

Discrete Stress-Field in two-dimensional Soap Foams

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Abstract. – Dry two-dimensional soap foams are very well described at the level of a single cell or of a vertex by the the laws of Laplace and Plateau. However there is a lack of understanding of their behavior at a macroscopic scale. Continuum mechanics, which relates stress and strain seems to fail for soap foams (as well as for other macroscopic systems such as granular media). After a brief discussion of the commonly used definition for the stress tensor, we introduce a discrete local stress field at the vertices of the foam. We show that an equilibrium relation is directly induced by the definition, and that the whole stress field can be determined for given boundary conditions, without any supplementary stress-strain relation. Macroscopic properties such as the shape of the foam and its free energy are directly given by the stress field.

Introduction. – Many materials with a macroscopic structure exhibit a complex behavior in rheology experiments [1]. Such is the case of liquid foams which present globally a viscoelastic behavior. Small deformations are reversible but when the strain exceeds a critical value, topological rearrangements (T1) occur and the deformation becomes irreversible [2]. It has been shown that the free energy and the stress drop down when these rearrangements appear. When a constant shear rate is applied, the repetition of these singularities cause stress fluctuation [2, 3] and the formation of shear bands [4]. Similar properties, stress fluctuation [5] and shear banding [6, 7], are reported for granular media, too. The continuum mechanical approach, where a constitutive relation relates unequivocally stress and strain, fails here [8, 9].

Continuum mechanics is based on scale separation. In classical materials, the microscopic length scale -the distances between the molecules or the length of their interaction - is much smaller than the characteristic length scale of macroscopic variations [10]. In this case a mesoscopic scale can be introduced. A volume at this intermediate scale includes a sufficient number of molecules to define such macroscopic quantities as temperature, pressure and stress, but is small enough to consider these quantities as constant inside this volume. Thermal fluctuations ensure that all configurations are accessible so that the microscopic details can be replaced by mean values.

The stress tensor σ is mathematically defined by the relation

$$d\mathbf{F} = \sigma d\mathbf{S} \tag{1}$$

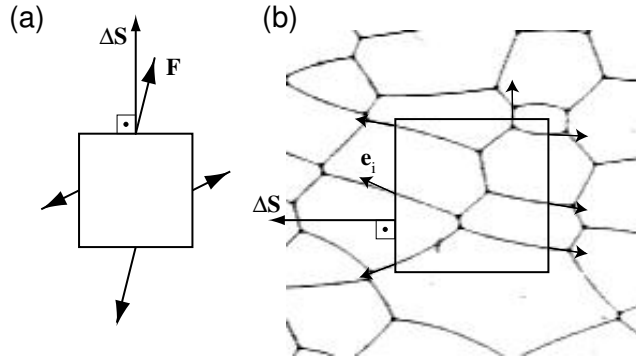


Fig. 1 - (a) A mesoscopic box. The force on the right side is the opposite of the force on the left side, the total torque is zero. (b) In soap foams, the force on a side of the box is the sum over the unity vectors carried by the intersecting cell walls. The size of the box does not satisfy a mesoscopic approximation because the forces on the opposite sides do not compensate each other.

where $d\mathbf{F}$ is the force on the surface element $d\mathbf{S}$. In practice however this relation concerns the mesoscopic scale and the infinitely small surface element $d\mathbf{S}$ has to be replaced by a finite surface $\Delta\mathbf{S}$ so that relation (1) becomes:

$$\mathbf{F} = \sigma\Delta\mathbf{S} \quad (2)$$

In two dimensions a box such as shown in fig. 1a is chosen and the forces on its sides are then determined. The mesoscopic approximation is satisfactory when the force on one side is balanced by the force on the opposite side and when the total torque is zero. Equation 2 can then be applied. Otherwise, the box is either too small and the fluctuations dominate, or too large and the long scale variations of the measured quantities become perceptible.

For ‘macroscopic’ materials such as granular media and liquid foams, the scale separation is weaker. The microscopic length scale (size of the grains or of the cells) is not much smaller than the length scale of the variations of the macroscopic quantities. It is thus difficult to define an intermediate scale. Some authors avoid this lack of a mesoscopic scale by using temporal averages [11, 12]. However, the characteristic energies in the macroscopic systems are much larger than the thermal energy kT , so that the systems are jammed, remaining trapped in states of local minima of their free energy landscape [1]. The use of temporal averages is therefore questionable.

In this letter we propose a definition for a microscopic stress tensor in the special case of dry, two-dimensional soap foams. This *microscopic* stress, in contrast to the usual *mesoscopic* stress, will be adapted to the geometry of the system. We will show that this proposed microscopic stress can be measured and leads to a well defined stress field at the macroscopic scale. With the knowledge of this field, all important macroscopic quantities such as free energy and shape of the foam can be deduced. The discrete stress field can be calculated for given boundary conditions.

Stress in the soap foam crystal. - A two dimensional soap foam can be produced by confining a three dimensional soap foam between two parallel (glass-) plates [2]. The distance e between the plates has to be smaller than the characteristic bubble size so that the cell walls

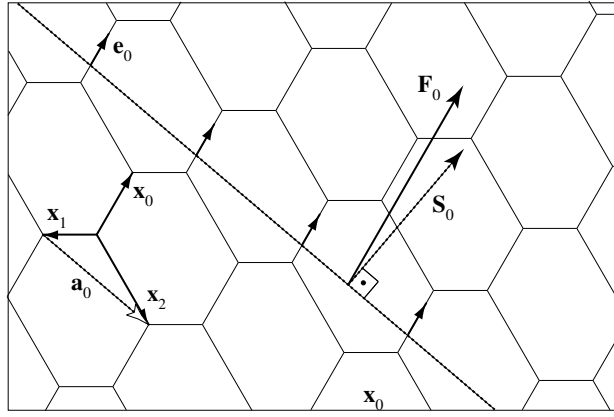


Fig. 2 – The foam crystal is characterized the three vectors $\{\mathbf{x}_i\}$. The primitive vector, which define the discrete translation symmetry are $\mathbf{a}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1}$. The stress of the crystal can be calculated by choosing surfaces which are parallel to the primitive vectors.

are perpendicular to the plates. If the volume fraction of the liquid phase is negligible and the thickness of the cell walls is close to zero, the foam will be called dry and can be represented by a 2D planar network.

Such a foam is an ideal model system since all the necessary laws are well established at the "microscopic" level. All physical quantities are given by its geometry and can be determined on a photograph (see fig. 1). The free energy of such a dry foam is proportional to the surface separating the gaseous phase and the liquid phase. It is given for a dry two-dimensional foam by $2\gamma e \sum l_i$ where l_i is the length of the cell wall i . The forces due to surface tension γ have thus a constant magnitude $2\gamma e$ and are carried by the cell walls. Its normal component on a curved cell wall has to be balanced by the pressure differences between the cells: $\Delta p = 2\gamma/R$ (Laplace's law). The mechanical equilibrium at a vertex is realized when the three cell walls intersect with $2\pi/3$ angles. The mechanical equilibrium presents a local minimum of the total free energy. In contrast to glasses, the foam can not evaluate spontaneously towards lower free energy states because the energy barriers are much higher than the thermal energy kT . To simplify the expressions in following text we set $\gamma = 1/2$ and $e = 1$.

Although there is in general no sufficient scale separation to introduce the intermediate mesoscopic scale, a particular, but experimental realizable, case exists, were the stress is well defined. This is the soap foam crystal composed of regular hexagonal cells [13]. We will use the obtained stress to introduce the a microscopic stress field that will be defined in a disorder case, too.

We characterize the crystal shown in fig. 2 by the three vectors $\{\mathbf{x}_i\}$, pointing from a given vertex to its neighbors. In the following the enumeration is always taken in a counterclockwise order and is taken modulo three. The pattern is invariant under translation by a primitive vector $\mathbf{a}_i = \mathbf{x}_{i+2} - \mathbf{x}_{i+1}$ (see fig. 2) This discrete translational symmetry makes the foams homogeneous at large scale. This is why, contrary to the disordered case, a mesoscopic scale exists. The stress can be calculated using its definition (eq. 2). The simplest way to do this, is to chose surfaces parallel to this primitive vector (fig. 2):

$$\mathbf{S}_i = N D_{\frac{\pi}{2}}(\mathbf{x}_{i+2} - \mathbf{x}_{i+1}) \quad (3)$$

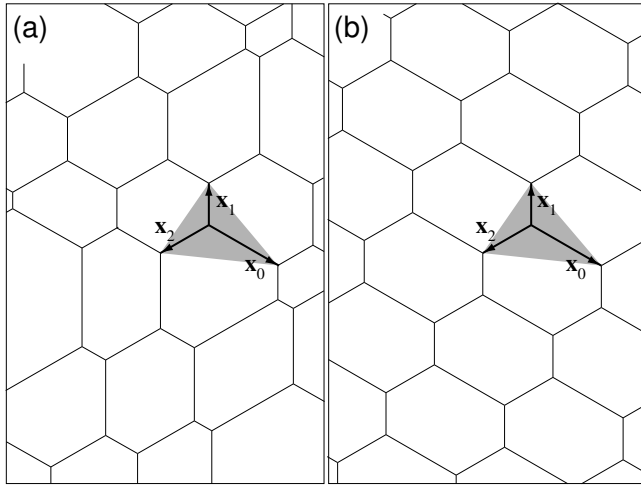


Fig. 3 – In order to define the stress of the marked vertex in the disordered foam (a), it is replaced by a solid triangle (gray). The solid triangle is placed in a crystal structure (b), where the stress can be calculated.

where N is the number of crossed cells and D_α is the operator which rotates a vector by an angle α :

$$D_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad (4)$$

The surface intersects with N cell walls \mathbf{x}_i . Each cell wall applies a force on the surface in the direction of the unit vector carried by it and with constant magnitude $2\gamma e = 1$. The total force is thus $\mathbf{F}_i = N\mathbf{x}_i/x_i$. This is compared with eq. 2 and we obtain:

$$\frac{\mathbf{x}_i}{x_i} = \sigma D_{\frac{\pi}{2}} (\mathbf{x}_{i+2} - \mathbf{x}_{i+1}) \quad (5)$$

These are three linear vectorial equations for the four components of σ . They are not independent: when we sum the three equations, the right hand side becomes zero by construction, the left hand side because of the $2\pi/3$ angles. Further, it can be shown in a straight forward calculation that σ is symmetrical and has therefore three independent components. The determinant and the trace of σ take the simple expressions:

$$\begin{aligned} \det(\sigma) &= \frac{1}{\sum_{i=0}^2 x_i x_{i+1}} = \frac{\sqrt{3}}{2V} \\ \text{tr}(\sigma) &= \frac{2}{\sqrt{3}} \frac{\sum_{i=0}^2 x_i}{\sum_{i=0}^2 x_i x_{i+1}} = \frac{L}{V} \end{aligned} \quad (6)$$

where V is the volume of a cell and L the wall length per cell.

The microscopic stress tensor. – We would like to generalize the previous results to disordered, heterogeneous foams. For this purpose we want to define a stress field by measuring a stress tensor at each vertex. This discrete stress field should be self consistent so that no further parameter (box size and position) has to be chosen and that an equilibrium condition can

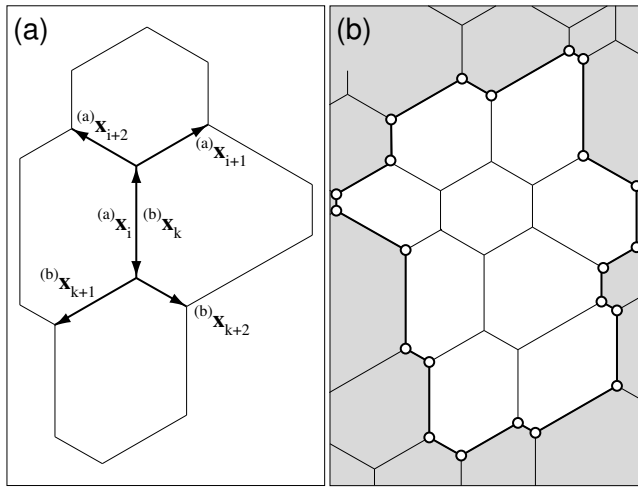


Fig. 4 – (a) Two neighboring vertices. The equilibrium condition can be presented in terms of σ . (b) A simple example for given boundary conditions. Either we consider the shape of the gray border as boundary (imposed deformation) or the stresses at the vertices marked by circles. In the two case the foam and the stress field inside can be reconstructed.

be written down. We limit ourselves here to foams with straight cell walls of the type shown in fig. 3a. We cannot do this by simple extrapolation of the mesoscopic procedure, choosing boxes that only include one vertex. As the stress would depend only on the chosen box and not on the vertex, we would obtain arbitrary results with no physical meaning. Nevertheless we can use the results obtained for the foam crystal after exerting a *gedankenexperiment*: let us replace a vertex and its closest neighbors in fig. 3a by a solid triangle. This operation will not have any influence on rest of the foam. The stress we are looking for will be assigned to this triangle. To do this we repeat the *gedankenexperiment* in reverse: we place the triangle in a crystal structure built up by periodic repetition of the pattern inside the triangle (fig. 3b). The physical properties of the solid triangle are the same in the crystal and the disordered network, its stress will not change. In the crystal structure we can calculate the stress using eq. 5: the stress for each vertex is found. It is given by its three cell wall vectors \mathbf{x}_i .

With this definition, no mesoscopic box must be defined, only the intrinsic properties of the foam are used. The stress has the same properties that were found in the foam crystal: the total force on a element is zero and the torque vanishes, because σ is symmetrical. Furthermore, σ is easy to measure on photographs of the foam and therefore appropriate for experimental investigation. Referring to eqs 6, we can write down the total length of the cell walls and therefore the free energy of the foam in terms of stress:

$$\frac{1}{2} \sum_k \sum_{i=0}^2 x_i^{(k)} = \frac{\sqrt{3}}{4} \sum_k \sum_{i=0}^2 \frac{Tr(\sigma^{(k)})}{\det(\sigma^{(k)})} \quad (7)$$

where (k) is the enumeration of the vertices.

In order to clarify the significance of the microscopic stress, we discuss the analogs of the continuum mechanical equilibrium condition and stress-strain relation. The equilibrium condition in continuum mechanics involves spatial differentiation:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (8)$$

Understanding differentiation as handling with infinity small differences, these equations are a kind of neighborhood relations. The equivalent of these relation exists in the discrete stress field, too. It is given directly by the definition (eq. 5). The left hand side term is the force carried by the cell wall i . The force on vertex (a) in fig. 4 applied by the cell wall i must be opposite to the force applied by the same cell wall on vertex (b). Thus:

$$\sigma^{(a)} D_{\frac{\pi}{2}} \left(\mathbf{x}_{i+2}^{(a)} - \mathbf{x}_{i+1}^{(a)} \right) + \sigma^{(b)} D_{\frac{\pi}{2}} \left(\mathbf{x}_{k+2}^{(b)} - \mathbf{x}_{k+1}^{(b)} \right) = 0 \quad (9)$$

The differentiation $\partial/\partial x$ is thus replaced by finite differences $\mathbf{x}_{i+2} - \mathbf{x}_{i+1}$. The two equilibrium conditions (eqs. 8) are replaced by one vectorial equation. Let us now inverse the definition of the stress (eq. 5):

$$\mathbf{x}_{i+2} - \mathbf{x}_{i+1} = D_{-\frac{\pi}{2}} \sigma^{-1} \frac{\mathbf{x}_i}{x_i} \quad (10)$$

Once the directions of the cell walls fixed, (it is sufficient to fix one absolute direction for the whole foam) the vector difference $\mathbf{x}_{i+2} - \mathbf{x}_{i+1}$ can be given in terms of σ , too. When we include the information that the angle between \mathbf{x}_{i+1} and \mathbf{x}_{i+2} is $2\pi/3$, we can obtain the cell wall vectors as functions of the stress. We abbreviate this by introducing the operator \mathbf{X}_i . When applied to σ , it gives the vector of the cell wall i

$$\mathbf{x}_i = \mathbf{X}_i(\sigma) \quad (11)$$

This operator can be understood as the equivalent of the stress-strain relation in continuum mechanics. It relates the stress field to the geometrical form of the network. We can use it to reformulate the equilibrium condition (eq. 9):

$$\sigma^{(a)} (\mathbf{X}_{i+2}^{(a)} - \mathbf{X}_{i+1}^{(a)}) + \sigma^{(b)} (\mathbf{X}_{k+2}^{(b)} - \mathbf{X}_{k+1}^{(b)}) = 0 \quad (12)$$

The most common exercise in continuum mechanics is to calculate stress and strain for given boundary conditions. We can use the previous results to determine the discrete stress field. Let us illustrate this with the example shown in fig. 4b. The boundary conditions are given by fixing the stress tensors on the vertices marked with circles. Only the topology is need as supplementary information to calculate the stress field and the geometry of the foam. Note that its shape, drawn with bold lines, can be directly obtained by using the \mathbf{X}_i operator. We have to determine the stress field. For this propose we have to calculate 8 stress tensors in the example of fig. 4b, this means 8 times three independent components. 16 cell walls are corresponding to 16 equilibrium conditions which are given by vectorial equations of type eq. 9. This are the 32 equations that are required to determine the 32 variables. This example can be generalized to an arbitrary number of vertices: The discrete stress field is entirely determined by its boundaries. If the stress field is known once, the free energy is given by eq. 7 and the geometrical structure is obtained by the use of the \mathbf{X}_i operator.

Since the macroscopic quantities, the shape of the foam, its free energy and the forces applied to it, can be deduced from the stress, the given description is complete. But as the boundaries fix completely the geometry, the volumes of the cells result of it. This is directly due to our basic assumption limiting ourselves to cells with straight walls. If - which is the physical case - the volume is fixed, we have to allow curved cell walls. Curved cell are needed for cells with more or less than six sides, too. An extension of the introduced definition will be necessary.

Conclusion. - The concept of stress and strain as continuous fields fails in soap foams. This is due to the weakness of the separation between small and large scales. We introduced

a discrete stress field in two-dimensional soap foams by a *gedankenexperiment*: a vertex and its neighbors is replaced by a solid triangle. This solid triangle can be placed in a foam crystal which is created by the repetition of the replaced object. The triangle will have the same properties in both foams, and can be calculated in the crystalline situation. We assign the same value σ to the vertex in the disordered foam. This procedure is limited to straight cell walls. We showed that (i) the geometry of the network can be directly given by the discrete stress field. As a result, (ii) all important quantities such as macroscopic shape and free energy are also determined. Furthermore we pointed out that (iii) there is an equilibrium condition, which directly results in the definition of σ . This equilibrium condition is a neighborhood relation which allows to (iv) calculate the stress field for given boundary conditions. Therefore, this definition of the stress provides a complete and self-consistent description of the mechanics of the foam. An important feature is that it is a measurable quantity which can be used for experimental or numerical investigations.

Finally, it is possible to imagine that the basic concept, which led us to the construction of σ , can be applied to several other systems such as solid foams of grains. It would then have to include the microscopic knowledge - the physics at the level of one vertex or one grain and the topology - into the construction of the stress field.

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REFERENCES

- [1] LUI, A.J. and NAGEL S.R., *Nature*, **396** (1998) 21.
- [2] WEAIRE D.L. and HUTZLER S., *The physics of foams* (Clarenton Press, Oxford) 1999
- [3] LAURIDSEN, J. and TWARDOS, J. and DENNIN, M., *Phys. Rev. Lett.*, **89** (2002) 09.
- [4] DEBRÉGEAS, G. and TABUTEAU, MELLIO, J.-M., *Phys. Rev. Lett.*, **87** (2001) 17.
- [5] POULIQUIEN O. and GUTFRAIND, *Phys. Rev. Lett.*, **53** (1996) 552.
- [6] MUETH, D. and DEBRÉGEAS, G. and KRACZMAR, G. and ENG, P.J. and NAGEL, S. and LAEGER, H., *Nature*, **406** (2000) 385.
- [7] TÓRÓK and KRISHNAMARTY, S. and KERTESZ J. and ROUX, S., *Phys. Rev. Lett.*, **84** (2000) 3815.
- [8] RAJCHENBACH, J. and CLÉMENT, E. and DURAN, J., *Fractal aspects of Materials*, edited by F. FAMILY *et al.*, Vol. **367** (M.R.S Symposium, Pittsburgh) 1995, p. 525.
- [9] RAJCHENBACH, J., *Physics of dry granular media*, edited by HERMANN, HOVI AND LUDING (Kluwer Academic, the Netherlands) 1998, p. 421-439.
- [10] LANDAU, L.D. AND LIFSCHITZ, E.M., *Theory of Elasticity* (Reed Educational and Professional Publishing, Oxford) 1986.
- [11] REINELT, D.A and KRAYNIK, A.M, *J. Rheol.*, **22** (2000) p. 1-22.
- [12] Y. JIANG, M. ASIPAUSKAS and J. A. GLAZIER and M. AUBOUY and F. GRANER, *Eurofoam 2000*, edited by P. ZITHA and J. BANHART and G. VERBIST (MIT Verlag, Bremen) 2000, p. 297-304.
- [13] PRINCEN, H.M., *J. Coll. Interf. Sci.*, **91** (1983) p. 160-175.