Supplementary Material for:
"Dynamic stability of crack fronts: Out-of-plane corrugations"

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The aim of this document is to provide additional technical details about the calculations of the complex functions $\overline{q}_{33}(s)$ in Eq. (11) and $\overline{q}_{11}(s)$ in Eqs. (13)-(14) in the manuscript.

I. FIRST ORDER PERTURBATIONS OF THE STRESS INTENSITY FACTOR

Willis and Movchan introduced a convolution identity (Eq. (3.12) of [S1]) that allows to calculate the change of the stress intensity factors due to a change in the crack surface, induced by a perturbation of its propagation front. Starting from an unperturbed state defined by a large crack submitted to a mode I loading, the first order perturbation terms of the stress intensity factors, $\overline{K}_I^{(1)}(\xi_2,\omega)$ and $\overline{K}_{II}^{(1)}(\xi_2,\omega)$ are given by (Eqs. (3)-(4) of the main text)

$$\overline{K}_I^{(1)}(\xi_2,\omega) = \left[\overline{Q}_{33}(\xi_2,\omega)K_I^{(0)} + \sqrt{\frac{\pi}{2}}A_3^{(0)}\right]\overline{\phi}(\xi_2,\omega)\ , \tag{A1}$$

$$\overline{K}_{II}^{(1)}(\xi_2,\omega) = \left[-\Theta_{13}\overline{Q}_{11}(\xi_2,\omega)K_I^{(0)} + i(\omega/v)\omega_{13}K_I^{(0)} - \Theta_{13}A_3^{(0)}\right]\overline{\psi}^*(\xi_2,\omega) + \overline{T}_{II}(\xi_2,\omega)\ . \tag{A2}$$

Explicit expressions for the quantities in Eqs. (A1)-(A2) can be found in [S2]. The "history" term T_{II} includes the contributions of the zero order traction terms on the broken surface convoluted with the corrugations of the crack surface $\psi(X<0,x_2,t)$. It is expressed as [S2]

$$T_{II}(x_2,t) = [U]_{11} * \langle P_1^{(1)}\rangle + [U]_{21} * \langle P_2^{(1)}\rangle - \langle U\rangle_{31} * [P_3^{(1)}]\ , \tag{A3}$$

where $[\cdot]$ and $\langle\cdot\rangle$ denote the jump and the average of the corresponding quantities through the unperturbed crack plane. Here $\mathbf{U}$ are the weight functions defined in [S2] and $\mathbf{P}^{(1)}$ are traction terms obtained by imposing the boundary conditions on the edges of the perturbed crack. The latter are given by

$$P_i^{(1)} = -\partial_\alpha\left(\sigma_{i\alpha}^{(0)}\psi\right) + \rho v^2 u_{i,XX}^{(0)}\psi\ . \tag{A4}$$

Also, one can show that $\overline{K}_I^{(1)}$ does not include such a term, i.e. that $T_I(x_2,t)=0$.

The functions $\overline{Q}_{ij}(\xi_2,\omega)$ are homogeneous of degree one and can be written as

$$\overline{Q}_{11}(\xi_2,\omega) = |\xi_2|\,\overline{q}_{11}(\omega/|\xi_2|), \qquad \overline{Q}_{33}(\xi_2,\omega) = |\xi_2|\,\overline{q}_{33}(\omega/|\xi_2|)\ , \tag{A5}$$

where $|\xi_2|$ is the modulus of the *complex* wavenumber ξ_2 and ω is a *complex* frequency. The functions $\overline{q}_{11}$ and $\overline{q}_{33}$ are given in [S2]

$$\overline{q}_{11}(s) = -i\xi_c^-(s) + \frac{i}{2}\xi_b^-(s) + \left(\frac{a_+a_- - b_+b_-}{a_+a_- + b_+b_-}\right) + \int_{C_-}\frac{d\xi_1}{2\pi}\ln(T(s,\xi_1,1))\ , \tag{A6}$$

$$\overline{q}_{33}(s) = -i\xi_c^-(s) + \frac{i}{2}\xi_a^-(s) + \int_{C_-}\frac{d\xi_1}{2\pi}\ln(T(s,\xi_1,1))\ , \tag{A7}$$

where

$$T(s,\xi_1,1) = \frac{\gamma^2[4\xi_\perp^2\sqrt{(s-v\xi_1)^2/c_d^2-\xi_\perp^2}\sqrt{(s-v\xi_1)^2/c_s^2-\xi_\perp^2} + ((s-v\xi_1)^2/c_s^2-2\xi_\perp^2)^2]}{R(v)((s-v\xi_1)^2/c_R^2-\xi_\perp^2)(\xi_1-s/v)^2}\ , \tag{A8}$$

$$a_+ = \frac{i-\xi_c^-(s)}{\sqrt{i-\xi_b^-(s)}}T_+(s,i,1)\ , \qquad a_- = \frac{c_s^2R(v)(-i-\xi_c^+(s))}{\beta v^2\sqrt{-i-\xi_b^+(s)}}T_-(s,-i,1)\ , \tag{A9}$$

$$b_+ = \sqrt{i-\xi_b^-(s)}\ , \qquad b_- = \beta\sqrt{-i-\xi_b^+(s)}\ , \tag{A10}$$

with $\xi_\perp^2 \equiv \xi_1^2 + 1$ and

$$T_\pm(s, \pm i, 1) = \exp\left(-\frac{1}{2\pi i}\int_{C_\mp} d\xi_1 \frac{\ln(T(s,\xi_1,1))}{\xi_1 \mp i}\right) . \tag{A11}$$

The contours $C_\pm$ encircle the branch points $\xi_a^\pm$ and $\xi_b^\pm$ respectively, with

$$\xi_a^\pm = -\frac{s\,v}{(c_d\alpha)^2} \pm i q_a; \qquad q_a = \frac{1}{\alpha}\sqrt{1 - \frac{s^2}{(c_d\alpha)^2}} , \tag{A12}$$

$$\xi_b^\pm = -\frac{s\,v}{(c_s\beta)^2} \pm i q_b; \qquad q_b = \frac{1}{\beta}\sqrt{1 - \frac{s^2}{(c_s\beta)^2}} , \tag{A13}$$

$$\xi_c^\pm = -\frac{s\,v}{(c_R\gamma)^2} \pm i q_c; \qquad q_c = \frac{1}{\gamma}\sqrt{1 - \frac{s^2}{(c_R\gamma)^2}} , \tag{A14}$$

with $\alpha^2 = 1 - v^2/c_d^2$, $\beta^2 = 1 - v^2/c_s^2$, $\gamma^2 = 1 - v^2/c_R^2$. The kinematic functions $\Theta_{13}(v)$ and $\omega_{13}(v)$ are defined in the main text.

Finally, using the homogeneity of the functions $\boldsymbol{U}$ and $\boldsymbol{P}^{(1)}$ and the time-independance of the zeroth order stresses, one can show that the function T_{II} can be written in Fourier space as

$$\overline{T}_{II}(\xi_2, \omega) = K_I^{(0)}\sqrt{\frac{|\xi_2|}{L}}\,\overline{t}_{II}(\omega/|\xi_2|, |\xi_2| L)\,\overline{\psi^*}(\xi_2, \omega) , \tag{A15}$$

where $\overline{t}_{II}$ is a *complex* function of degree zero and L is the geometrical length scale introduced in the main text. Here we used the fact that to leading order in ϵ one has $\psi(X, x_2, t) = \psi(X + vt, x_2)$.

II. CALCULATIONS

For a real s and $\sqrt{s^2+v^2} < c_R$, Eqs. (A6)-(A7) reduce to

$$\Re\left[\overline{q}_{11}(s)\right] = -q_c + \frac{1}{2}q_b + \left(\frac{a_+a_- - b_+b_-}{a_+a_- + b_+b_-}\right) + \Re\left[F_+(s)\right] , \tag{A16}$$

$$\Im\left[\overline{q}_{11}(s)\right] = \frac{s\,v}{(c_R\gamma)^2} - \frac{1}{2}\frac{s\,v}{(c_s\beta)^2} + \Im\left[F_+(s)\right] , \tag{A17}$$

$$\Re\left[\overline{q}_{33}(s)\right] = -q_c + \frac{1}{2}q_a + \Re\left[F_+(s)\right] , \tag{A18}$$

$$\Im\left[\overline{q}_{33}(s)\right] = \frac{s\,v}{(c_R\gamma)^2} - \frac{1}{2}\frac{s\,v}{(c_d\alpha)^2} + \Im\left[F_+(s)\right] , \tag{A19}$$

where

$$F_+(s) = \int_{C_-}\frac{d\xi_1}{2\pi}\ln(T(s,\xi_1,1)) , \tag{A20}$$

The functions a_+a_- and b_+b_- are given by

$$b_+b_- = 1 + \sqrt{1 - H^2/c_s^2} , \tag{A21}$$

$$a_+a_- = \frac{c_s^2 R(v)\left(1 + \sqrt{1 - H^2/c_R^2}\right)^2}{v^2\gamma^2(1 + \sqrt{1 - H^2/c_s^2})}\exp[G_+(s) + G_-(s)] , \tag{A22}$$

with

$$H^2 = s^2 + v^2 , \tag{A23}$$

and

$$G_{\pm}(s) = -\int_{C_{\mp}} \frac{d\xi}{2\pi i} \frac{\ln(T(s,\xi,1)}{\xi \mp i} \ . \tag{A24}$$

The main calculations consist in computing the integrals $F_+(s)$ and $G_{\pm}(s)$. The algebraic manipulation of $F_+(s)$ was performed in [S3, S4] and is given by

$$\begin{aligned} F_+(s) \ &= \ \frac{1}{2\pi}\int_{c_s^2}^{c_d^2} dJ \frac{2v^2 J - H^2(J+v^2)}{\sqrt{J(J-H^2)}(J-v^2)^2} \tan^{-1}\left[\frac{4\sqrt{1-J/c_d^2}\sqrt{J/c_s^2-1}}{(2-J/c_s^2)^2}\right] \\ &- \ \frac{i}{\pi}\int_{c_s^2}^{c_d^2} dJ \frac{s\,v}{(J-v^2)^2} \tan^{-1}\left[\frac{4\sqrt{1-J/c_d^2}\sqrt{J/c_s^2-1}}{(2-J/c_s^2)^2}\right] \ . \end{aligned} \tag{A25}$$

It is now established that the energy balance of the in-plane crack front dynamics leads to non-dispersive waves with real velocity s_c for all crack front propagation speeds [S4]. Using Eq. (11) of the main text, one then has

$$\Re\left[\overline{q}_{33}(s_c)\right] \ = \ 0 \ , \tag{A26}$$

$$\Im\left[2\overline{q}_{33}(s_c) - i s_c \partial_v \left[\ln(\alpha(1-\beta^2)/R(v))\right]\right] \ = \ 0 \ . \tag{A27}$$

Eq. (A25) shows that $\Im[F_+]$ depends linearly on s. Consequently, the crack front wave solution s_c is a solution of Eq. (A26) while Eq. (A27) is an identity which is satisfied for every real s and v. This result can be checked through direct computation of the integral in (A27). Moreover, one has

$$\Im\left[\overline{q}_{11}(s)\right] - \Im\left[\overline{q}_{33}(s)\right] = \frac{1}{2}\frac{s\,v}{(c_d\alpha)^2} - \frac{1}{2}\frac{s\,v}{(c_s\beta)^2} \ . \tag{A28}$$

Combining this result with Eq. (A27) one gets

$$\Im\left[\overline{q}_{11}(s)\right] = \frac{s}{2}\partial_v\left[\ln(\beta(1-\beta^2)/R(v))\right] \ . \tag{A29}$$

In the following, we present the computation of $G_{\pm}(s)$ following the same steps as for the computation of F_+ [S3]. Let us introduce the following change of variables in the above defined integrals

$$J = \left(\frac{s - v\xi}{\xi_{\perp}}\right)^2 = \frac{(s-v\xi)^2}{1+\xi^2} \ . \tag{A30}$$

We then have

$$T(J) = \frac{\gamma^2 v^2[(2-J/c_s^2)^2 - 4\sqrt{1-J/c_d^2}\sqrt{1-J/c_s^2}]}{R(v)J(J/c_R^2-1)} \ , \tag{A31}$$

and

$$\xi_{\pm}(J) = \frac{-s\,v \pm i\sqrt{J(J-H^2)}}{J-v^2} \ , \tag{A32}$$

leading to

$$J(\xi_a^{\pm}) = c_d^2 \ , \qquad J(\xi_b^{\pm}) = c_s^2 \ , \tag{A33}$$

and

$$\xi'_{\pm}(J) = \frac{d\xi_{\pm}}{dJ} = \frac{s\,v}{(J-v^2)^2} \pm \frac{i}{2}\frac{[H^2(J+v^2)-2v^2 J]}{\sqrt{J(J-H^2)}(J-v^2)^2} \ . \tag{A34}$$

Note that we search for solutions for which $v^2 < H^2 < c_R^2$. Thus, the branch cut runs from c_s to c_d and the contribution of $\ln T(J)$ to the contour integral does not come from the denominator of $T(J)$ which is positive. Then, one has

$$G_{\pm}(s) = \frac{1}{\pi}\int_{c_s^2}^{c_d^2} dJ \frac{\xi'_{\mp}(J)}{\xi_{\mp}(J) \mp i} \tan^{-1}\left[\frac{4\sqrt{1-J/c_d^2}\sqrt{J/c_s^2-1}}{(2-J/c_s^2)^2}\right] \ , \tag{A35}$$

and thus

$$G_+(s) + G_-(s) = \frac{2}{\pi} \int_{c_s^2}^{c_d^2} dJ \Re\left[\frac{\xi'_-(J)}{\xi_-(J) - i}\right] \tan^{-1}\left[\frac{4\sqrt{1 - J/c_d^2}\sqrt{J/c_s^2 - 1}}{(2 - J/c_s^2)^2}\right] , \tag{A36}$$

with

$$2\Re\left[\frac{\xi'_-(J)}{\xi_-(J) - i}\right] = -\frac{\left[2v^2(J + s^2) - H^2(J + v^2)\right]/(J - v^2)}{(s\,v)^2 + (\sqrt{J(J - H^2)} + J - v^2)^2} - \frac{\left[2v^2 J - H^2(J + v^2)\right]/\sqrt{J(J - H^2)}}{(s\,v)^2 + (\sqrt{J(J - H^2)} + J - v^2)^2} . \tag{A37}$$

With these algebraic manipulations and simplifications at hand, the numerical computation of $\Re[\overline{q}_{33}(s)]$ and $\Re[\overline{q}_{11}(s)]$ is straightforward.

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