A formal approach to market organization: choice functions, mean field approximation and maximum entropy principle

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Abstract

We present a formal, although simple, approach to the modeling of a buyer behavior in the type of markets studied in [11]. We compare possible buyer's choice functions, such as linear or logit function. We study the resulting behaviour, showing that they depend on some convexity properties of the choice function. Our results make use of standard Statistical Physics concepts and techniques. In particular we use the "mean field approximation" to derive the long term behaviour of buyers, and we show that the standard "logit" choice function can be justified from a general optimization principle, leading to an *exploration-exploitation* compromise.

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1 Introduction

In the bounded rationality approach to agents behaviour, individuals are not supposed to have a perfect knowledge of the economic system in which they participate. The simplest view is that their economic choices are based on very rudimentary rules. Of course the question arises as to the origin of the chosen rules. Obviously, these rules reflect some "knowledge" about the world, whether innate or learned. Learning involves using past or present information, but in most cases agents have a lot of available information and a first

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step in learning is simply coding this information by some set of variables. In a neural net approach for instance, coding is achieved by architecture and synaptic weights, in time series analysis by the linear coefficients of an ARMA model etc. As we show here and as proposed in other economic models such as the discrete choice theory of product differentiation[1], coding can be very direct and simple, such as a list of weighted profits or utility functions. The second step of a choice theory is the definition of a choice function which describes the choice of the agents based on the coded available information. Discrete choice theory for instance uses probabilistic choice among possible actions, such as purchasing a given product or visiting a given shop, where the probability is a fraction of a function of the coded information divided by the sum of such functions corresponding to the all available choices. The simplest possible function is the linear function, but we will show here that behavior of agents based on such a linear function is rather peculiar.

In this paper we address the question of the role and interpretation of choice functions. We limit the discussion to the case where agents only use past private information and have no access to public information, concerning for instance the behavior of other agents. After defining the general framework in section 2, in section 3, we make use of the *mean field approximation*[7] in order to compare the qualitative behaviour obtained from different choice functions. We show that for some of them, such as the so-called logit function[1, 5, 10], *phase transitions* can be observed. Next, in section 4, we show that the logit choice function can be derived from an optimization principle, analogous to the maximum entropy principle used in physics an inference theory. We discuss the possible meanings of such an optimization scheme. Possible avenues for future research are given in the Conclusion.

2 A general framework for the study of buyer's dynamics

We are interested in the modeling of buyers'behaviour. In this paper we consider that each buyer makes use of previous experience to select a seller. Since we want to emphasise the role of the individual buyers' choice functions, we assume that there is *no direct interaction between buyers*. We also assume that information on a seller is only obtained on the occasion of a transaction with him (there are no "posted" prices; see [11] for a real instance of such a condition). The general framework, illustrated in figure 1, is thus the following. Each time a buyer makes a transaction with a seller, he acquires some information about what he can expect from this particular seller (quality of goods, profit,...). This information will be encoded in some way which updates the previously acquired information about sellers. This stored information is the input to the (possibly probabilistic) choice (or decision) rule used by the buyer in order to select a seller for the next transaction.

information (observations) → $\boxed{\text{coding}}$ → $\boxed{\text{choice function}}$ → decision (seller selection)

Figure 1: The general model

Let us illustrate this general model with simple specific examples. Considering one given buyer, we will denote by $\mathbf{J} = \{J_j, j = 1, ..., N\}$ the stored information, J_j being the information concerning the jth seller. In the simplest case, J_j is a scalar. For instance, J_j may be the profit obtained the last time the buyer dealt with the jth seller; or it may be some moving average value of past profits from seller j, e.g.,

$$J_j(t) \;=\; \gamma \; J_j(t-1) \;+\; (1-\gamma) \; \pi_j(t) \tag{1}$$

where $\pi_j(t)$ is the actual profit at time t if j is the seller visited at time t, and $\pi_j(t) = 0$ otherwise. The parameter γ is smaller than 1: events far in the past are progressively forgotten. The normalization in (1) is such that for a time independent profit $\pi_j(t) = \pi$, if j is always chosen, one has at each time $J_j = \pi$. An updating rule such as (1) is an example of a coding scheme. One may consider more involved rules, taking into account not only the mean profit obtained from each seller, but also some information on the frequency of visits to each seller. In the following, we will only consider the case of a single variable J_j stored for each seller. We note, however, that our approach can be easily generalised to more complicated situations.

The choice rule, to be denoted by $P(j \mid \mathbf{J})$, is the probability that the buyer choose the jth seller based on his knowledge of the stored information $\mathbf{J}$. A large class of model encountered in the literature corresponds to a decision rule defined by

$$P(j \mid \mathbf{J}) \;=\; \frac{f(J_j)}{\sum_{k=1}^{N} \; f(J_k)} \tag{2}$$

where $f(.) \; \geq 0$ is some a priori chosen function - to be called below the *choice function*. If, as above, J_j is a scalar, f is a real valued function of a single variable. Typical choices for f are, a linear or affine function [9], or an exponential function[1, 5], in which case the choice rule is called the *logit* rule.

3 Choice functions and phase transition

3.1 Mean field approximation

In this section we consider, for a general choice function, the mean field approximation as used in [11] for the logit case (the mean field approach has been also applied to other economic problems, see e.g. [2, 6]). We consider a buyer whose choice rule is as defined in (2), and the coding rule as in (1) (the discussion can be easily generalised to other coding rules). Moreover, we assume for simplicity a constant, seller independent, profit π from each transaction (this imply in particular that the transaction is always possible and realized between the buyer and the chosen seller). Hence we have:

$$\begin{aligned} \pi_j(t) \;&=\; \pi \;\text{ if } j \text{ is chosen,} \\ &=\; 0. \;\text{ otherwise} \end{aligned} \tag{3}$$

The Mean Field Approach[7] consists in replacing randomly fluctuating quantities by their expectation, thus neglecting fluctuations. Averaging (1), one gets

$$J_j(t) \;=\; \gamma J_j(t-1) \;+\; (1-\gamma) \; \pi \; P(j \mid \mathbf{J}(t-1)) \tag{4}$$

In the large time limit, one gets the fixed point (mean field) equations:

$$J_j \;=\; \pi \; \frac{f(J_j)}{\sum_k \; f(J_k)} \tag{5}$$

In the above equation we have replaced $P(j \mid \mathbf{J})$ by its expression (2).

Let us study now the solutions of the mean field equations. More precisely, the equations (5) are fixed point equations of a dynamical process: among the solutions, only the stable ones are meaningful, so that we will have to study the stability of the solutions.

As a preliminary remark, summing over j the fixed point equations (5) one sees that any solution $\mathbf{J}$ satisfies

$$\sum_j J_j \; = \; \pi. \tag{6}$$

Obviously,

$$J_j \; = \; \frac{\pi}{N} \quad j = 1, ..., N \tag{7}$$

is always a solution. Developing (4) at the vicinity of this symmetric fixed point (7), one finds that it is stable if the quantity α defined by

$$\alpha \; \equiv \; \left. \frac{d \ln f(x)}{d \ln x} \right|_{x=\frac{\pi}{N}} \tag{8}$$

is smaller than 1. Otherwise, that is if

$$\alpha \geq 1 \tag{9}$$

the symmetric solution (7) is unstable: there must exist other, stable, solutions.

To simplify the discussion, let us consider the simplest case of two sellers, $N = 2$. In that case, we can work with the single variable J_1, since according to (6) the other one J_2 is equal to $\pi - J_1$. Then the mean field equations becomes simply

$$J_1 \; = \; \pi \, \frac{f(J_1)}{f(J_1) + f(\pi - J_1)} \; \equiv \; g(J_1) \tag{10}$$

In fact it is clear that if J_1 is a solution, then $\pi - J_1$ is also a solution. Hence we have at least two stable solutions. Since we have $J_2 = \pi - J_1$, each pair of solutions can be written $\{J_1, J_2\}$. To keep the discussion simple, we will restrict the discussion below to the simplest case of a unique stable pair of solutions (hence one unstable and two stable solutions). Geometrically, a solutions J_1 of (10) is given, in the plane $\{x, y\}$ by the intersection of the straight line $y = x$ with the curve $y = g(x)$. One can show that the parameter α defined above is here equal to the slope of g at that value $\frac{\pi}{2}$ of J_1. Hence the condition for having the symmetric point unstable is

$$\alpha \; = \; \left. \frac{dg(x)}{dx} \right|_{x=\frac{\pi}{2}} \; \geq \; 1. \tag{11}$$

Remark: if $f(0) = 0$, it is easily seen that there are always (at least) three solutions, the symmetric point (7), and the pair $\{0, \pi\}$. Performing the stability analysis one finds that the non symmetric solutions $\{0, \pi\}$ are stable if

$$\frac{\pi f'(0)}{f(\pi)} < 1. \tag{12}$$

3.2 Interpretation

If the only stable solution of the equilibrium equations is $J_j = \frac{\pi}{N}$, the frequencies of visits to any seller are equal. The probabilities of visiting any seller simply fluctuate without any stable preference for one seller emerging.

If there are other stable solutions $J_j \neq \pi/N$, one frequency of visit is larger than the others. The buyer has a stable preference for one seller. According to the above discussion, the qualitative behaviour of the buyer depends on the choice function $f(.)$, the number of sellers N and the profit π only through the quantity α defined in equation (8). If the buyer modifies his choice strategy, or if his profit varies, in such a way that his α changes, an abrupt change of behaviour will be observed if α crosses the critical value 1. This is analogous to a second order phase transition in physical systems, where the parameter α has the meaning of the inverse of the temperature. If one starts with a small value of α, the stable solution $\{J_j = \frac{\pi}{N},\ j = 1, ..., N\}$ remains valid until α reaches 1. Just above the transition, $\mathbf{J}$ starts to depart from the symmetric solution, with

$$\left| J_j \; - \; \frac{\pi}{N} \right| \sim \sqrt{\alpha - 1} \tag{13}$$

Now it is reasonable to assume that the buyers of a market have different choice strategies, or/and make different profits, so that they have different values of α. When there exists a wide range of α values, distributed around the critical value 1, one will observe two categories of buyers: the ones who choose randomly the seller they will visit and the other who have strong preferences. We say that the distribution is bimodal.

3.3 Specific choice functions

The linear and affine cases

Let us consider the simplest case, that is an affine choice function. As can be seen from the definition (2) of the choice rule, $f(x)$ and $af(x)$, for any $a > 0$, give the same choice rule. Without loss of generality, an affine choice function can thus be defined by

$$f(J_j) \;=\; \beta\, J_j \;+\; 1 \tag{14}$$

with $\beta \;\geq\; 0$. For that case the quantity α is

$$\alpha = \frac{1}{1 \;+\; \frac{N}{\beta\,\pi}} \tag{15}$$

The purely linear case, $f(J_j) = J_j$ (studied in [9]), is obtained for $\beta \to \infty$. For that case $\frac{1}{\beta} = 0$, the number of solutions is infinite: every $\mathbf{J} = \{J_j, j = 1, ..., N\}$ such that $\sum_j \; J_j = \pi$ is a stable solution. This is analogous to what happens in the classical model of Blackwell's urns. If $\beta < \infty$, this degeneracy does not subsist. There is only one solution, the symmetric one, $\{J_j = \frac{\pi}{N}, j = 1, ..., N\}$ (which is indeed stable: $\alpha < 1$). In any case, that is whatever β, there will exist no transition.

The power law case

Let us consider the power law case, a simple generalization of the affine case:

$$f(J_j) \;=\; (\beta\, J_j)^n \;+\; 1 \tag{16}$$

with $n > 0$ and $\beta \geq 0$. For this choice function α is given by

$$\alpha \; = \; \frac{n}{1 \; + \; \left(\frac{N}{\beta\,\pi}\right)^n} \tag{17}$$

For $n = 1$ one recovers the results for the linear and affine cases: α is always smaller than 1 for $\beta < \infty$, and equal to 1 if $\frac{1}{\beta} = 0$. For $n < 1$, α is always smaller than 1, there is no transition as found in [9].

For $n > 1$, there exists the possibility of observing a transition, hence a bimodal situation: α is larger than 1 for $\frac{\beta\,\pi}{N} \; > \; (n-1)^{-\frac{1}{n}}$.
Remark: in the particular case $\beta \to \infty$, that is for $f(J_j) = (J_j)^n$, one has $f(0) = 0$. Since $n > 1$, $f'(0) = 0$, so that, according to (12), the non symmetric solutions $\{0, \pi\}$ are stable.

To conclude, in this case (16), the convexity of f is a necessary condition for observing a bimodal behaviour.

The exponential case

The standard logit case corresponds to an exponential choice function:

$$f(J_j) \;\; = \;\; \exp(\; \beta \; J_j \;). \tag{18}$$

In that case α is simply given by

$$\alpha \;\; = \;\; \frac{\beta\,\pi}{N} \tag{19}$$

The symmetric point is unstable if $\alpha = \frac{\beta\,\pi}{N} > 1$.

A last remark concerning the possible occurrence of bimodality. In all the case considered α is an increasing function of $\frac{\beta\,\pi}{N}$, where β parametrises the buyer's choice function. Let us assume that the distribution of β values among buyers is not (or only weakly) dependent on their profits. It follows that buyers making large profit are more likely to have $\alpha > 1$ than buyers making small profits. This is in agreement with what is observed in the fish market of Marseille, where big buyers develop fidelity to one particular seller, whereas small buyers do not. A more detailed discussion of this market is presented in ref. [11].

4 Derivation of the logit function from an optimization principle

4.1 An exploration-exploitation compromise

In the previous section we studied the qualitative behaviour that we can expect for a general choice function. The next question is then: in what sense a given choice function is efficient? One attractive feature of a choice function is that it may represent some sort of "best behaviour" with respect to some criterion. Here we will show that the logit function can be derived from an optimization strategy. In particular we argue below that, for modeling the buyer's strategy, one can define a maximization principle, formally identical to the so call *maximum entropy principle*[3] considered in statistical physics - and to be shortly presented later on for comparison.

Let us assume that the buyer wants to find a compromise between getting the best profit at the next transaction, and keeping the best possible knowledge of a market in order to be able to make good choices in the future: the market can vary in time because of external events or because the sellers strategies moves. This requires that he will visit every seller as frequently as possible (he can only get information about a seller by making transactions with this seller). If p_j is the probability of visiting seller j, $p_j = p_j^0 \equiv 1/N$ would correspond to maximum information. A proper measure of the similarity between this uniform distribution $\{p_j^0\}_{j=1}^N$ and the actual distribution $\{p_j\}_{j=1}^N$ is the entropy $\mathcal{S}$,

$$\mathcal{S} = - \sum_j p_j \ \ln p_j. \tag{20}$$

The entropy is a measure of uncertainty in the occurrence of the events $j = 1, ..., N$. In the context of Information Theory[4], it is the minimal *amount of information* (measured in bits if the logarithm in (20) is taken in base 2) required in order to code the set of events.

One may thus want to choose the p_js from a compromise between the maximization of the entropy and a maximization of the immediate profit. Taking the (moving) average J_j as an estimate of the profit to be obtained from seller j, we thus maximise

$$\mathcal{C} \equiv \mathcal{S} + \beta \sum_j p_j \ J_j \tag{21}$$

over all possible p_j's. The quantity $\frac{\ln 2}{\beta}$ is equal to the amount of profit considered to be equivalent to one bit of information.

Introducing a Lagrangian multiplier λ in order to impose the normalization constraint $\sum_j p_j = 1$, one finally maximises

$$\mathcal{C} \ = \ \mathcal{S} \ + \ \beta \sum_j \ p_j \ J_j \ - \ \lambda \left(\sum_j p_j \ - \ 1 \right) \tag{22}$$

Taking the derivative of $\mathcal{C}$ with respect to one p_j, one gets

$$- \ 1 \ - \ \ln p_j \ + \ \beta \ J_j \ - \ \lambda \ = \ 0 \tag{23}$$

which gives precisely

$$p_j \ = \ \frac{1}{Z} \ \exp \ \beta \ J_j \tag{24}$$

with $Z = \sum_j \ \exp \ \beta \ J_j$.

The logit strategy is thus obtained as a consequence of the optimization of a cost function which expresses the compromise between short term profit and preservation of information for long term profits.

4.2 Link with physics and inference theory

The exponential family of probability distributions plays a central role in statistical physics, where it is derived from the *maximum entropy principle*[3]. The maximum entropy principle is more generally a tool for making inferences. In fact, it has already been used in economics in order to justify the choice of an exponential distribution (see e.g. [10, 12]).

For completeness we restate here this inference principle. One constructs a probability distribution $\{p_j, j \ = \ 1, ..., N\}$, based on some prior knowledge, in such a way that the resulting probability law does not contain more information than what can be gained from this prior knowledge. The measure of uncertainty in the occurrence of the events is given by the entropy $\mathcal{S}$ of the probability distribution, as defined in (20). If we know some mean value E of an observable quantity E_j, we estimate the p_js by maximizing the entropy $\mathcal{S}$ under the constraint that E is given. This leads to

$$p_j \;=\; \frac{1}{Z}\;\exp\;-\beta\;E_j \tag{25}$$

where Z is the normalization constant (the "partition function"). For a physical system, $T \equiv \frac{1}{\beta}$ is the temperature, and E is the energy. If one works at a given value of β (instead of a given value of E), one sees that as T goes to zero (β goes to ∞) the system will choose the states with the smallest possible values of the energy. In our model of buyer's strategy, the quantity which play the role of the energy is thus *minus* the mean profit (since the profit has to be maximised). With the maximum entropy principle one predicts the probability distribution without making any hypothesis on the dynamics. The resulting probability distribution is the best guess based on the knowledge we have about the system: the logit function can be understood as the best description of the buyer's strategy based on the knowledge of the mean profit he obtains.

The specificity of Statistical Physics is that the application of this *inference* principle leads precisely to the correct *physical* description - the law of thermodynamics. Clearly, there is no reason *a priori* for expecting such a success in the context of economics. Nevertheless, there are several approaches tending to show that the exponential family may play also a fundamental role in economy, as discussed in particular in [10]. What we have shown in this paper is that the maximum entropy principle has an appealing "physical" interpretation in the context of the search for an exploitation/exploration compromise.

A last remark is in order. One should note that to derive a choice function from an optimization principle does *not* imply that one assumes the buyer to be aware of optimizing some criterion. An analogy can be made with living systems evolving according to past experiences. One of the main approach to the modeling of Evolution in nature assumes the optimization of some cost function, the survival fitness. Clearly, no genetic system is aware of what is really going on, and only mutation rules can be observed at the level of individuals. Similarly, it is commonly believed that the brain organization is *optimally* fitted to the tasks it has to solve, through evolution and adaptation. It is not unreasonable to expect that a buyer follows some empirical rule, the rule itself being chosen according to some kind of cultural knowledge, based on past experiences possibly including those of previous generations, in such a way that, implicitly, the rule implements the optimization of some cost function.

5 Conclusion

In this paper, we have focussed on two aspects of the modeling of a buyer's choice of seller in a market. We have been concerned with the case in which buyers only learn from their own private experience. We have used the mean field approach to show that a phase transition may occur in individual behaviour - even in the extremely simple case where he gets the same profit from any seller. Previous work[11] provides empirical evidence from the Marseille wholesale fish market for the behaviour predicted by the present formal model.

Then, we have shown that the logit function (which generates phase transitions) can be obtained from maximizing a cost function which expresses a compromise between exploration - keeping information about the market - and exploitation - making the largest profit at the next transaction. We have discussed the possible meanings of such an optimization process.

Our results can be easily extended or adapted to other situations: models with other coding schemes, with interactions between buyers, etc.. In particular one can consider models in which individuals also receive information about the experience of others (see [8]), and can thus be related to standard models of evolutionary games in which, for example, players know the frequencies of choices for the whole population. In such a case where an agent has access to information about other agents'choices, phase transitions will exist as well, although the related market dynamics might be different.

Another application is the case in which buyers have access to public information that they code by some utility function, see for instance [9], such as the utilities of different brands, or different strategies. The transition which will be observed in such case will be a transition between coexistence of different brands in the market in the case of low α, and supremacy of one brand which sells better than the others for large values of α.

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