Fracture Spacing in Layered Materials

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We perform an elastostatic analysis of a periodic array of cracks under constant loading. We give an analytical solution and show that there is a limitation to the fracture spacing, due to a transition from an opening to a compressive loading. For this configuration, the threshold of the fracture spacing depends on neither the applied strain nor the elastic parameters of the material. This result shows that, in the general case of layered materials, the physical mechanism that is responsible for the limitation in the density of fractures is related mainly to the geometry of the problem. This is in agreement with observations and with recent numerical results.

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Parallel open mode brittle fractures, or joints, in layers are common structures in Nature (see Fig. 1). They are currently observed in the Earth's crust such as in sedimentary rocks [1]. They are also present in laminated engineering materials [2]. In most cases, it is observed that the ratio s of the crack spacing to the layer thickness cannot decrease below a certain threshold value [3], although the physical intuition suggests that the spacing should decrease with increasing applied loading. Indeed, since the joints are stopped by the neighboring layers, fracturing new joints would be the only way to dissipate the stored energy. Therefore, as the tension increases, it seems that there is no limitation to the density of cracks.

A recent finite element analysis has shown that there is a limitation to the density of fractures [4]. This threshold was explained by the change from an opening to a compressive mode at the middle of the spacing (at half the wavelength). So a new fracture cannot occur. This analysis is in agreement with other simulations and experiments on the permeability of joints in the geophysics literature [5]. However, these numerical treatments do not allow one to set the control parameters which fix the spacing bound. In the following, we propose an exact treatment of this crack problem in a model situation, where the different layers have the same elastic properties, which is also the case considered in the numerical simulations of [4]. We show the existence of the instability from tension to compression as the spacing decreases. Moreover, in our model the spacing threshold is of order 1, and does not depend on either the applied loading or the elastic parameters of the material layers. It turns out that this elastic instability is a generic feature which is related mainly to the geometry of the problem. This result suggests that for layered materials with different elastic properties, the physical mechanism that is responsible for the limitation to the density of fractures is purely geometrical. However, the spacing threshold in the general case will depend on the elastic mismatch parameters between the layers.

In our approach, we consider a material sample with an infinite array of parallel fractures equally separated by a distance λ. We choose half the wavelength as the length unit. The crack spacing s is then given by $1/a$, where $2a$ is the length of the cracks in dimensionless units (see Fig. 2). Fixing the ends of the fractures is a way to mimic the effect of the neighboring layers, since the observed cracks do not cross the neighboring interfaces. We assume that the sample is loaded in the y direction by an average strain ϵ_∞ which represents the tension supported by the layer. We perform a classical elastostatic analysis and show that the fracture spacing threshold does not depend on either the applied strain or the elastic parameters of the material, which are the Young modulus E and the Poisson ratio ν.

Under plane stress conditions, the two-dimensional strain tensor $\overline{\overline{\epsilon}}$ is related to the stress tensor $\overline{\overline{\sigma}}$ by

$$\sigma_{ij} = \frac{2}{1 - \nu^2}[(1 - \nu)\epsilon_{ij} + \nu\epsilon_{kk}\delta ij]. \qquad (1)$$

The plane strain configuration is recovered by a suitable change of the Poisson ratio. For convenience, all the quantities in Eq. (1) are dimensionless: $\overline{\overline{\sigma}}$ is scaled by

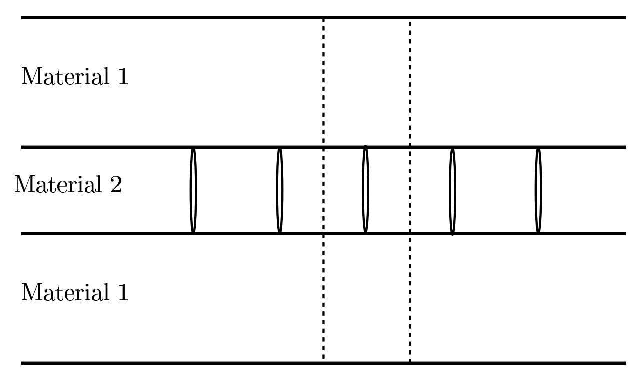

FIG. 1. Schematic representation of a layered system with a periodic array of cracks generated in the less compliant material. The dotted region represents the unit cell that will be studied.

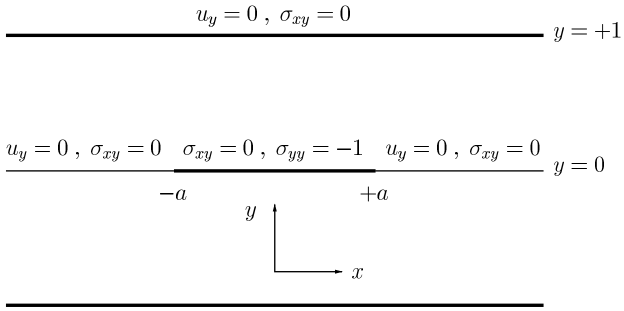

FIG. 2. Schematic representation of the elastostatic problem.

$E\epsilon_\infty/(1 - \nu^2)$ and $\overline{\overline{\epsilon}}$ is scaled by $2\epsilon_\infty/(1 - \nu^2)$. The body is loaded by means of a uniform remote tension of magnitude $\sigma_\infty = 1$. Since the cracks' faces are traction-free, it is convenient to superimpose this solution with the one where the cracks are subject to compressive stresses of the same magnitude. Moreover, due to the periodicity of the configuration, it is enough to solve the problem for the stress field in the region $0 \leq y \leq 1$.

The equilibrium equations in the absence of body force are given by

$$(1 - \nu)\Delta u_i + (1 + \nu)\frac{\partial^2 u_j}{\partial x_i \partial x_j} = 0, \tag{2}$$

where $\vec{u}$ is the displacement field. The boundary conditions for this problem are simply given by

$$\sigma_{xy}(x, 1) = u_y(x, 1) = 0, \tag{3}$$

$$\sigma_{xy}(x, 0) = 0, \tag{4}$$

$$u_y(|x| > a, 0) = 0, \tag{5}$$

$$\sigma_{yy}(|x| < a, 0) = -1. \tag{6}$$

The conditions on the displacement and stress fields in Eq. (3) are imposed by the periodicity of the configuration, while Eqs. (4) and (5) are imposed by the symmetry of the opening mode loading. Finally, Eq. (6) comes from the fact that the total solution of the present problem has to satisfy traction-free boundary conditions on the cracks' faces. At this stage, the problem depends only on the Poisson ratio ν and on the dimensionless crack length a.

The strip geometry and the boundary conditions suggest the use of Fourier sine and cosine transforms [6]. Because of the symmetry of the problem one can write the displacement field in the form

$$u_x(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(k, y)\sin kx\, dk, \tag{7}$$

$$u_y(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(k, y)\cos kx\, dk. \tag{8}$$

Also, we define the Fourier transform of $\sigma_{yy}(x, y)$ by

$$\sigma_{yy}(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty s(k, y)\cos kx\, dk. \tag{9}$$

The equations satisfied by the functions $f(k, y)$ and $g(k, y)$ are readily derived from Eq. (2)

$$(1 - \nu)f'' - 2k^2 f - (1 + \nu)kg' = 0, \tag{10}$$

$$2g'' - (1 - \nu)k^2 g + (1 + \nu)kf' = 0, \tag{11}$$

where the derivatives are with respect to y. After some algebraic manipulations, one finds that

$$\frac{f(k, y)}{g(k, 0)} = \frac{1 + \nu}{2}\left\{\left[ky - \frac{1 - \nu}{1 + \nu}\coth k + \frac{k}{\sinh^2 k}\right]\cosh ky + \left[\frac{1 - \nu}{1 + \nu} - ky\coth k\right]\sinh ky\right\}, \tag{12}$$

$$\frac{g(k, y)}{g(k, 0)} = \frac{1 + \nu}{2}\left\{\left[\frac{2}{1 + \nu} + ky\coth k\right]\cosh ky - \left[ky + \frac{2}{1 + \nu}\coth k + \frac{k}{\sinh^2 k}\right]\sinh ky\right\}. \tag{13}$$

Written under this form, one can verify that the functions $f(k, y)$ and $g(k, y)$ satisfy the bulk equations (10) and (11) and the boundary conditions (3) and (4). We also obtain, using Eq. (1),

$$\frac{s(k, y)}{g(k, 0)} = -\frac{k}{2\sinh^2 k}[ky\cosh k(2 - y) + k(2 - y)\cosh ky + \sinh k(2 - y) + \sinh ky]. \tag{14}$$

Note that $s(k, y)$, and thus $\sigma_{yy}(x, y)$, does not depend on the Poisson ratio. So it is completely independent of the material properties. In the following, the two main functions that will be manipulated are $s(k, 0)$ and $s(k, 1)$. They are easily calculated from Eq. (14):

$$s(k, 0) = -[k + F(k)]g(k, 0), \tag{15}$$

$$s(k, 1) = -G(k)g(k, 0), \tag{16}$$

where

$$F(k) = \frac{k}{\sinh^2 k}[k + e^{-k}\sinh k], \tag{17}$$

$$G(k) = \frac{k}{\sinh^2 k}[k\cosh k + \sinh k]. \tag{18}$$

Note that $\sigma_{yy}(x, 1)$ is the key quantity of this problem, since the sign of $[1 + \sigma_{yy}(x, 1)]$ indicates if the middle of the spacing between two fractures is under tension or compression. If it is under tension, one can expect nucleation of a new fracture, which will be responsible for defining a new wavelength (half the previous one). However, if it is under compression, the breaking process stops and the elastic energy must be dissipated according to a different scenario. It is generally believed that this elastic energy is used for the opening of the preexisting cracks and induces a full compression of the horizontal layer in the y direction [5].

In order to solve the problem completely, the determination of the function $g(k,0)$ is needed. This is done by using the boundary conditions (5) and (6), which can be expressed as

$$\sqrt{\frac{2}{\pi}}\int_0^\infty g(k,0)\cos kx\,dk = 0, \qquad |x| > a, \tag{19}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty [k + F(k)]g(k,0)\cos kx\,dk = 1, \qquad |x| < a. \tag{20}$$

This is a set of dual integral linear equations whose analytical solutions are not available. A compilation of known solutions of such equations can be found in [7]. The condition (19) is automatically satisfied if the function $g(k,0)$ is given by

$$g(k,0) = \sqrt{\frac{\pi}{2}}\int_0^a \Phi(t)J_0(kt)\,dt, \tag{21}$$

where J_0 is the Bessel function, and $\Phi(t)$ is a yet unknown function. Replacing $g(k,0)$ as given by Eq. (21) in Eq. (20) leads to an integral equation for Φ:

$$\frac{d}{dx}\int_0^x \frac{\Phi(t)}{\sqrt{x^2 - t^2}}\,dt + \int_0^a dt\,\Phi(t)\int_0^\infty dk\,F(k)J_0(kt)\cos kx = 1, \qquad |x| < a. \tag{22}$$

This equation can be simplified by using Abel inversion transforms [6]. One obtains a Freedholm integral equation for $\Phi(t)$, given by

$$\Phi(t) = t - t\int_0^a H(t,u)\Phi(u)\,du, \qquad 0 < t < a, \tag{23}$$

where

$$H(t,u) = \int_0^\infty F(k)J_0(kt)J_0(ku)\,dk. \tag{24}$$

Despite many attempts, we did not succeed in finding an analytical solution for Eq. (23). However, the numerical resolution of this integral equation is straighforward.

Once the integral equation is solved for each value of a, one can determine the displacement and stress fields at any location. As an example, one can calculate the stress intensity factor, which is a quantity of interest in the field of fracture mechanics. In this case, the dimensionless stress intensity factor K_I is given by [6]

$$K_I(a) = \sqrt{\frac{\pi}{a}}\,\Phi(a). \tag{25}$$

Figure 3 shows the variation of the stress intensity factor as a function of the crack length a.

The quantity of interest in the present problem is $\sigma_{yy}(0,1)$, the stress at the middle spacing between two successive cracks. It is simply given by

$$\sigma_{yy}(0,1) = -\int_0^a dt\,\Phi(t)\int_0^\infty dk\,G(k)J_0(kt). \tag{26}$$

The sign of $1 + \sigma_{yy}(0,1)$ determines whether there is a tensile or a compressive loading. It is clear that for $a = 0$,

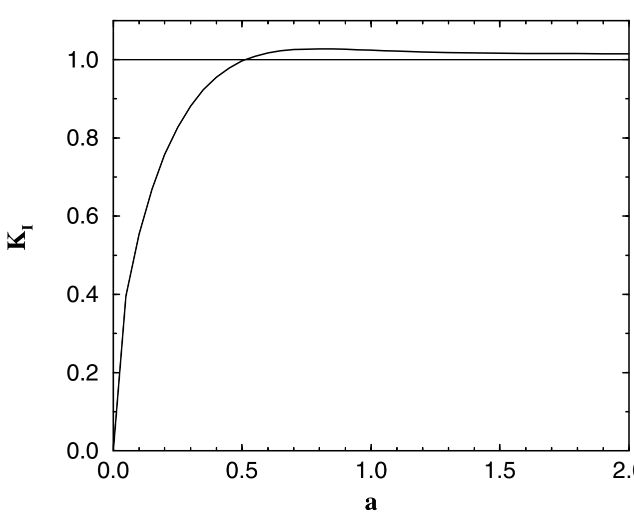

FIG. 3. The dimensionless stress intensity factor K_I as a function of the crack length a.

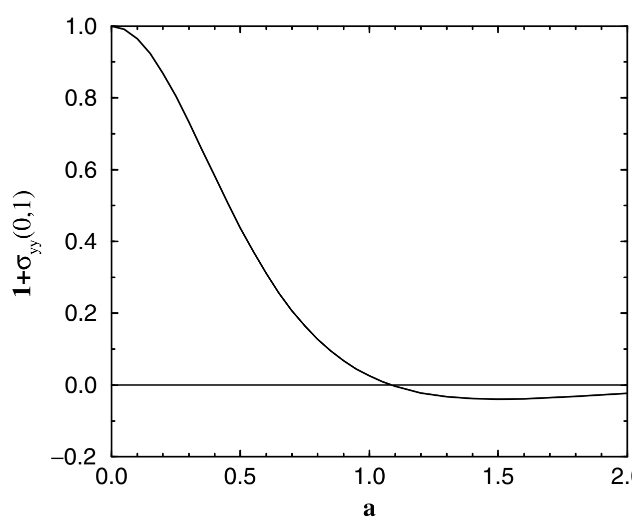

FIG. 4. The magnitude of the total stress at the point $x = 0$ and $y = 1$ as a function of the crack length a.

$\sigma_{yy}(0,1) = 0$ [see Eq. (26)], and for $a \gg 1$, the sample is completely broken, so one must have $1 + \sigma_{yy}(0,1) \rightarrow 0$. This behavior is in agreement with Fig. 4, which represents the total stress at the middle of the spacing as a function of the crack length a. Figure 4 also shows that at $a \approx 1$, or equivalently $s \approx 1$, the stress changes effectively from an opening to a compresssive mode, as has been found numerically for a particular case [4]. The transition point does not depend on any material parameter. This threshold does not even depend on the Poisson ratio in our model. This is an intrinisic instability which is due only to the geometry of the loading.

This simple model which can be solved exactly shows a well-known feature of fracture in layered materials. Our numerical value for the spacing applies for identical layers. The physical origin of the instability lies in the exchange of the elastic energy from fracturing to opening of the existing cracks. Note that introducing layers of different elastic constants will not modify the generic feature of the instability. Evidently, the spacing threshold will depend on the ratios of the elastic constants but the instability should always occur, since its origin lies in the geometry of the cracks' patterns.

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