From dynamical mechanisms in the physical space to a statistical description of turbulent energy cascade

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A new approach of turbulence is developed, based on the simultaneous analysis of the dynamical mechanisms governing the Lagrangian evolution of velocity increments and of their statistical effects. The corresponding Fokker-Planck equation is derived, including the three inertial effects (amplification of rotation by stretching, negative feedback of rotation on stretching and scale stretching), pressure, viscous diffusion and forcing. Their influence on the cascade is then investigated and discussed.

One of the most striking features of 3D turbulent flows is to induce a very efficient dissipation of energy whereas the viscosity ν, responsible for this dissipation, becomes inefficient in this regime at the scales at which energy is injected. To solve this paradox, Richardson [1] proposed the existence of a mechanism which would transfer the energy from large scales where kinetic energy is injected in the flow, to small scales where it is dissipated by viscosity. Expressed in statistical terms, this energy cascade corresponds to the existence, on the average, of an energy flux $\varepsilon(\ell)$ in scale space. In the idealised case of an homogeneous and isotropic turbulence, Kàrmàn and Howarth [2] have expressed $\varepsilon(\ell)$ as a function of the statistics of velocity increments $\delta\vec{v}$ and Kolmogorov [3] derived some important consequences of this relation (when there exists a range of scales for which dissipation is negligible compared to energy transfer ($\varepsilon(\ell) \simeq \varepsilon$)). Although these two papers had been followed by an incredible amount of litterature (see [4] for a review of these works), two problems of fundamental importance have remained open up to now. Firstly, what are, in the physical space, the dynamical mechanisms which govern the energy cascade? Second, what determines the sign of the cascade? How does the energy flux $\varepsilon(\ell)$ evolve in particular during transients? The aim of this letter is to shed some light on these two questions by developping a new framework to analyse turbulent flows both instantaneously, in the physical space, and statistically. This is done by establishing the equations governing the Lagrangian evolution of velocity increments, and by deriving the corresponding Fokker-Plank probability equation.

Since pioneering work by Richardson [1] and Taylor [5], the energy cascade has been ascribed to the vortex stretching mechanism. However, half of the problem is forgotten if the dynamics of the stretching is not considered simultaneously. It has been shown experimentally by Andreotti et al. [6,7] that there exists a negative feedback of rotation on stretching. To introduce these two mechanisms (amplification and feed-back) into a statistical description of turbulent flows, we will consider simultaneously the two components of velocity increments:

$$\delta\vec{v} = \vec{v}\left(\vec{r}+\ell/2\vec{e}_{\|}\right) - \vec{v}\left(\vec{r}-\ell/2\vec{e}_{\|}\right) = u_{\|}\vec{e}_{\|} + u_{\perp}\vec{e}_{\perp} \quad (1)$$

The longitudinal component $u_{\|}$ is the velocity at which the two particles move away one from each other, and the transverse component $u_{\perp}$ is the rotation speed around the middle point (Fig. 1).

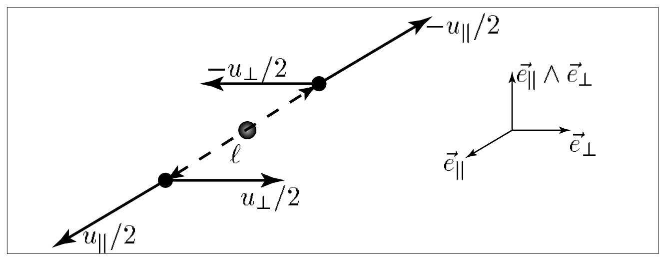

FIG. 1. Velocity increment represented in the middle point frame of reference. $u_{\perp}/\ell$ is the rotation rate around the middle point and $u_{\|}/\ell$ the local stretching rate.

Starting from the Lagrangian equation of motion for the two particules (Fig. 1), the equations governing the evolution of the increment three characteristics, ℓ, $u_{\|}$ and $u_{\perp}$, are obtained by projection on the basis $(\vec{e}_{\|}, \vec{e}_{\perp}, \vec{e}_{\|} \wedge \vec{e}_{\perp})$:

$$\frac{d\ell}{dt} = u_{\|} \quad (2)$$

$$\frac{du_{\perp}}{dt} = -\frac{u_{\|}u_{\perp}}{\ell} + \vec{e}_{\perp}\cdot\vec{f} \quad (3)$$

$$\frac{du_{\|}}{dt} = \frac{u_{\perp}^2}{\ell} + \vec{e}_{\|}\cdot\vec{f} \quad (4)$$

where $\vec{f}$ denotes the force difference (pressure and viscosity) between the two points: $\vec{f} = \delta\left(-\vec{\nabla}p + \nu\Delta\vec{v}\right)$.

The transfer of kinetic energy from one scale to another comes simply from the fact that the scale ℓ changes and ℓ evolves due to the stretching of material lines (eq. 2) at the time rate $u_{\|}/\ell$. If $u_{\|} < 0$ the energy flux is locally orientated from large to small scales; if $u_\parallel > 0$ the cascade is locally inverse. The first (inertial) terms of eqs. (3) and (4) precisely corresponds to the amplification of rotation due to the streching (eq. 3), and to the negative feedback (eq. 4). They can also be interpreted as Coriolis and centrifugal forces, respectively.

We can now derive from the Langevin equations (2-4), the equation of conservation of probability (Fokker-Planck). The basic quantity to be considered is the probability $P^*(u_\parallel, u_\perp, \ell \mid t)$ that the increment defined by two particles chosen at random in the control volume be characterised by ℓ, $u_\parallel$ and $u_\perp$. For the sake of simplicity, we will only consider the case of a 3D unbounded flow. In that case, the probability that two points be separated by ℓ scales as ℓ^2, due to incompressibility, so that the joint PDF conditioned by the scale ℓ reads:

$$P(u_\parallel, u_\perp \mid \ell, t) \propto \ell^2 P^*(u_\parallel, u_\perp, \ell \mid t) \tag{5}$$

Roughly speaking, $P(u_\parallel, u_\perp)$ corresponds both to an averaging in space (through the random trial of the two particles) and to an averaging over a set of possible initial conditions. It is of fundamental importance to note that the normalisation of P to 1 is not insured automatically by the Fokker-Planck equation, but results from incompressibility [8]. It can be shown from elementary Fourier analysis that incompressibility leads to four invariants [9] which should be preserved by the Fokker-Planck equation:

$$\begin{aligned}
\chi_1 &= < 1 > -1 = 0 \\
\chi_2 &= < u_\parallel >= 0 \\
\chi_3 &= \ell \frac{\partial < u_\parallel^2 >}{\partial \ell} + 2 < u_\parallel^2 > - < u_\perp^2 >= 0 \\
\chi_4 &= \ell \frac{\partial < u_\parallel^3 >}{\partial \ell} + < u_\parallel^3 > -3 < u_\parallel u_\perp^2 >= 0
\end{aligned} \tag{6}$$

Each term in the Langevin equations (2-4) (each dynamical mechanism) leads to a separate term in the PDF equation :

$$\partial_t P = I[P] + \Pi[P] + F[P] + V[P] \tag{7}$$

$I[P]$ corresponds to the inertial effects (first terms in eqs. (2, 3, 4), $\Pi[P]$ to pressure, $F[P]$ to the forcing and V[P] to the viscous dissipation. An elementary Langevin/Fokker-Planck transform [10] of inertial terms allows to find the exact expression of $I[P]$:

$$I[P] = -\frac{1}{\ell}\left(\ell \frac{\partial P}{\partial \ell} + 2 u_\parallel P + \frac{\partial u_\perp^2 P}{\partial u_\parallel} - \frac{\partial u_\parallel u_\perp P}{\partial u_\perp}\right) \tag{8}$$

The kinetic energy per unit mass $E_<(\ell, t)$ "contained" in the scales smaller than ℓ can be constructed as:

$$E_<(\ell) = \frac{1}{4}\left(< u_\parallel^2 > + < u_\perp^2 >\right) \tag{9}$$

Its time derivative due to inertial effects i.e. the energy flux $\varepsilon(\ell)$ through scale ℓ follows from eq. (8):

$$\varepsilon(\ell) = -\frac{1}{4}\left(\frac{2}{\ell} + \frac{\partial}{\partial \ell}\right)\left[< u_\parallel^3 > + < u_\parallel u_\perp^2 >\right] \tag{10}$$

Note that the energy flux exclusively proceeds from the scale stretching (eq. 2) and not from the interaction between stretching and rotation (eqs. 3 and 4) which is a conservative process.

Pressure is constrained by incompressibility. The inertial term $I[P]$ preserve three of the four incompressibility relations (6):

$$\begin{aligned}
\partial_t \chi_1)_{I[P]} &= -\left(\frac{\partial}{\partial \ell} + \frac{2}{\ell}\right)\chi_2 = 0 \\
\partial_t \chi_2)_{I[P]} &= -\frac{1}{\ell}\chi_3 = 0 \\
\partial_t \chi_3)_{I[P]} &= -\left(\frac{\partial}{\partial \ell} + \frac{2}{\ell}\right)\chi_4 = 0
\end{aligned} \tag{11}$$

The fourth one $\partial_t \chi_4)_{I[P]}$ is not null and thus gives a constraint on the pressure term. The pressure gradient being a regular force, only its local component i.e. its average conditionned by the local properties of the flow appears in the Fokker-Planck equation [11]. From a physical point of view, the pressure gradient mainly balances part of the inertial forces, namely their potential part. The pressure gradient being null on the average, this suggests the following form,

$$\begin{aligned}
< -\vec{e}_\parallel \cdot \delta \vec{\nabla} p \mid u_\parallel, u_\perp >= &+ \frac{\alpha(\ell)}{\ell}\left(u_\parallel^2 - < u_\parallel^2 >\right) \\
&- \frac{\beta(\ell)}{\ell}\left(u_\perp^2 - < u_\perp^2 >\right)
\end{aligned} \tag{12}$$

which turns out to be a generalisation of the pressure Poisson equation: in a rotating region of the flow, the pressure is locally minimum and balances part of the centrifugal force while in a stretching dominated region, the pressure is locally maximum. Both α and β should thus be positive. The pressure gradient cannot balance the solenoidal part of inertial terms in particular the effect of stretching on rotation (conservation of angular momentum) [12]. On the average, we thus get:

$$< -\vec{e}_\perp \cdot \delta \vec{\nabla} p \mid u_\parallel, u_\perp >= 0 \tag{13}$$

Finally, within this assumption, the pressure term $\Pi[P]$ reads:

$$\Pi[P] = \frac{\partial}{\partial u_\parallel}\left[\beta\left(u_\perp^2 - < u_\perp^2 >\right) - \alpha\left(u_\parallel^2 - < u_\parallel^2 >\right)\right]\frac{P}{\ell} \tag{14}$$

The pressure gradient conserves energy. Thus, $\alpha(\ell)$ and $\beta(\ell)$ should verify:

$$\alpha(\ell) < u_\parallel^3 >= \beta(\ell) < u_\parallel u_\perp^2 > \quad (15)$$

Moreover, pressure should insure incompressibility. The first two incompressibility invariants (6) are automatically preserved. χ_3 is invariant, due to relation (15). The fourth one, χ_4, gives the second independant equation on $\alpha(\ell)$ and $\beta(\ell)$, which can be easily explicited if needed:

$$\partial_t \chi_4)_{I[P]} + \partial_t \chi_4)_{\Pi[P]} = 0 \quad (16)$$

Figure 2 shows $\alpha(\ell)$ and $\beta(\ell)$, computed for a Gaussian field whose energy distribution $E_<(\ell)$ was taken from experimental data. We observe that the two functions $\alpha(\ell)$ and $\beta(\ell)$ continuously decrease from small to large scale where they vanishes. This means that the mean compensation of inertial forces by pressure is much more efficient at small scale than at large one, due to the progressive decorrelation of the pressure field when the scale increases.

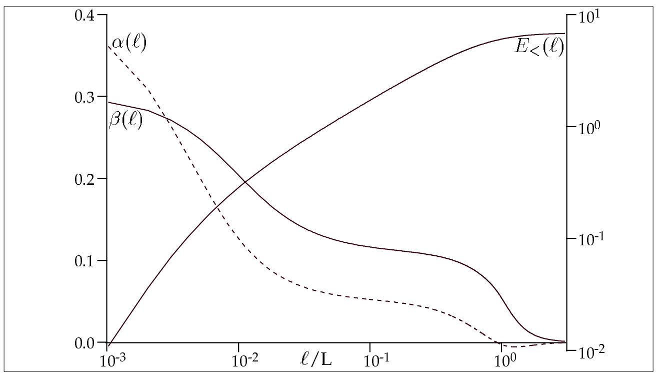

FIG. 2. Functions $\alpha(\ell)$ (dotted line) and $\beta(\ell)$ (solid line) plotted for an initial gaussian field whose energy distribution $E_<(\ell)$ is plotted in log scale (right axis).

There is a fundamental problem in modelling the forcing term $F[P]$: no mechanical injection of energy can occur in an infinite, unbounded flow. The evolution of the energy contained in an arbitrary volume V_a is given by

$$\partial_t \int_{V_a} \frac{v^2}{2} dV\Big)_F = \int_{S_a} \left(\frac{v^2}{2}(\vec{v}-\vec{v}_a) + p\vec{v}\right) \cdot d\vec{S} \quad (17)$$

where $\vec{v}$ is the material velocity and $\vec{v}_a$ is the velocity of the surface S_a which encloses the volume V_a. If S_a corresponds to the cell boundary, the controle volume is material ($\vec{v}_a = \vec{v}$) so that the energy injection exclusively comes from pressure. On the contrary, if we consider a small eulerian volume ($\vec{v}_a = 0$) in the flow bulk, the energy injection essentially comes from the material transport of energy through the boundary. A simple trick to model this forcing is to introduce a scale dependant noise in the Lagrange equations (3) and (4):

$$F[P] = 2\varepsilon_\parallel \frac{\partial^2 P}{\partial u_\parallel^2} + \varepsilon_\perp \left(\frac{\partial^2 P}{\partial u_\perp^2} - \frac{\partial}{\partial u_\perp}\left(\frac{P}{u_\perp}\right)\right) \quad (18)$$

where $\varepsilon_\parallel(\ell)$ and $\varepsilon_\perp(\ell)$ are the powers injected at scales smaller than ℓ in stretching and in rotation, respectively. $\varepsilon_\parallel(\ell)$ and $\varepsilon_\perp(\ell)$ can be expressed using the second incompressibility invariant (6):

$$\varepsilon_\parallel = \frac{1}{3} f(\ell)\varepsilon \quad (19)$$

$$\varepsilon_\perp = \frac{1}{3}(2f(\ell) + \ell f'(\ell))\varepsilon \quad (20)$$

as functions of the mean power ε and of the non dimensional function $f(\ell)$ which characterizes the distribution in scale of the injected power. By definition of ε, $f(\ell)$ should tend to 1 when the scale ℓ becomes larger than the integral scale L. For scales much smaller than L, $f(\ell)$ should scale as ℓ^2. The energy globally injected at scales smaller than ℓ is then:

$$\partial_t E_<(\ell))_{F[P]} = \left(\frac{2}{\ell} + \frac{d}{d\ell}\right) \frac{f(\ell)\varepsilon\ell}{3} \quad (21)$$

Using either the Langevin formulation of Navier-Stokes [8] or the usual eulerian one, it is easy to show that the powers dissipated by viscosity in longitudinal and transverse motions are respectively:

$$D_\parallel(\ell) = \nu \left[\frac{1}{2}\frac{\partial^2 < u_\parallel^2 >}{\partial \ell^2} + \frac{2}{\ell}\frac{\partial < u_\parallel^2 >}{\partial \ell}\right]_0^\ell \quad (22)$$

$$D_\perp(\ell) = \nu \left[\frac{1}{2}\frac{\partial^2 < u_\perp^2 >}{\partial \ell^2} + \frac{1}{\ell}\frac{\partial < u_\perp^2 > - < u_\parallel^2 >}{\partial \ell}\right]_0^\ell \quad (23)$$

Using the fact that viscous diffusion preserve the gaussianity of the velocity field and the incompressibility, we can write the viscous term as:

$$V[P] = 2D_\parallel \frac{\partial^2 P}{\partial u_\parallel^2} + D_\perp \left(\frac{\partial^2 P}{\partial u_\perp^2} - \frac{\partial}{\partial u_\perp}\left(\frac{P}{u_\perp}\right)\right) \quad (24)$$

Note that the injection and the dissipation terms ($F[P]$ and $V[P]$) are of the same form.

The power dissipated at scales smaller than ℓ follows from the viscous term,

$$\partial_t E_<(\ell))_{V[P]} = 2\nu \left[\left(\frac{2}{\ell} + \frac{\partial}{\partial \ell}\right)\frac{\partial E_<}{\partial \ell}\right]_0^\ell \quad (25)$$

so that that the equation of energy conservation (eqs. 10, 21 and 25) is now closed. If a stationnary state is achieved, the latter simplifies into:

$$6\nu \frac{d^2 E_<}{d\ell^2}\Big)_{\ell=0} = \varepsilon \quad (26)$$

$$< u_\parallel^3 >= -\frac{4}{5} g(\ell)\varepsilon\ell + 6\nu \frac{dE_<}{d\ell} \quad (27)$$

$g(\ell)$ being a non-dimensional function linked to $f(\ell)$ by the differential equation:

$$g(\ell) + \frac{1}{5}\ell g'(\ell) = 1 - f(\ell) \tag{28}$$

This fully generalises the Kàrmàn-Howarth relation [2] to any unbounded turbulent flow with an arbitrary injection spectrum, the price to pay being the space averaging.

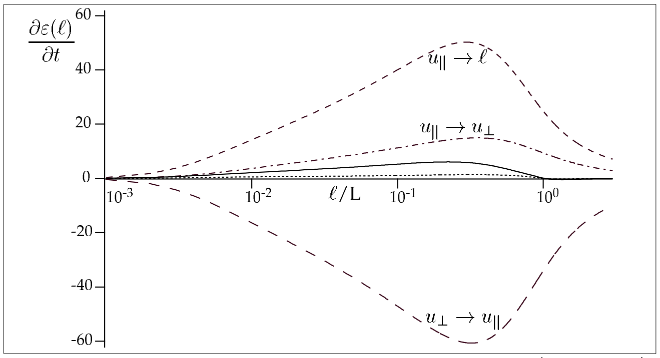

FIG. 3. The time derivative of the energy flux (solid line), starting from the gaussian field shown in figure 2, is decomposed according to the different dynamical mechanisms: scale stretching (dashed line), rotation stretching (dot-dashed line), retroaction of the rotation on stretching (long-dashed line) and pressure (dotted line)

Using the Gaussian field whose energy distribution is plotted on fig. 2, we have computed the time derivative of the energy flux to investigate the departure from gaussianity (fig. 3. It appears to be positive in the whole range of scale, indicating the creation at short times of a direct energy cascade. It turns out that stretching tends to create a positive energy flux (from large to small scales) by both its action on scale and on rotation. On the contrary, the negative feedback of rotation on stretching tends to generate an inverse energy cascade (from small to large scales). The pressure induced flux is slightly positive but negligible in front of the inertial mechanism. An important point is that the scale stretching (and only it) depends on the fact that the space be 3D and not 2D, and is much less efficient in the later case. This of course suggests that the inverse cascade in 2D turbulence could be modelled and explained by a simple translation of the model.

We have proposed in this letter an approach of turbulence problem based on the simultaneous investigation of dynamical mechanisms in the physical space and of their statistical effects, giving thus a physical alternative to multifractal formalism. The first step was here to build a Fokker-Planck equation on the basis only of conservations laws and of physical mechanisms, and to point out their effect on the cascade. What remains to be done is the integration of the model and the comparison with experiments. In particular, the question of intermittency will be examined in a forthcoming article.

[1] L.F. Richardson, *Weather Prediction by Numerical Process.* Cambridge University Press, Cambridge (1922).

[2] T. von Kàrmàn and L. Howarth , *Proc. R. Soc. Lond.* A **164**, 192-215 (1938).

[3] A.N. Kolmogorov, *Dokl. Akad. Nauk. SSSR* **30** 9 (1941).

[4] U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov*, Cambridge University Press (1995).

[5] G. I. Taylor *Proc. Roy. Soc.* A **164**, 15-23, (1938).

[6] B. Andreotti, S. Douady and Y. Couder *Advances in Turbulence VII*, 251-254, Kluwer Academic Publishers (1998).

[7] B. Andreotti, S. Douady and Y. Couder *Turbulence modeling and vortex dynamics*, 92-108, Springer Verlag (1997).

[8] Incompressibility is already related to the PDF normalisation in the Langevin formulation of Navier-Stokes equations:

$$\frac{d\vec{r}}{dt} = \vec{v} + \sqrt{\nu}\vec{\Gamma}(t) \qquad \frac{d\vec{v}}{dt} = -\vec{\nabla}p$$

where the viscous diffusion appears as a stochastic noise $\vec{\Gamma}(t)$. If the fluid is incompressible, the probability of observing a particle at a position $\vec{r}$ depends neither on $\vec{r}$ nor on time: the Eulerian probability $P(\vec{v} \mid \vec{r})$ is proportional to the Lagrangian probability $P^*(\vec{r}, \vec{v})$. Thus, P verifies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} + \frac{\partial v_i P}{\partial r_i} = \frac{\partial p}{\partial r_i}\frac{\partial P}{\partial v_i} + \nu\frac{\partial^2 P}{\partial r_i^2}$$

If the velocity field is perfectly known, $P(\vec{v} \mid \vec{r})$ is a delta function. The continuity equation is simply the integral of the Fokker-Planck equation, and the equation of motion is the integral of the same equation multiplied by v_j. For complementary informations on Langevin diffusion processes, see
A. Einstein, *Ann. Physik* **17**, 549 (1905) and **19**, 371 (1906).
P. Langevin, *Comptes rendus* **1146**, 530 (1908).

[9] It is worth noting that this relations are valid independently of any assumption on homogeneity and isotropy (see [2]). The price to pay for these general validity is evidently the integration over space.

[10] H. Risken, *The Fokker-Planck equation*, Springer-Verlag (1989).

[11] F.O. Minotti and C.Ferro Fontàn, *European J. Mech.* B **17**, 505-518 (1998).

[12] This can be shown assuming that the transverse component of the pressure gradient is of the form

$$< -\vec{e}_{\perp} \cdot \delta\vec{\nabla}p \mid u_{\|}, u_{\perp} >= \frac{\gamma(\ell)}{\ell} u_{\|} u_{\perp}$$

and using both the third incompressibility invariant χ_3 and the conservation of energy.

[13] While completing this letter, we became aware of a work by B.Chertkov, A.Pumir and B. Shraiman, to appear in *Phys. Fluids*. It shares the same general formalism although done using a Lagrangian tetrad. Among the main differences, we stress here the specificity of rotation versus stretching and the link between incompressibility statistical invariants and pressure.