3
Social opinion dynamics

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3.1
Introduction

First of all, we will here discuss very simplified models of opinion dynamics. At this level of description, the same set of models are used concerning the diffusion of opinions and the resulting decisions, although this distinction might of importance in many circumstances of real life.

Opinion dynamics is a central topics in sociology, especially in socio-dynamics. Decision dynamics under social influence have been under scientific scrutiny for many decades (see for instance the book Diffusion of Innovations by Everett Rogers, 1962, [43]). In fact, the issue of how opinions are made and decisions adopted by crowds is mentioned even by historians from the Antiquities: Livius for instance wonders what were the processes that made the Plebs to retire on the Aventine hill.

Opinions and decisions dynamics under social influence occur in economic, political, or even in more personal contexts. An important field of study with socio-political implications is the diffusion of working practices. The earlier empirical studies were concerned the choice of corn varieties in the US (by Bryce Ryan and Neal C. Gross, rural sociologists at Iowa State University 1943,[44]) and the diffusion of antibiotics prescription by doctors (by Coleman, James S., Elihu Katz, and Herbert Menzel. 1957,[45]). Later literature discusses the empirical and normative aspect of the diffusion of innovations in agriculture (the "Green revolution" or environmental friendly practices,[47]) or in the organisation of the economy (the study of the 90's transition in Eastern Europe[48]).

The basic process that we here hypothetise is Gabriel Tarde's (Les lois de l'imitation (1890), [46]) imitation principle "do as the others do". The rationale for this imitation behaviour are of two kinds:

- Bounded rationality: in situation when agents lack objective information, such as choosing a new movie or an unknown restaurant, they

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might suppose that the other agents whom they observe have made a documented choice;

- Externalities (an expression from economics): in some cases doing as the majority might bring some advantage, such as sharing knowledge and getting a better service in the case of technological equipment for instance.

Among the important simplifications that we introduce by only considering social processes (influence, imitation etc.) we neglect the importance of:

- Knowledge: Some objective knowledge might exist which would bias opinions;
- Individual cognitive processes which would for instance modify opinions according to direct personal experience.

All the dynamic processes that we discuss here privilege social interactions as the main vector for opinion change. The general framework is then as follows:

- At any given time a distribution of opinions exists in the population;
- At each time step some group of individuals interact and as a result, one or several opinions are shifted, generally towards some consensus among the group.
- As a result of these interactions, some form of clustering of opinions is observed.

In the spirit of complex system studies, we focus on the attractor of the dynamics and their spatio-temporal structure because the characterisation of attractors are the generic properties of the dynamics [49][51] . The models that we her discuss are based on such strong simplifications of the processes occurring in the real world that the eventual conclusions that may come out of these models are only the generic properties of the dynamics, to be compared with what economists call stylised facts.

We will in general be interested in:

- the nature of the attractors, consensus or diversity of opinons;
- their number;
- possible patterns on social networks
- regime transitions, and so on ...

The specifics of the different variants further described concern:

- The interaction space: are all interactions a priori possible (the "well mixed case") or do interactions only occur via a neighborhood structure such as a "social network"?
- The "dimension" of the opinion; the most standard studies referred to binary opinions or binary choices: good/bad, buy/not buy, for/against. The generalisation to continuous opinions, how much is this worth?, or to vector opinions as in some models of cultural dynamics, yields surprisingly different results.
- The detailed results might depend upon the specific of the iteration modes, parallel/sequential, the heterogeneity of the agents, the reversibility of the decision.

Not surprisingly, the role of the dimensions of the interaction space and of the opinion variable reminds us of the universality classes of the renormalisation group approach [50] .

The present contribution first recalls the main results obtained with the binary choice models. The next section is devoted to continuous opinion dynamics. We then discuss vector opinion models. The exposition in each section starts with well-mixed case and continues with generalisations to social networks. We examine in the conclusion section the possible connections with empirical data and the experimental approach of social psychology, and some "normative" aspects: how to use some understanding of opinion dynamics to plan a campaign to convince customers, firms or fellow citizens. We have chosen to present simulations rather than the formalisms which sometimes yields exact results, but the bibliographical references to formal methods are given. We also tried to give some indications about connections with Social Sciences literature.

3.2 Binary opinions

Binary opinions and binary choices are the most standard approach to opinion dynamics. The basic model is that at each time steps one agent samples a small group of other agents and chooses the majority opinion. In the simplest version, closely parallel to epidemiology (Susceptible-Infected models), one starts from a large majority of agents with opinion 0. Whenever they encounter agents with opinion 1, they are irreversibly converted to opinion 1.

When all binary encounters are a priori possible, a differential equation approach is possible when the number of agents tends towards infinity; it yields exact results.

3.2.1
Full mixing

Let us call N the total number of agents, and n the number of agents with opinion 1. When N is large we can postulate that the probability of encountering agents with any opinion 0 or 1 is equal to their proportion in the population. If we suppose that only agents in state 0 can change their opinion, the dynamics of n can be written:

$$\frac{dn}{dt} = a \cdot (N - n) \cdot n \tag{3.1}$$

where a is a kinetic coefficient. This equation is readily integrated and gives the S shaped solution represented on figure 1:

$$n = \frac{N}{1 + \alpha exp(\beta t)} \tag{3.2}$$

where α and β relate respectively to initial conditions and to the kinetic coefficient.

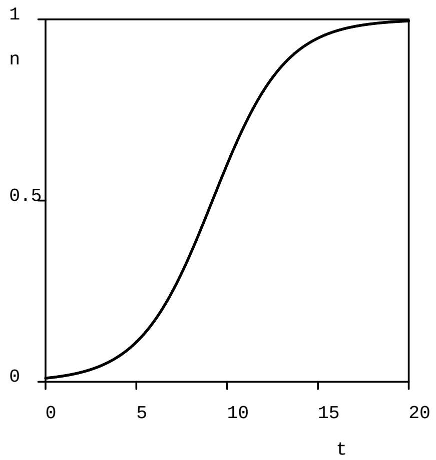

Fig. 3.1 Time evolution of the fraction of agents with opinion 1 in a full-mixing topology.

The S curve, and the integrated bell curve, are standard material in textbooks on marketing, representing the sales curves of new products. The S curve is also the rate adoption curve of new technologies. This simple model when applied to the adoption of new products or techniques supposes that the transition from choice 0 to choice one is always beneficial: each transition is thus asymmetrical. The status of the initial adopters supposes that they had some special reason to adopt as opposed the later adopters who adopted because of social influence.

The precise set-up that we used, namely an infinitely large number of agents and asymmetry of opinions is important. Other set-ups can give different outcome.

For instance, the two restaurants problem[36], where newcomers would visit restaurant A or B with a probability proportional to the number of agents already inside each restaurant, is a standard problems among economists discussing "herd behaviour". The set-up is symmetry with respect to either choice, irreversibility of choice but a finite number of customers. A first customer facing the two empty restaurants would choose randomly among them. New customers would choose randomly with a probability proportional to the number of customers present in each restaurant. The first steps in the series of customer choices are crucial in determining the ultimate choices of later customers. It can be shown that attractors of the dynamics of the fraction of customers in one restaurant (say A) are distributed in the interval 0,1 depending upon the initial customers choice: the problem is equivalent to Polya's urn. The term of information cascade is often used by economists to describe these dynamics [28], [35].

3.2.2
Lattices as surrogate social nets

Even the earliest empirical studies revealed that social influence occurs on some social network. Human agents are not influenced by unknown individuals (as implied by a full mixing topology); they rather trust some selected individuals with whom they have stronger connections: according to the problem they have to solve, they might use family connections, seek professional advice, or check the opinion of their peers. The topics of social network has always been very active and received a lot of attention in the last ten years. For instance, small worlds [52] and scale free networks [29] were proposed as topological structures more relevant to social dynamics than lattices or random networks.

We nevertheless base our exposition on periodic lattices because :

- They are easy to visualise;
- They display a high degree of betweenness, a property shared with real social networks.

(Betweenness is the proportion of your neighbours which are themselves neighbours; the property is important in the models that we will use). Most of the concepts that we here develop apply to structures more disordered than lattices.

3.2.3 Cellular automata

The lattice version of binary dynamics is expressed by cellular automata "counter" rules (so-called because the new state of the automaton is obtained by counting how many neighbours are in state 1). The name voter rules applies when a majority rule is applied. Each cell i of the network represents an agent and its binary state S_i represents an opinion. Cells are connected to some neighbourhood of z "neighbours". Cells can be in state 0 or 1. At any time step, a cell i updates its state S_i taking into account the number of its neighbours S_j in state 1 according to:

$$\text{if} \quad \sum_{j=1}^{z} S_j > \theta_i \quad \text{then} \quad S_i = 1 \tag{3.3}$$

$$\text{else} \quad S_i = 0 \tag{3.4}$$

where θ_i is a fixed integer threshold.

These models are formally equivalent to the Ising model of ferromagnetism at 0 temperature with an external field $h_i = z/2 - \theta_i$, where z is the number of neighbours of cell i.

In the "socio-economic" interpretation of counter automata, the S variable represents the binary decision, for instance buy/not buy. The threshold θ represents the (economic) utility of choice S, including eventually the price as a negative component. Alternatively, in a buy brand A or buy brand B interpretation, the threshold θ represents the difference in utility of the two brands A and B. The sum term represent the social influence of the neighborhood on the choice of agent i.

As such, this simple set-up can result in many different dynamics according to updating rules, choice of thresholds, probabilistic versus deterministic transition rules etc.

Let us first discuss the most standard case of cellular automata rules:

- Parallel iteration: the state of all cells is updated simultaneously.
- Homogeneity: all thresholds θ_i are identical.
- We use a square lattice with von Neumann neighborhood: each cell is connected to four neighbours (see the figure 3.2.3).

3.2.3.1 Growth

The necessary specifications of "counter" automata rules are completed by the choice of a neighbourhood and of a threshold. Several neighbourhoods have been proposed on square lattices, von Neumann neighbourhood which include the four neighbours N, S, E and W; and Moore neighbourhood with 8

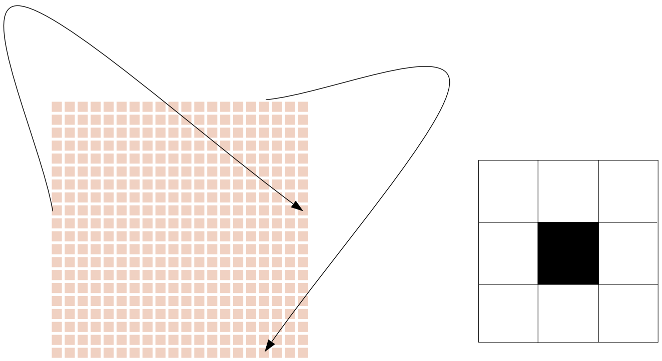

Fig. 3.2 Interaction geometry: Periodic boundary connections are displayed on the left. On the right, depending upon the neighborhood choice, the black cell in the center is influenced by four or eight neighbours (the white cells).

neighbours, N, NW, W, WS, S, SE, E and NE. In fact the *i* site itself may also be included; the standard choices respecting the symmetry of the square lattice are 4, 5, 8 or 9 neighbours.

Let us describe possible dynamics for the 5 neighbours case; the results are easily generalised to other neighborhoods. One classify "counter" automata by the attractors that are reached when the initial configuration consists in a few automata in state 1 surrounded by a "sea" of automata in state 0. Small thresholds favour the growth of contiguous regions of cells in state 1, while larger thresholds ($\theta_i > 3$) favour the growth of contiguous regions of cells in state 0. But as discussed below, there are limits to growth and some "local" conditions have to be fulfilled to allow growth[53].

When θ_i is 0 or less, growth of a 1's region accross the whole lattice occurs under any initial condition in one time step.

For a threshold of 1, any initial configuration with at least one cell in state 1 generates clusters of 1's which grow to fill the entire lattice. A possible interpretation of the case of threshold 1 is that state 1 correspond to the situation when the agent is "informed" while agents in state 0 are not yet "informed". One important variant concern the case when not all lattice sites are occupied by an automaton: this is the percolation problem (more later on this issue).

The next couple of figures display the possible evolution when the threshold is two neighbours. When the density of ones is low, figure 3.2.3.1 display the evolution of isolated small clusters of 1's.

Larger initial clusters evolve towards their convex hull after possible trimming of corners as seen on figure 3.2.3.1

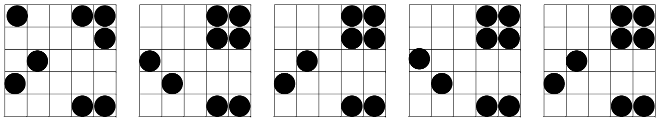

Fig. 3.3 Evolution in five time steps from left to right of small initial clusters of 1's for a threshold of 2 neighbors out of 5. Black circles represent cells in state 1, cells in state 0 are empty. Isolated 1 disappear, horizontal (and vertical) pairs are stable, oblique pairs oscillate.

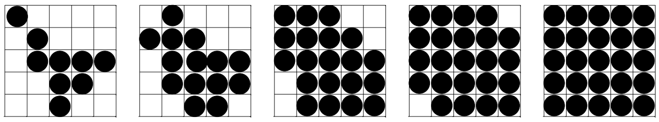

Fig. 3.4 Growth from an initial cluster (left pattern) to its convex hull (right pattern) in 5 time steps for a threshold of 2 neighbors out of 5. Growth proceeds along cluster edges and stops when all are completed.

The evolution of the system thus depends on the initial conditions; low densities of initial 1′s end up in isolated clusters of 1′s; larger densities might result in filling up the lattice with 1′s with the exception of a few clusters of 0′s. In any case coarsening of the clusters is obtained: small clusters disappear, and boundaries between clusters are vertical or horizontal lines.

When the threshold for growth is 3 neighbours, coarsening occurs for initial densities of close to 0.5 and large domains can be observed. But unbalanced initial densities bias the evolution of the system towards larger densities of 0′s or 1′s.

Larger thresholds from 4 to 6 yield symmetrical behaviour with respect to 0′s and 1′s regions

The description given above for threshold automata with a neighborhood of 5 is easily generalised to other neighborhood (4,8 or 9 neighbours). The important conclusions still hold. Except for the limit cases of extreme thresholds which favour homogeneous conditions:

- The dynamics favour coarsening of the initial domains;
- Attractor configurations depend upon initial conditions;
- Growth necessitates "seed" configurations to occur; it occurs at the boundaries of clusters.

3.2.4 INCA

The term INCA INhomogeneous Cellular Automata has been proposed by H. Hartman and G. Vichniac [53] and applies to a variety of models. The idea is that all automata on the lattice obey the same "kind" of rules, but with different parameters. INCAs belong to a large class of systems described by theoretical physicists as disordered systems. The "philosophy" of disordered systems is to replace structural disorder observed in natural systems such as glasses or diluted magnetic alloys, by interaction disorder on regular lattices.

A general expression can be written:

$$\text{if} \qquad \sum_j J_{ij} S_j > \theta_i \qquad \text{then} \qquad S_i = 1 \tag{3.5}$$

$$\text{else} \qquad S_i = 0 \tag{3.6}$$

where interaction strength J_{ij} and threshold θ_i are local variables. These models are formally equivalent to Ising spin glasses at 0 temperature, and to neural nets.

Let us discuss shortly a few examples:

- The percolation problem [54] can be interpreted as a cellular automaton with homogeneous threshold 1 when a fraction of the cells are empty. Percolation models display abrupt transitions in the diameter of the largest cluster (at equilibrium) of cells at state one as a function of the fraction of empty cells.
- A number of models of social influence proposed by the "Polish school" [2],[3],[5],[1] are based on INCA's. A simple version proposed by Kacperski and Holyst[1] discuss the role of a strong leader with a large influence on his neighborhood in the case of homogeneous threshold. Most of these papers were written to describe the "transition" in Eastern Europe from a socialist to a free market economy: state 0 correspond to the choice of socialism by economic actors and state 1 to free market choice. The threshold is interpreted as some external "global" signal applied uniformly to all actors. These systems display first order transitions and hysteresis when the threshold is varied[1].

More generally, networks of threshold automata are also called neural nets, and are widely use as models in cognitive science[49].

One application of this formalism to coalition formation in social and political sciences has been proposed by R. Axelrod[21]. In this model the J_{ij} are computed according to common traits to social actors in a version quite close to Hebb's rule in cognitive science. J_{ij} are constructed starting from 0 and adding +1 for each common trait and -1 for each different trait. J_{ij} can then be negative. The signal S_i corresponds to which coalition is chosen: one example

1) First order transitions and hysteresis are not specific of INCA's and also happen with homogeneous cellular automata when the threshold is varied

used by Axelrod is WWII with two coalitions Allies and Axis represented by for instance by $S_i = 1$ and $S_i = -1$. No spatial structure is involved and all interactions are a priori possible. The attractors of the dynamics correspond to stable coalitions. Axelrod applied his model of coalition formation from XXth century political situation to the adoption of different UNIX standards among computer firms.

Analytical solutions derived from analogies with disordered physical systems have been proposed for both the percolation model (by the Renormalisation Groupe approach)[54] and the inhomogeneous threshold automata model (by the Mean Field formalism)[1].

3.2.5
Probabilistic dynamics

In these above models, the disorder is quenched: it only concerns the initial conditions, plus eventually the sequence of which cells are chosen to be updated when sequential updating is used. Most of these models have also been studied in their probabilistic (or finite temperature version). The expressions are easier written with variables $S = 1$ or $S = -1$:

$$\Delta E_i = -2 \sum_j J_{ij} S_j - \theta_i \tag{3.7}$$

$$S_i = 1 \qquad \text{with probability} \qquad \frac{1}{1 + exp(\beta \Delta E_i)} \tag{3.8}$$

$$\text{else} \qquad S_i = -1 \tag{3.9}$$

where β is the inverse temperature coeeficent, and ΔE_i the difference in "energy" between states 1 and -1. The J_{ij} coefficients represent the intensity of the influence of agents j on agent i. Larger values of β yield behaviour similar to those observed for the deterministic case described above. Low values of β yield disordered behaviour with cells randomly changing states. A second order phase transition is observed for some intermediate β_c value depending upon interaction and threshold distribution. The analogy with magnetic systems at finite temperature and Glauber dynamics is obvious.

In the probabilistic versions of cellular automata with large values of β, the limits to growth and the necessity of having favourable initial conditions loose their compulsory character, but simply appear as rate limiting steps to growth. For instance in the case of threshold equal to four out of 8 neighbours, convex hull of ones are not limiting growth for ever: they rather slow down the growth of "faceted macro crystals". The same is true for the seed configurations allowing growth for lower thresholds: one observes metastable configurations which evolve towards uniform configurations of ones after the long awaited apparition of seeds. These delay phenomena are typical of first order phase transitions.

Finally, let us note that this transcription from Boltzmann statistics to opinion dynamics is already familiar to economists[27] since the 70's: they use utility functions where physicists use energy function, although with a minus sign. The Boltzmann factor in expression 3.8 is often called the logit function by economists[42]. Various interpretation of this factor have been proposed from a noise term to optimisation between exploitation vs exploration compromise[40]

3.2.6
Groups processes

In the above discussion, we only considered processes when one agent would survey other agent(s) (voters models), and choose a new opinion accordingly. Some authors (Sznajd-Weron etal[22], Galam[14]) considered the possibility of elementary processes involving more than one agent, agents are engaged in small discussion groups and eventually changing opinions accordingly. Both group of authors had in mind modeling votes in a political context. Galam model[14], is discussed in the present volume by its author.

The idea of Sznajd-Weron etal[22] is based on the formula: " United we stand divided we fall ", hence the proposed acronym for their model USDF. USDF is a model for "propaganda", where pairs of sites in the same state convince their neighbours to share their views. On the opposite, pairs of opposite views propagate opposite views to their neighbours. The model was originally proposed with binary states on a 1D line, and give attractors which are either homogeneous with all nodes at state 0 or 1 (equivalent to ferromagnetism) or alternation of zero and 1 sites (equivalent to anti-ferromagnetism). Later variants by Stauffer[23] and his "Brazilian connection" have generalised to larger dimensions and integer opinions. They can be compared to the models described in the next sections.

3.3
Continuous Opinion Dynamics

The rationale for binary versus continuous opinions might be related to the kind of information used by agents to validate their own choice:

- the actual choice of the other agents, a situation common in economic choice of brands: "do as the others do";
- the actual opinion of the other agents, about the "value" of a choice: "establish one's opinion according to what the others think or at least according to what they say".

One often encounters situations when opinions concern quantities rather than two options:

- How much does this worth?
- How should we share?

etc.

On the empirical side, there exist well documented studies about social norms concerning sharing between partners. Henrich *et al.* [19] compared through experiments shares accepted in the ultimatum game and showed that people agree upon what a "fair" share should be, which can of course vary across different cultures. Young and Burke[20] report empirical data about crop sharing contracts, whereby a landlord leases his farm to a tenant laborer in return for a fixed share of the crops. In Illinois as well as in India, crop sharing distributions are strongly peaked upon "simple values" such as 1/2-1/2 or 1/3-2/3. The clustering of opinions about "fair shares" is the kind of stylised fact that continuous opinions models try to reproduce.

The model discussed here was introduce [26] to interprete decisions made by farmers about accepting or refusing grants to change their agricultural practices in favour of environmental-friendly practices; according to the survey results, they first evaluate the economic outcome of new practices and compare them to their old practice in order to take their decision.

Modeling of continuous opinions dynamics was earlier started by applied mathematicians and focused on the conditions under which a panel of experts would reach a consensus, (Stone [18], Chatterjee and Seneta[15], Cohen[16] etal, Hegselmann and Krause[17], further referred to as the "consensus" literature).

The purpose of this section is to present results concerning continuous opinion dynamics subject to the constraint that convergent opinion adjustment only proceeds when opinion difference is below a given threshold. The rationale for the threshold condition is that agents only interact when their opinions are already close enough; otherwise they do not even bother to discuss. The reason for refusing discussion might be for instance lack of understanding, conflicting interest, or social pressure. The threshold would then correspond to some openness character. Another interpretation is that the threshold corresponds to uncertainty: the agents have some initial views with some degree of uncertainty and would not care about other views outside their uncertainty range.

3.3.1
The basic case: Complete Mixing and one fixed threshold[26]

Let us consider a population of N agents i with continuous opinion x_i. We start from an initial distribution of opinions, most often taken uniform on [0,1] in the computer simulations. At each time step any two randomly chosen agents meet: they re-adjust their opinion when their difference in opinion is smaller in magnitude than a threshold d. Suppose that the two agents have opinion x and x'. Iff $|x - x'| < d$ opinions are adjusted according to:

$$x = x + \mu \cdot (x' - x) \tag{3.10}$$
$$x' = x' + \mu \cdot (x - x') \tag{3.11}$$

where μ is the convergence parameter whose values may range from 0 to 0.5.

In the basic model, the threshold d is taken as constant in time and across the whole population. Note that we here apply a complete mixing hypothesis plus a random serial iteration mode[2].

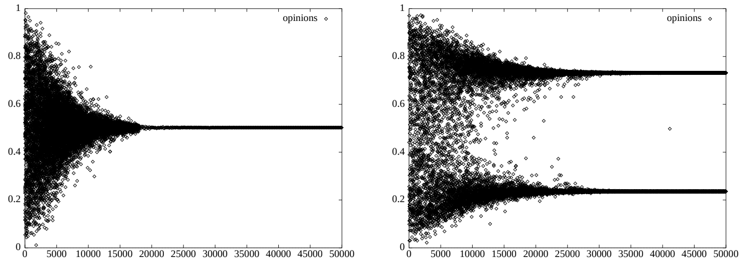

Fig. 3.5 Time charts of opinions for different interaction thresholds ($d = 0.3$ on the left, $d = 0.2$ on the right). Each dot correspond to a sampled opinion at a time given by the x axis postion. One time unit corresponds to sampling a pair of agents. In both cases $\mu = 0.5$, $N = 2000$ for the right chart and $N = 1000$ for the left one.

- For large threshold values ($d > 0.3$) only one cluster is observed at the average initial opinion as seen on the left pattern of figure 3.3.1, which represents the time evolution of opinions starting from a uniform distribution of opinions.
- For lower threshold values, several clusters can be observed (see the right pattern of figure 3.3.1). Consensus is then NOT achieved when thresholds are low enough.

2) The "consensus" literature most often uses parallel iteration mode when they suppose that agents average at each time step the opinions of their neighbourhood. Their implicit rationale for parallel iteration is that they model successive meetings among experts.

Obtaining clusters of different opinions does not surprise an observer of human societies, but this result was not a priori obvious since we started from an initial configuration where transitivity of opinion propagation was possible through the entire population: any two agents however different in their opinions could have been related through a chain of agents with closer opinions. The dynamics that we describe ended up in gathering opinions in clusters on the one hand, but also in separating the clusters in such a way that agents in different clusters don't exchange anymore.

The number of clusters varies as the integer part of $1/2d$: this is to be further referred to as the "1/2d rule" (see figure 3.3.1).

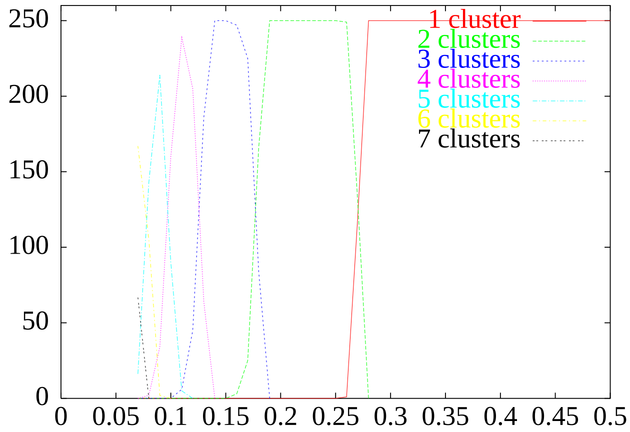

Fig. 3.6 Statistics of the number of opinion clusters as a function of d on the x axis for 250 samples ($\mu = 0.5, \quad N = 1000$).

An alternative method to directly runing simulations is to solve the master equations describing the time evolution of populations of bins of opinions. This approach is used for instance by Redner and co-authors [7] to finely describe the structure of the attractors as a function of the interaction threshold d.

Rather than working on continuous opinions, Stauffer and other authors[24] have proposed to use integer opinions: in that case convergence is rapidly obtained and easily tested.

3.3.2
Social Networks

The literature on social influence and social choice also considers the case when interactions occur along social connections between agents[27] rather than randomly across the whole population.

Apart from the similarity condition, we now add to our model a condition on proximity, i.e. agents only interact if they are directly connected through a pre-existing social relation.

We then started from a 2 dimensional network of connected agents on a square grid. Any agent can only interact with his four connected neighbours (N, S, E and W). We used the same initial random sampling of opinions from 0 to 1 and the same basic interaction process between agents as in the previous sections. At each time step a pair is randomly selected among *connected agents* and opinions are updated according to equations 1 and 2 provided of course that their distance is less than d.

At equilibrium, clusters of homogeneous opinions appear. The opinions themselves are not very different from those observed with non-local opinion mixing as described in the previous section,

For the larger values of d (e.g. $d = 0.3$ as seen on the left pattern of figure 3.3.2), the lattice is filled with a large majority of agents who have reached consensus around $x = 0.5$ while a few isolated agents have "extremists" opinions closer to 0 or 1. The importance of extremists is the most noticeable difference with the full mixing case described in the previous section.

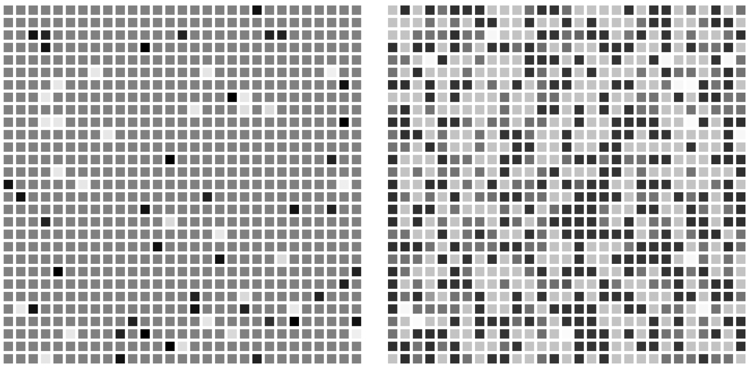

Fig. 3.7 Display of final opinions of agents connected on a square lattice of size 29x29. Opinions between 0 and 1 are coded by gray level (0 is black and 1 is white). The left pattern is obtained for a larger interaction threshold, $d = 0.3$. Note the percolation of the large cluster of homogeneous intermediate opinion and the presence of isolated "extremists". The right pattern is obtained for a lower interaction threshold, $d = 0.15$. It displays many small clusters.

Interesting differences are noticeable for the smaller values of $d < 0.3$ as observed on the right pattern of figure 3.3.2: several opinion clusters are observed and none percolates across the lattice. Similar opinions, but not identical, are shared across several clusters. The differences of opinions between groups of clusters relate to d, but the actual values inside a group of clusters fluctuate from cluster to cluster because homogenisation occurred independently among the different clusters: the resulting opinion depends on fluctuations of initial opinions and histories from one cluster to the other. The same increase in fluctuations compared to the full mixing case is observed from sample to sample with the same parameter values.

The clustering effects observed with full mixing and lattice topologies, and the observation of outliers in lattice are maintained for disordered topologies such as scale free networks of the Barabasi-Albert type[29]. Simulations were run by D. Stauffer and H. Meyer-Ortmanns[25], and by Weisbuch[6]. To summarise some of the results obtained on scale free networks:

- One does observe clustering effects, and the opinions in the main clusters do not differ much from what they are for equivalent tolerance thresholds in lattices.
- As a result of the scale free distribution of node connectivity, well connected nodes are influenced by other nodes and are themselves influential. Most of them belong to the big cluster(s) after the convergence process.
- Even more outliers nodes are observed, and most of them are among the less connected nodes.

3.3.3 Extremism

The above results where obtained when all agents have the same invariant threshold. Several models were tried with distributions of thresholds and eventually thresholds which vary according to the agents' experience[26] . One of the most interesting variant concerns modeling populations with large numbers of open minded "centrist" agents and a small proportion of extremists.

The issue of whether extremist or moderate opinions are adopted in committees is thoroughly discussed by social psychologist see for instance Moscovici and Doise [11]. A connection with statistical physics and the Ising model was soon proposed, for instance by Galam [12] who collaborated with Moscovici. They considered binary opinions as in the vast majority of the literature on binary social choice.

Fascinating results were obtained in the "extremism" model of Deffuant etal [31]. When interaction thresholds are unevenly distributed, and in particular when agents with extreme opinions are supposed to have a very low threshold for interaction, extremism can prevail, even when the initially extremist agents are in very small proportion. The so-called "extremist model" can be applied to political extremism, and a lot of the heat of the discussion generated by these models relates to our everyday concerns about extremism. But we can think of many other situations where some "inflexible" agents are more sure about their own opinion than others. Inflexibility can arise for instance:

- Because of knowledge; some agents might know the answer while others only have opinions; think of scientific knowledge and the diffusion of new theories;
- Some agents might have vested interests different from others.

Although the model has some potential for many other applications, we will here use the original vocabulary of extremism.

The model[3] for extremism introduce by Deffuant etal [31] is based on two more assumptions than the original continuous opinion model.

- A few extremists with extreme opinions at the ends of the opinion spectrum and with very low threshold for interaction are introduced.
- Whenever the threshold allows interaction, both opinions and threshold are readjusted according to similar expressions.

$Iff\ |x - x'| < d$

$$x = x + \mu \cdot (x' - x) \tag{3.12}$$

$$d = d + \mu \cdot (d' - d) \tag{3.13}$$

A symmetrical condition and equations apply to the other agent of the pair with opinion x' and tolerance d' but when thresholds are different the influence can be asymmetric: the more "tolerant" agent (with larger d) can be influenced by the less tolerant (with smaller d) while the less tolerant agent is not. This “effective” asymmetry is responsible for the outcome of “extremist” attractors.

Let us demonstrate the demonstrate the issue in the very simple case of a single extremist in the presence of a large majority of centrist agents [32]. We check opinion and tolerance dynamics by time plots of single simulations.

The time plots display different dynamical regimes according to the eventual predominance of the extremists: sometime they remain isolated and most agents cluster as if there were no extremist (e.g as represented on figure 1 left frame); otherwise extremism prevails and most agents cluster in the neighborhood of one (e.g as represented on figure 1 right frame) or both extreme.

Still, in both cases, a phase of convergence towards average opinion of most initially centrist agents is observed (for roughly 10 updating per agents). The initial convergence towards the center is due to the much larger number of

3) In fact we here give a simplified version (called bounded confidence) of the model actually used by Deffuant etal in their original model (relative agreement model)

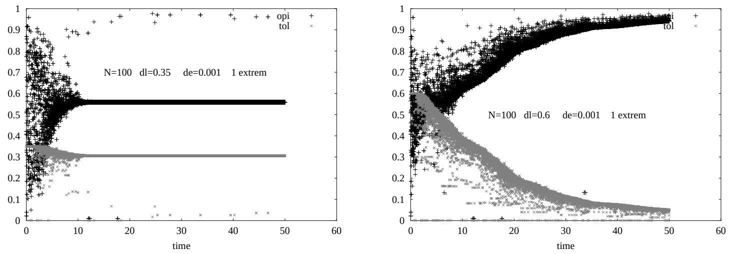

Fig. 3.8 Time plots of opinion (black '+') and tolerance (grey 'x') dynamics exhibiting a "centrism" attractor on the left frame and a single extremist cluster on the right frame. Time is given in the average number of updating per agent. The number of agents is $N = 100$, extremists' tolerance is $de = 0.001$ and two opposite extremists are initially present. Any pair of agents can a priori interact. The centrism attractor was obtained when the initial centrist tolerance was $dl = 0.35$ and the extremist attractor when $dl = 0.6$

centrists as compared to extremists. After this preliminary phase, the center clustered agents can either slowly evolve towards extremism if they still feel their influence, in other words when their tolerance is larger than their distance to extremists ($dl > 0.5$); otherwise (when $dl < 0.5$) they are not anymore under extremists influence and remain at the center. Due to the random character of the initial opinion distribution and pair sampling, the $dl = 0.5$ is not a sharp boundary. but rather indicates a dynamical crossover.

Convergence characteristic time then differs: convergence is fast for the centrism attractor and slow for the extremism attractor. The ratio in convergence time is approximately the ration in the initial fraction of centrists and extremists.

More generally, which attractor is reached depends mainly upon the parameters of the simulation (number and initial tolerance of extremists, and the initial tolerance dl of the other agents). A sketchy conclusion is that some kind of extremism prevails for larger values of the tolerance of initially non-extremist agents when $dl > 0.5$, and centrism when $dl < 0.5$. In other words, the outcome of the dynamics is largely determined by the tolerance of the non-extremists agents. But systematic studies show *co-existence* parameter regions where several attractors can be reached depending upon the specific initial distribution of opinions and upon the specific choice of updated pairs.

3.4
Diffusion of Culture

Axelrod[10] proposed in 1997 a model for dissemination of cultures based on the idea that cultures were sets of cultural traits which could exist under different specifications. For instance cultural traits would be described by how major challenges are answered; how are house built, cooking, language, hunting habits and so on. Axelrod's model inspired by the analogy between memes (cultural traits) and genes which exist as different alleles, goes beyond more classical models of cultural dynamics based on population genetics, because the dynamics of cultural traits are not taken as independent from each other. Axelrod model involves space modeled by a lattice. We here start with a simpler model based on binary traits and full mixing.

3.4.1
Binary traits

Usually people have opinions on different subjects, which can be represented by vectors of opinions. A typical case are political opinions: citizens can have positive or negative opinions on m topics such as social security, free trade or protectionism, pro or anti-nuclear energy, abortion etc. In some sense a traditional view about politics such as the standard view in France about single dimension position on a right/left axis would corresponds to continuous opinions, while the more pragmatic anglo-saxon view of political platforms would correspond to binary opinion vectors.

In accordance with our previous hypotheses, we suppose[26] that agents interact according to their distance in opinions. In order to simplify the model, we use binary opinions. An agent is characterised by a vector of m binary opinions about the complete set of m topics. We use the notion of Hamming distance between binary opinion vectors (the Hamming distance between two binary opinion vectors is the number of different bits between the two vectors). We only treat the case of complete mixing; any pair of agents might interact and adjust opinions according to how many opinions they share.

The adjustment process only occurs when agents agree on at least $m - d$ subjects (i. e. they disagree on $d - 1$ or fewer subjects). The rules for adjustment are as follows: among the binary traits which are different for the two agents, one is selected and the two agents take the same opinion (randomly selected among the two possible) with probability μ. Obviously this model has connections with population genetics in the presence of sexual recombination when reproduction only occurs if genome distance is smaller than a given threshold. Such a dynamics results in the emergence of species (see Higgs and Derrida [34]). We are again interested in the clustering of opinion vectors. In fact clusters of opinions here play the same role as biological species in evolution.

3.4.2
Results

μ and N only modify convergence times towards equilibrium; the most influential factors are threshold d and m the number of topics under discussion. Most simulations were done for $m = 13$. For $N = 1000$, convergence times are of the order of 10 million pair iterations. For $m = 13$:

- When $d > 7$, the radius of the hypercube, convergence towards a single opinion occurs (the radius of the hypercube is half its diameter which is equal to 13, the maximum distance in the hypercube).
- Between $d = 7$ and $d = 4$ a similar convergence is observed for more than 99.5 per cent of the agents with the exception of a few clustered or isolated opinions distant from the main peak by roughly 7.
- For $d = 3$, one observes from 2 to 7 significant peaks (with a population larger than 1 per cent) plus some isolated opinions.
- For $d = 2$ a large number (around 500) of small clusters is observed.

The same kind of results are obtained with larger values of m: two regimes, uniformity of opinions for larger d values and extreme diversity for smaller d values, are separated by one critical d_c value for which a small number of clusters is observed (e.g for $m = 21$, $d_c = 5$. d_c seems to scale in proportion with m).

Figure 10 represents the populations of the different clusters at equilibrium (iteration time was 12 000 000) on a log-log plot according to their rank-order (Zipf plots). No scaling law is obvious from these plots, but we observe the strong qualitative difference in decay rates for various thresholds d.

The main difference between continuous opinion and vector opinion dynamics is in the sharpness of the transition and the scaling of the number of clusters. Let us recall than for continuous opinions, the number of clusters follow a staircase: everytime the quantity $int(1/2d)$ increases, this number increases by one. For the low threshold values in the case of vector opinion dynamics the number of clusters scale increases exponentially in m in the limit of large N: A finite fraction of the sites of the hypercube $\{0, 1\}^m$ is occupied by clusters. The transition from consensus to an exponentially number of clusters is abrupt.

3.4.3
Axelrod model of cultural diffusion

In Axelrod model[10], cultural features in some small number (d) can take different integer values (from 1 to q). Agents are situated at the nodes of a 2 dimensional lattice. Interactions only occur among neighbours; each agent has

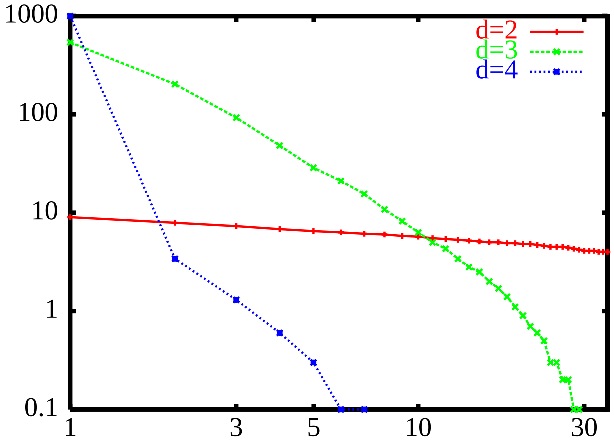

Fig. 3.9 Log-log plot of average populations of clusters of opinions arranged by decreasing order for $N = 1000$ agents ($\mu = 1$).

4 nearest neighbours. At each time step, one pair of neighbours is selected and they interact with a probability proportional to their overlap, i.e. their relative number of common features. As a result of interaction one trait, among those which are different, is copied from one agent to the other.

When starting from random initial conditions, one observes a sharp transition between cultural homogeneity and the persistence of clusters with many different cultures depending upon the values of q and d. Cultural homogeneity is observed for smaller q values, and cultural diversity when q is above a threshold which increases with d and the network size.

Castellano etal[9] derived master equations describing the evolution of population of links P_m between sites which share m common traits. Numerical integration of the master equation yields the same transition in behaviour as directly observed on the original system. Several variants were described by Klemm and co-authors[33], including a probabilistic version.

3.5 Conclusions

3.5.1 Range and limits of opinion dynamics models

All the models that we presented, whether binary, continuous or vector opinions were based on simple hypotheses about imitation processes.

Unfortunately, we still miss a solid experimental and empirical basis to validate these hypotheses Too few experiments[4] have been conducted in social psychology which could validate a choice among rationality, seen as individuals only taking into account knowledge or direct experience, versus complete conformism, not to mention in the case of conformism a possible distinction between bounded confidence or total openness.

The above skepticism is not shared in the agent based models community; computer scientists often take as granted the "psychological" hypotheses that describe their agents individual behaviour and envision a possible validation against empirical data. The sociophysics approach is more prone to admit the shortcomings of the simple agent description and search for generic properties rather than for a "Popperian" validation.

One can still hope, that modeling will provoke experiments, in the same way as the General Equilibrium Model inspired the empirical work of Tversky and Kahneman[37] to check its assumptions.

Many empirical studies have been done on social networks: who influences whom under what conditions, see for instance Lazega[39]. Empirical evidence about elections[56] have been proposed to validate the results of some variant of the Sznajd model. Some empirical data might comfort the predictions of binary opinions models: consensus of either type, everyone choosing one solution or the other. For instance one observation of the adoption of environmental-friendly contracts by farmers in northern Italy in two neighbouring regions, showed that the adoption rate was nearly zero in Piemonte and maximum in Lombardia[13]. A book, “How hits happen”[38] is entirely devoted to the occurence of these very contrasted success rate in the media and toy industry for instance.

3.5.2
How to convince

Understanding opinions dynamics, or just believing that you understand it, leads to strategies to convince: producers, sellers, political leaders, government agencies are faced with the issue:

- Producers and designers have to decide the specifics of a product, including price, in view of the buyers expectations, including the acceptation dynamics of the product. Externalities and social factors are of prime importance in computers and information technology, but also in the media industry.

4) references to experimental work include [2],[11],[4].

- Political science often discuss politicians strategies in terms of game theory; but choosing a political platform, including those topics on which to campaign, also relates to the approaches that we here discussed.
- The importance of social dynamics for the acceptation of reforms by the stakeholders has been long discussed by the academics working on policy design[47]; how much of it is used used by international institutions such as IMF or World Bank is another story.

Marketing specialists could of course dream of designing a campaign with a full knowledge of the social network and of the local processes which would convince most individuals at the lowest cost. We are far from such a knowledge and this might be a good thing!

But we can still observe sellers tricks' using networking knowledge to convince prospects to change their cars or to buy insurance policies. Such tricks are also use by politicians to convince voters or public agencies to convince citizens to adopt new practices more favourable to the environment.

The multiplicity of attractors leading to strong differences in market shares or uptake rates of proposals is sometimes perceived as a source of undeterminacy by scientists. But it is an asset for marketing people: it implies that they do have openings to interfere and modify the outcome of the dynamics. This would not be the case if there were only one attractor.

3.5.3
How to make business

Several papers were written on coupled seller/buyers dynamics: how one seller should adjust his prices taking into account opinion dynamics? or what would happen to a seller who would re-adjust her prices as a function of the volume of sales ignoring opinion dynamics? The hypotheses made by sociophysicists are often in contradiction with the perfect knowledge hypothesis of the General Equilibrium theory; and not surprisingly they give rise to totally different predictions.

When using the formalism of networks of "counter" automata, the S variable represent the binary decision buy/not buy or eventually buy brand A or buy brand B. The threshold θ represents the (economic) utility of choice S, including eventually the price as a negative component. A rule such as equation 3.1 is interpreted as buy when social influence (the sum term) is larger than differences in utility (the threshold).

We have seen in binary choice models that because of interactions, the fraction of buyers undergoes sharp transitions as a function of the threshold distribution. In terms of economics, the aggregated demand as a function of prices is not convex as hypothetised by economists, and hence the standard equilibrium analysis does not apply.

Let us take Solomon etal[55] social percolation model as an example of the sociophysics approach. It starts from a lattice or a random graph, with sites occupied by agents i susceptible to go and see a given movie. The quality q of the movie is only known to agents who viewed it. It would determine whether an agent knowing about it would go and see it.

Each of the agents i initially informed about the movie decides to go and see it, if and only if the quality q of the movie is larger than his/her personal preference p_i, i.e. if $q > p_i$.

However, the initial agents which decided to go and see the movie become themselves sources of information about the movie: their first neighbours j are informed about the movie and decide (according to the $q > p_j$ criterion) whether to see it or not. One continues until the procedure stops by itself i.e. until all the neighbours of all the agents which went to the movie up to now, either went to the movie already or decided already not to go.

Since the agent preferences p_i are frozen, the present model is a classical percolation problem. For instance, if the personal preferences p_i on various sites i take independent random values distributed uniformly between 0 and 1, then the average probability for an agent to go to the movie, once one of its neighbours went, is the movie quality q. Consequently, if a movie happens to have a quality q lower than the percolation threshold p_c, i.e. $q < p_c$, then after a certain short time its diffusion among the public will stop. The movie will then be a flop and will not reach any significant percentage of its potential public (which is a fraction q of the lattice).

On the other hand, when the movie quality q is larger than p_c (and not too close to p_c), the movie will reach most of its potential viewers as the islands of interest will percolate. The movie will be viewed by roughly a fraction q of the entire viewers population. Up to here the model predicts the existence of a percolation transition regime for some values of the quality q and preferences p_i.

When the possible moves of the provider of the goods, in the above case the movie producer, are taken into account, the resulting dynamics can be very different from the supply/demand equilibrium predicted by economists.

- Let first suppose a "myopic" behaviour of the producer, who adjusts her price according to a "Walrasian tatonnement": she increases price when she has too many customers, and decreases it when she have too few. In the presence of a sharp discontinuity in the demand/price curve, she would never be able to adjust price to a desidered production level. Depending upon the adjustment rate of the producer versus the propagation of information among customers several dynamical regimes can be observed including Self Organised criticality or limit cycles (see Solomon etal [55] for details).

- Gordon, Nadal and co-workers [41] have studied the case of a more "strategic" producer facing a crowd of buyers which imitate each other. The preferences of the buyers, the θ_i in our formalism, are distributed according to the derivative of a logit distribution (see equation 3.8) with coefficient β. All are biased by the selling price set by the producer. Buyers imitate each other with the same influence coefficient J. For some values of β and J coefficients, around a first order transition in the market share versus price, there are two possible attractors of the opinion dynamics. The seller is interested in selling more. On the other hand, he might make more profit by selling less at a higher price. He is then facing two possible strategies in the transition region as seen on figure 3.5.3.

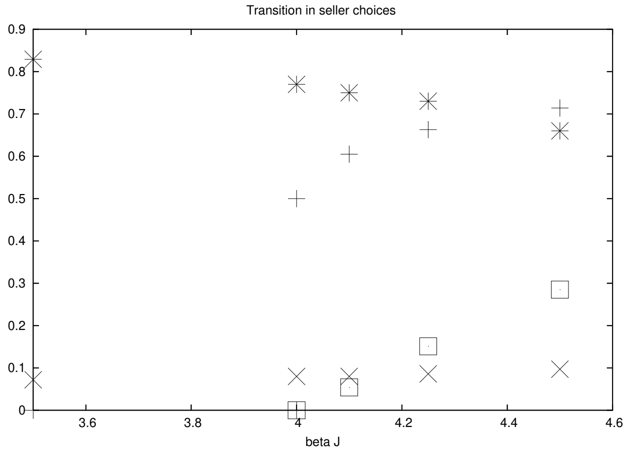

Fig. 3.10 Variation of prices and market shares of a producer facing a crowd of buyers which imitate each other, as a function of the intensity of imitation βJ. Around the transition ($\beta J = 4.18$) two solutions coexist: a low price (squares), large market share ('+') solution and a high price ('*'), small market share ('x') solution. Farther from the transition only one solution is stable: for $\beta J < 4$ the high price, small market share solution and for $\beta J > 5$ the low price, large market share solution. (By courtsey of J-P Nadal).

3.5.4 **Final conclusions**

The issues which we addressed using physics concepts and methods are fondamental issues in the social sciences:

> " If people tend to become more alike in their beliefs, attitudes and behavior when they interact, why do not all differences eventually disappear?"

Axelrod (1997)[8]

Although physicists models are too simple to give any accurate prediction concerning social systems, they are good enough to provide an appropriate framework to formulate questions. Notions such as dynamics, consensus versus diversity attractors, the importance of seeds and edges in opinion dynamics, scaling laws, etc. shed a new light on old issues in Social Sciences.

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