Path Prediction of Kinked and Branched Cracks in Plane Situations

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Using the asymptotic expansion of the stress field ahead a curved extension of a straight crack, some general results on the paths selected by kinked and branched cracks are derived. When dealing with the dynamic branching instability of a single propagation crack, the experimentally observed shape of the branches is recovered without introducing any adjustable parameter. It is shown that the length scale introduced by the curved extension of the branches is given by the geometrical length scale of the experiment. The theoretical results agree quantitatively with the experimental findings.

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The field of fracture mechanics is concerned with the quantitative description of the mechanical state of a deformable body containing a crack or cracks. The continuum theory of fracture mechanics studies the nucleation of cracks, the conditions for which they propagate and their dynamics [1,2]. In the framework of continuum theory of brittle fracture, the relationship between internal stress and deformation and the pertinent balance laws of physics dealing with mechanical quantities do not include the possibility of material separation. Indeed, the "equation of motion" of the crack tip is based on additional statements for crack growth. The most popular criterion for crack propagation in a two dimensional elastic body consists of two parts; the Griffith hypothesis and the principle of local symmetry.

The Griffith energy criterion [1,2] states the intensity of the loading necessary to promote propagation through $\mathcal{G} = \Gamma$, where $\mathcal{G}$ is the energy release rate, which is defined as the rate of mechanical energy flow into the crack tip per unit crack advance, and Γ is the fracture energy of the material. The principle of local symmetry states that the crack advances such that the in-plane shear stress in the vicinity of the crack tip vanishes. This rule was first proposed for quasistatic cracks [3], and generalized to rapidly moving cracks [4]. Moreover, it was shown that the two criteria rise from the same physical origin [4]. The energy release rate is the component, F_1, of the driving force along the direction of crack motion. The Griffith energy criterion can thus be reinterpreted as a material force balance between F_1 and a resistance force to crack advance per unit length of the crack front; $F_1 = \Gamma$. However, this equation of motion is not sufficient to determine the trajectory of a crack. If one assumes that material force balance holds at the crack tip, one should impose that the component of the material force perpendicular to the direction of crack propagation vanishes. This condition is identically satisfied if the loading in the vicinity of the crack tip is purely tensile.

The Griffith criterion and the principle of local symmetry predict adequately the path and the stability of slowly propagating cracks [5]. Controlled experiments on quasistatic cracks confirm the theoretical results [6]. In the case of fast crack propagation, the experiments on PMMA (poly-methyl-methacrylate) and glass samples [7–9] have identified a dynamic instability of a propagating crack which is related to a transition from a single crack to a branched crack configuration. Some aspects of this dynamic instability were described within the theory of brittle fracture mechanics [10]. However, the subsequent shape of the branches has not been explained yet.

The main purpose of the present study deals with this aspect of the branching instability. Following the analysis of [11,12], the asymptotic expansion of the static stress field ahead of a curved extension of a crack tip is presented. Using these exact results, some features of the paths selected by kinked and branched cracks are derived. As a main result, the experimentally observed shape of the branches is recovered without introducing any additional parameter. Moreover, in the case of the experiments in [7–9], the length scale introduced by the curved extension of the branches is found to be the width of the sample. The present study shows that both the branching instability threshold, the branching angle and the subsequent paths of the branches are predicted within the continuum theory of brittle fracture mechanics.

Stress field ahead of a curved extension of a crack.— Consider an elastic body containing a straight crack with a kinked curved extension of length s and a kink angle $\lambda\pi$ (see Fig. 1). Let xOy denote the coordinate system

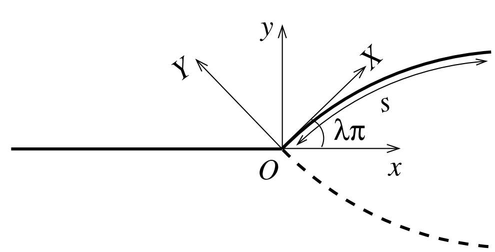

FIG. 1. Schematic representation of a straight crack with a kinked (or symmetrically branched) curved extension.

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with the Ox axis directed along the initial straight crack, and let XOY denote the coordinate system with the OX axis directed along the tangent to the extension at the point O. These two coordinate systems are related by

$$X = x\cos\lambda\pi + y\sin\lambda\pi, \tag{1}$$

$$Y = y\cos\lambda\pi - x\sin\lambda\pi. \tag{2}$$

Using minimal assumptions, it was shown in [11] that the asymptotic shape of the crack extension is necessarily given by

$$Y = aX^{3/2} + O(X^2), \tag{3}$$

where a is a curvature parameter whose dimension is $(\text{length})^{-1/2}$. Moreover, the expansion of the static stress intensity factors $K'_l(s)$ $(l = 1, 2)$ at the crack tip in powers of s obeys the general form

$$K'_l(s) = \sum_{m=1,2} F_{lm}(\lambda)K_m + \sum_{m=1,2} [G_m(\lambda)T\delta_{lm} + aH_{lm}(\lambda)K_m]\sqrt{s} + O(s). \tag{4}$$

In this expansion, K_l and T are the static stress intensity factors and the nonsingular stress in the universal expansion of the stress field at the original crack tip O without the kinked extension, which is given by

$$\sigma_{ij}(r,\theta) = \sum_{l=1,2} \frac{K_l}{\sqrt{2\pi r}}\Sigma_{ij}^{(l)}(\theta) + T\delta_{ix}\delta_{jx} + O(\sqrt{r}), \tag{5}$$

where (r, θ) are the polar coordinates with $r = 0$ located at the point O, and $\Sigma_{ij}^{(l)}$ are known functions describing the angular variations of the stress field components [2]. The functions F_{lm}, G_l, and H_{lm} are universal in the sense that they depend neither on the geometry of the body nor on the applied loading. They depend on the kink angle only and their computation was performed in [12].

The asymptotic expansions as given by (3) and (4) must necessarily be considered if the extension of the crack is obtained by actual propagation of the crack and not simply by arbitrary machining of the body [11]. Because of the linearity of the problem, the expressions (3) and (4) can be predicted from dimensional arguments. Since the K_l's scale as stress $\times \sqrt{\text{length}}$ and T scales as stress, the first order expansion of the stress intensity factors in (4) must involve an additional parameter whose dimension is $1/\sqrt{\text{length}}$. This parameter is provided by the asymptotic expansion (3) of the kinked extension.

It is straightforward to extend these results to quasi-static branched cracks. The crack tip of each branch extension must obey similar asymptotic expansions, with different universal functions F_{lm}, G_l, and H_{lm}. The functions F_{lm} for a symmetrically branched configuration have been computed in [10], while the computation of G_l and H_{lm} can be carried out using the same approach as for the kinked crack problem [12].

The detailed expansion of the stress intensity factors being available, it remains to combine it with a propagation criterion for crack path prediction. The Griffith energy criterion [1,2] and the principle of local symmetry [3] impose that the advance of the crack tip is controlled by the following equations

$$\mathcal{G}'_1(s) \equiv \frac{1}{2\mu}K_1'^2(s) = \Gamma, \tag{6}$$

$$K'_2(s) = 0, \tag{7}$$

where μ is the Lamé shear coefficient of the material. Note that Eq. (7) imposes the symmetry of the stress field in the vicinity of the crack tip which in turn restricts the crack direction of propagation. Therefore, the crack path is mainly selected by the principle of local symmetry, while Eq. (6) controls the intensity of the loading necessary to the crack propagation. In the following, the stability of a tensile crack and the path selection of branched cracks will be discussed in the light of these general results.

Response of a tensile crack to a shear perturbation.— Cotterell and Rice [13] analyzed the stability of an initially straight crack under tensile loading in the presence of a small shear perturbation. They found that the non-singular stress T governs the stability mechanism. The so called T-criterion states that when $T > 0$, the straight crack propagation is unstable and the crack path grows exponentially. While when $T < 0$, the straight crack propagation is stable and the crack path behaves as $y(x) \sim \sqrt{x}$.

Equation (7) states that each coefficient in the expansion (4) of $K'_2(s)$ in terms of s must vanish. Therefore, in the presence of a small shear loading ($|K_2| \ll K_1$) the extension of the initial straight crack must satisfy

$$\lambda \approx -\frac{2}{\pi}\frac{K_2}{K_1}, \tag{8}$$

$$a \approx \frac{8}{3}\sqrt{2\pi}\lambda\frac{T}{K_1}, \tag{9}$$

where the expansions of the functions F_{lm}, G_l, and H_{lm} in terms of $\lambda \ll 1$ have been used [12]. Equation (8) fixes the kink angle that develops due to the presence of the shear loading, while Eq. (9) determines the subsequent curvature of the crack path. Moreover, Eqs. (8) and (9) show that the signs of λ and (a/λ) are governed by the signs of K_2 and T, respectively ($K_1 > 0$). Therefore, when $T > 0$ the perturbation induced by the shear loading is amplified and the departure from straight crack propagation increases. On the contrary, when $T < 0$ the instability of the crack path induced by the kinking process is decreased and tends to stabilize the straight crack advance (see Fig. 2). These results are consistent with the T criterion [13]. However, the subsequent paths followed

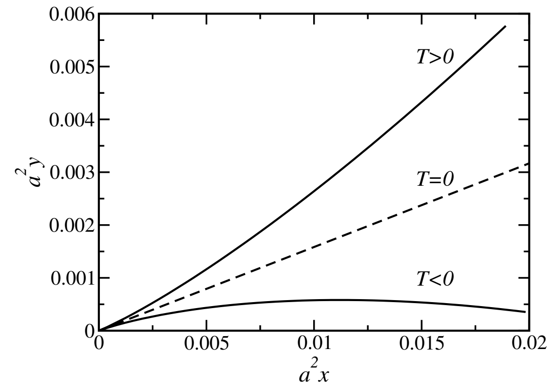

FIG. 2. Subsequent paths of an initially straight crack subjected to a small shear perturbation such that $\lambda = 0.05$.

by the cracks in either the stable or unstable case clearly differ from those predicted in [13]. This discrepancy results from the fact that the perturbation method developed in [13] is inadequate for cracks that are not smooth [12].

The condition $T < 0$ insures the stability of a straight crack propagation but it is not a necessary condition. Effectively, the straight crack might be stable even when $T > 0$, because of the presence of other stabilizing effects. A popular case study that confirms the limitations of the T criterion concerns the stability of a straight crack in a heated strip [6]. The analysis of a smooth wavy perturbation around a straight crack has shown that the crack path stability is governed by the competition between a stabilizing effect; the finite width of the strip, and a destabilizing effect; the heterogeneous thermal field [5]. The threshold of instability was found to be larger than the one predicted from the T criterion.

Shape of branched cracks.—Experiments in glass and PMMA [8,9] have established that the bifurcation of a crack tip into two branches results from the dynamic instability of a single propagating crack. When the crack speed exceeds a critical velocity v_c, a single moving crack is no longer stable and a repetitive process of microbranching occurs, which changes the crack dynamics. Although the lengths of the microbranches are broadly distributed, their functional form is well defined [9]. Once formed, the branch follows a trajectory of the form $y(x) \simeq 0.2x^{0.7}$. Furthermore, this scaling behavior does not hold at distances below 5 μm from the branching point, where branching angles of approximately 30° have been reported [9].

The problem for determining the in-plane dynamic stress intensity factors immediately after branching was formulated in [10]. It was shown that the in-plane elastic fields immediately after branching exhibit self-similar properties, and the corresponding stress intensity factors do not explicitly depend of the velocity of the single crack tip before branching. These properties are similar to the antiplane crack branching problem, which was solved exactly in [14]. This analogy allowed for the conclusion that under plane loading configurations, the jump in the energy release rate due to branching is maximized when the branches start to propagate quasistatically. Consequently, the branching of a single propagating crack under tensile loading is found to be energetically possible when its speed exceeds a threshold value [10]. Moreover, for a velocity dependent fracture energy, the critical velocity for branching, v_c, decreases with increasing $\Gamma(v_c)/\Gamma(0)$. The theoretical results for the critical velocity and the branching angle agree fairly with both experimental [8,9] and numerical results [15].

Since the branches start their propagation at vanishingly small speed [10], a quasistatic approximation is suitable for the determination of the subsequent paths followed by the branches. At a first approach, the influence of the crack tip velocity may be discarded. Consider a straight crack subjected to a tensile loading ($K_2 = 0$), that bifurcates into two symmetric branches. The two new branches propagate by satisfying the principle of local symmetry. Therefore at each crack tip, each coefficient in the expansion (4) of $K_2^l(s)$ in terms of s must vanish, which gives

$$F_{21}(\lambda) = 0, \tag{10}$$

$$a = -\frac{G_2(\lambda)}{H_{21}(\lambda)} \frac{T}{K_1}. \tag{11}$$

Equation (10) imposes $\lambda = 0.13$, corresponding to a branching angle of 24° [10], which is close to the experimental one estimated in [8,9]. Equation (10) determines the departure from a straight propagation of the branches. The functions G_l and H_{lm} are not available. However, we will assume that for $\lambda = 0.13$, the quantity $G_2(\lambda)/H_{21}(\lambda)$ is positive and of order unity. Ultimately, the computation of these functions will be a crucial step to confirm the following results.

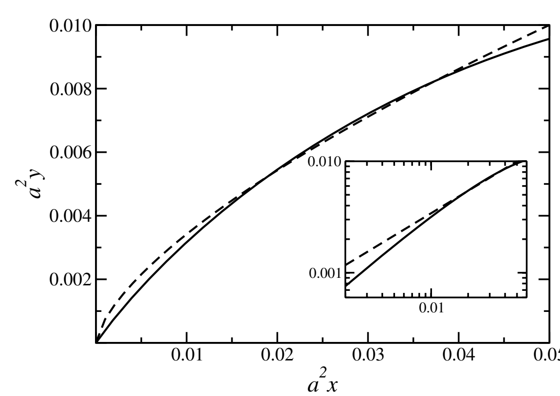

FIG. 3. Subsequent path of a crack after branching. The dashed curve corresponds to the function $y(x) = 0.074(x/|a|)^{2/3}$. The inset shows the same plots in Logarithmic scales.

Figure 3 shows the path followed by the branch as given by Eqs. (10) and (11), with the assumption that $a < 0$. The shape of the branch is very close to a power law form given by $y(x) = 0.074(x/|a|)^{2/3}$, which would correspond to a branched configuration with a branching angle of 90°. In order to compare with the experimental results, it is convenient to replace the power law behavior obtained in [9] by a more "realistic" function given by $y(x) = \alpha x^{2/3}$, with $\alpha \simeq 0.15$–$0.25\ \mathrm{mm}^{1/3}$. Therefore, the experimental and theoretical shape of the branch coincide if one takes $|a| \simeq (0.074/\alpha)^{3/2} \simeq 0.16$–$0.35\ \mathrm{mm}^{-1/2}$.

Furthermore, the curvature parameter a can be determined from the loading conditions and the geometry of the experiment. Effectively, it can be easily shown that the boundary conditions and the strip geometry of the experiments in [7–9] impose

$$\frac{T}{K_1} = \frac{\kappa - 2}{\sqrt{2(\kappa - 1)}} \frac{1}{\sqrt{W}}, \tag{12}$$

where $\kappa = (c_d/c_s)^2 \simeq 3$ is a material constant and W is the width of the strip. Here, c_d (c_s) is the dilatational (shear) wave speed of the material. Equations (11) and (12), show that the curvature parameter a scales as $1/\sqrt{W}$. The widths of plates used in experiments [8,9] were between 50 and 200 mm, so that $(T/K_1) \simeq 0.04$–$0.07\ \mathrm{mm}^{-1/2}$. If one takes $G_2(\lambda)/H_{21}(\lambda) \simeq 4$, the estimate of parameter a from the loading conditions is very close to the one evaluated directly from the comparison between the theoretical and the experimental shape of the branch. This result confirms that the length scale which governs the curvature of the branches is the geometrical length scale of the experiment. The introduction of a length scale which is related to some nonlinear process in the vicinity of the crack tip is not needed for describing the shape of the branches.

Conclusion.—In this Letter, some general results of the paths selected by kinked and branched cracks were derived. In the case of the response of a tensile crack to a small shear perturbation, it is shown that the T criterion is directly recovered from the asymptotic expansion of the stress field ahead a curved extension of a straight crack. However, the subsequent path followed by the kinked crack in either the stable or unstable case differs from those predicted in [13]. Concerning the dynamic branching instability, the shape of the microbranches was recovered and the length scale introduced by the curved extension of the branches was determined. The quantitative comparison with the experimental data of [8,9] was performed successfully. Although most of the present analysis was confined within a quasistatic approximation, the results should persist for dynamic cracks. A discussion on the relevance of the zero velocity immediately after branching can be found in [10]. The present results are in favor of the continuum theory of fracture mechanics combined with both the Griffith criterion and the principle of local symmetry. This framework provides the minimal ingredients for the prediction of the branching instability threshold, the branching angle and the subsequent path of the branches.

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