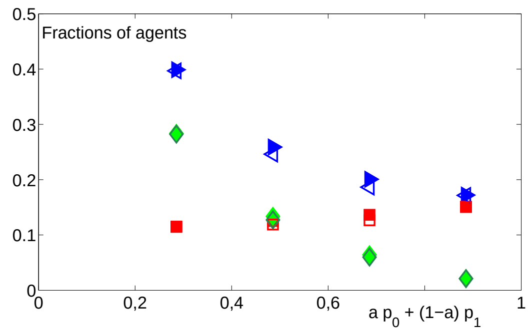

FIGURE 7. Stationary mean fractions of agents committing a crime (triangles), and punished (squares), and not punished (diamonds), as function of $ap_0 + (1-a)p_1$, which gives the distance to the critical line ($ap_0 + (1-a)p_1 = p_c$, with here $p_c = 1/2$, $a = 0.1442$). Protocol: sequential increase (full symbols) then decrease (empty symbols) of p_0 and p_1. Simulation with 5 levels of honesty, 100 agents, same parameters as for figures 3-6. All shown quantities exhibit a smooth behaviour with no hysteresis.

5 Simulations

We have performed numerical simulations, with results for the case of non interacting agents illustrated on Figures 3 to 7. These simulations correspond to the case (2.1), (2.2), (2.7), with $L = 5$ honesty levels, $\epsilon_+ = \epsilon_- = 1/2$, $\beta_0 \equiv \beta S_0 = 20$, $\gamma = S_0 L \ln(1/\alpha_0) = 5$. We considered $N = 100$ agents and applied the following protocol: starting with $p_0 = 0.2, p_1 = 0.3$, p_0 and p_1 increase each by $\delta p = 0.2$ every 100 time steps (1 time step = N elementary events, one event being the selection of an agent at random and determining whether he commits or not a crime and whether he is or not punished) up to $t = 500$; then they decrease by $\delta p = -0.2$ every 100 time steps. Results of simulations with global social influence (with same parameters and same protocol as for non interacting agents) are illustrated on Figures 8 to 10 for both variants, **B1** and **B2**.

In contrast with what is observed in Semeshenko et al. (2009), there is no hysteresis. Additional ingredients have to be taken into account in order to recover the hysteresis. In Semeshenko et al. (2009), the economic characteristics of the population are affected by the criminal activity. In the present model, the equivalent effect would be a slow evolution of the distribution of S as the criminal activity varies. This would clearly lead to hysteresis effects.

An interesting aspect which can be seen on Fig. 8 (right) and 9 (bottom-left) is that, as the punishment probabilities increase according to the protocol described above, the number of punished agents remains almost constant.

The results for the two variants of the model with social influence are quite similar, except for the location of the transition line, which depends on $p_c = \epsilon_-/(\epsilon_- + \epsilon_+)$ in the model **B1** and on θ in the model **B2**, two independent parameters.

6 Discussion

We have presented a simple model of criminal behavior in a society where the agents are characterized by a "honesty index" that evolves according to whether the crimes are punished. The probability that an agent commits a crime is influenced both by his honesty index and by the probability of being punished. The model allows to reproduce the main features observed in a more elaborated model discussed in Gordon et al. (2009), Semeshenko et al. (2009).

We have studied the dynamics of the honesty indices under different simplifying assumptions. We have shown that there exists a critical line in the space of parameters of the punishment probability: above this line the latter is large, the honesty indices of the overall population increase with time so that the crime-rates decrease reaching eventually a stationary level. Below the critical line crime-rates soar because the honesty indices decrease with time. On the critical line, the distribution of honesty index is uniform. Since enforcing the law is costly, it is tempting to conjecture that many societies adapt police and justice ressources (in particular through the adaptation of p_0 and p_1) in order to just avoid the collapse into the criminal state. This would lead to a self-organized critical state, with thus a more or less uniform distribution of honesty levels in the population.

In the future we plan to explore variants of the model taking into account more realistic features. In particular the case evoked at the end of section 2, where the offending

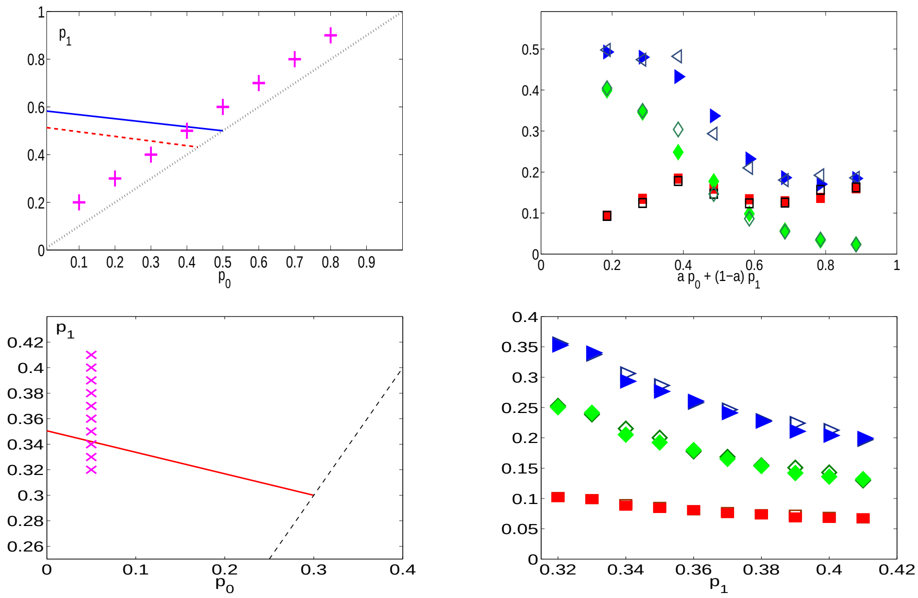

FIGURE 8. Case with social influence. Top: model **B1**, Bottom: model **B2** with $\theta = 0.3$. Left: In the half plane $(p_0 \leqslant p_1)$, critical line (solid line) and values of (p_0, p_1) (crosses) chosen for the simulations. In the case of model B1, the dashed line shows the critical line for the same parameters, without social influence. Right: Stationary mean fractions of agents committing crime (triangles), and punished (squares), and not punished (diamonds) – in the case of **B1**, the abcissa is $ap_0 + (1-a)p_1$, which gives the distance to the critical line. Simulations with 5 levels of honesty, same protocol as in Fig. 7 (full symbols: increasing the punishment probabilities; empty symbols: decreasing the punishment probabilities).

probability depends on a balance between risk and expected profit. In such case, the critical line in the plane p_0, p_1 will no more be a straight line. One may still write the critical condition as (4.6), however the parameter a defined by (4.7) depends now on p_0 and p_1: the critical line is thus distorted, and it will be interesting to see how.

One may also consider local (instead of global) social influences. Glaeser et al. (1996) study a simple case with next nearest neighbor interactions betwen agents "living" in a one dimensional space. Although this allows them to make some comparison with empirical data, the $1d$ case has specific properties which make it non generic. In Barthelemy et al. (2010) some consequences of social influence are explored in a related model but with agents located on a 2-dimensional lattice (the willingness to commit a crime of an agent being influenced by the behavior of his 4 neighbors). This allows to discuss the stability of an "island" of criminal activity. Other type of connectivity patterns between agents should be explored, such as small-world networks (see e.g. Newman (2000)), a structure frequently observed for social networks.

We have studied a dynamics in which, in the long run, the initial value of each agent's honesty index is forgotten. A different approach would be to assume an idiosyncratic reference value for the honesty index, which does not evolve in time – or evolves with a very slow dynamics. In the absence of any kind of deterrence policy or of social influence,

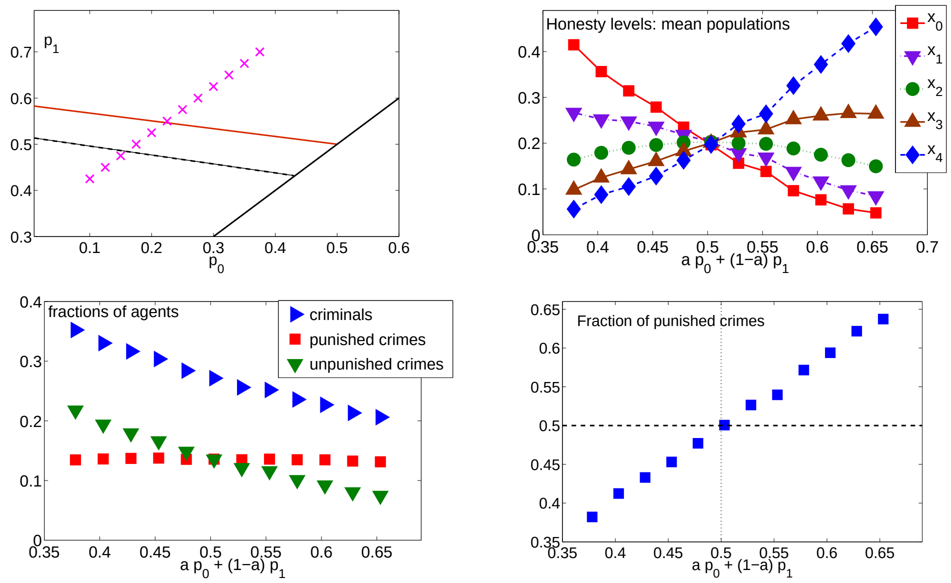

FIGURE 9. Case with instantaneous social influence (**B1**). Top, left: in the half plane ($p_0 \leqslant p_1$), critical line (solid line) and values of (p_0, p_1) (crosses) chosen for the simulations presented here. Top, right: fractions of agents with given honesty levels. Bottom, left: mean fractions of agents committing a crime, being punished and not punished. Bottom, right: fraction of punished crimes. This fraction is 1/2 at the transition.

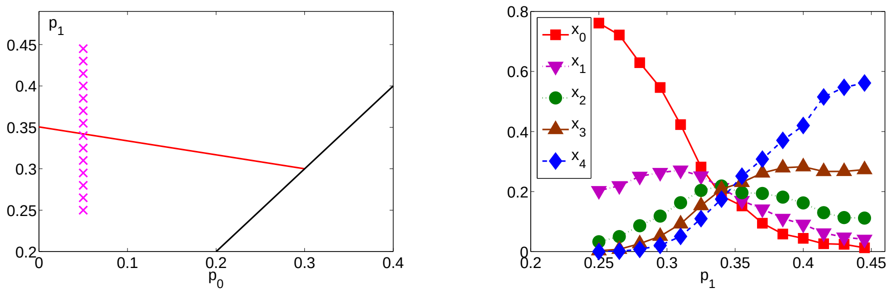

FIGURE 10. Case with delayed social influence and threshold (**B2**). Left: critical line (solid line) and values of (p_0, p_1) (crosses) chosen for the simulations whose results are presented on the right panel. Threshold $\theta = 0.3$, other parameters as for the case **B1**. Right: fractions of agents with given honesty levels.

the honesty index would relax towards this idiosyncratic value. Such a case is studied in this issue by Berestycki & Nadal (2010).

As mentioned in section 2, a more refined model should consider both the attractiveness and the setting instead of the composite variable S. Yet, as shortly discussed at the end of section 2.1, the model presented here already allows for some discussion of the role of the setting and attractiveness distributions. As proposed by Rosenfeld & Messner (2009), an upturn in economic conditions may explain the decrease in criminal activity observed in the US and European burglary rates. In the context of the present model, a change in the economic conditions can be translated into a change in the distribution of the attractiveness distribution – the likeliness that a target appears as attractive becomes smaller –, as well as in the setting distribution – e.g. more parents are in position to exert control on their children, and more ressources can be allocated to prevention. As a result, the distribution of the setting value S becomes more peaked near the origin, which lowers the critical line, that is the non criminal state exists at smaller values of the punishment probability. We draw this conclusion from the analysis done section 4.1.2, but more work is needed in order to see whether this is a generic property or not.

In the model discussed here, the main working hypothesis is that depending on whether an agent is punished or not, his honesty index decreases or increases. However it is known that juveniles who go through the criminal justice system may have a higher criminal activity than those who are not caught (see e. g. Smith (1999)). Within our framework one can study many different dynamical rules for the honesty index, by simply choosing other rules than those postulated in section 2.2. However, taking into account empirical facts such as the effects of imprisonment may require to complexify further the model – in particular by introducing different types of punishments, and the associated risks of having a higher propensity to offend after punishment.

Clearly the presented results are preliminary, but the framework introduced in this contribution allows to test different scenarios by considering many possible variants. Some of them, like the ones discussed here, are simple enough to allow for a rigorous mathematical analysis.

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