The aim of this document is to provide additional technical details about the calculations of the complex functions $\mathcal{Q}_{33}(s)$ in Eq. (11) and $\mathcal{Q}_{11}(s)$ in Eqs. (13)-(14) in the manuscript.

I. FIRST ORDER PERTURBATIONS OF THE STRESS INTENSITY FACTOR

Willis and Movchan introduced a convolution identity (Eq. (3.12) of [S1]) that allows to calculate the change of the stress intensity factors due to a change in the crack surface, induced by a perturbation of its propagation front. Starting from an unperturbed state defined by a large crack submitted to a mode I loading, the first order perturbation terms of the stress intensity factors, $\mathcal{K}_1^{(1)}(\xi_2, \omega)$ and $\mathcal{K}_{11}^{(1)}(\xi_2, \omega)$ are given by (Eqs. (3)-(4) of the main text)

$$\mathcal{K}_1^{(1)}(\xi_2, \omega) = \left[ \mathcal{Q}_{33}(\xi_2, \omega)K_0^{(0)} + \sqrt{\frac{\pi}{2}}A_3^{(0)} \right] \overline{\psi}(\xi_2, \omega), \quad (A1)$$

$$\mathcal{K}_{11}^{(1)}(\xi_2, \omega) = \left[ -\Theta_{13}\mathcal{Q}_{11}(\xi_2, \omega)K_0^{(0)} + i(\omega/e)\omega_13K_1^{(0)} - \Theta_{13}A_3^{(0)} \right] \overline{\psi}(\xi_2, \omega) + \mathcal{P}_{11}(\xi_2, \omega). \quad (A2)$$

Explicit expressions for the quantities in Eqs. (A1)-(A2) can be found in [S2]. The “history” term $\mathcal{P}_{11}$ includes the contributions of the zero order traction terms on the broken surface convoluted with the corrugations of the crack surface $\psi(X < 0, x_2, t)$. It is expressed as [S2]

$$\mathcal{T}_{11}(x_2, t) = [U]_{11} \ast \{P_1^{(1)} \ast [U]_{21} \ast \{P_2^{(1)} \ast [U]_{31} \ast \{P_3^{(1)} \} \} \}, \quad (A3)$$

where $\ast$ and $\langle \rangle$ denote the jump and the average of the corresponding quantities through the unperturbed crack plane. Here $\textbf{U}$ are the weight functions defined in [S2] and $P_i^{(1)}$ are traction terms obtained by imposing the boundary conditions on the edges of the perturbed crack. The latter are given by

$$P_i^{(1)} = -\partial_\alpha \left( \sigma^{(0)}_{i\alpha} \psi \right) + \rho \omega^2 u^{(0)}_{i,XX} \psi. \quad (A4)$$

Also, one can show that $\mathcal{K}_1^{(1)}$ does not include such a term, i.e. that $\mathcal{T}_1(x_2, t) = 0$.

The functions $\mathcal{Q}_{11}(\xi_2, \omega)$ are homogeneous of degree one and can be written as

$$\mathcal{Q}_{11}(\xi_2, \omega) = |\xi_2|\mathcal{Q}_{11}(\omega/|\xi_2|), \quad \mathcal{Q}_{33}(\xi_2, \omega) = |\xi_2|\mathcal{Q}_{33}(\omega/|\xi_2|), \quad (A5)$$

where $|\xi_2|$ is the modulus of the complex wavenumber $\xi_2$ and $\omega$ is a complex frequency. The functions $\mathcal{Q}_{11}$ and $\mathcal{Q}_{33}$ are given in [S2]

$$\mathcal{Q}_{11}(s) = -i\xi_2^{-}(s) + \frac{i}{2}\xi_2^+(s) + \left( \frac{a_+a_- - b_+b_-}{a_+a_- + b_+b_-} \right) + \int \frac{d\xi}{2\pi} \ln(T(s, \xi_1, 1)), \quad (A6)$$

$$\mathcal{Q}_{33}(s) = -i\xi_2^{-}(s) + \frac{i}{2}\xi_2^+(s) + \int \frac{d\xi}{2\pi} \ln(T(s, \xi_1, 1)), \quad (A7)$$

where

$$T(s, \xi_1, 1) = \frac{\gamma^2[4\xi_1^2 \sqrt{(s - v\xi_1)^2 / c_s^2} - \xi_1^2 \sqrt{(s - v\xi_1)^2 / c_s^2} - \xi_1^2 + ((s - v\xi_1)^2 / c_s^2 - 2\xi_1^2)^2]}{R(v)((s - v\xi_1)^2 / c_s^2 - \xi_1^2)(\xi_1 - s/v)^2}, \quad (A8)$$

$$a_+ = \frac{i - \xi_2^-(s)}{\sqrt{i - \xi_2^-(s)}}, \quad a_- = \frac{\xi_2^+(s)}{\beta v^2 \sqrt{-i - \xi_2^+(s)}}, \quad (A9)$$

$$b_+ = \frac{i - \xi_2^-(s)}{\sqrt{i - \xi_2^-(s)}}, \quad b_- = \beta \sqrt{-i - \xi_2^+(s)}. \quad (A10)$$
with $\xi_1^2 \equiv \xi_1^2 + 1$ and

$$T(\pm,s,\pm i,1) = \exp \left( -\frac{1}{2\pi i} \int_{C_\pm} d\xi_1 \ln(T(s,\xi_1,1))/\xi_1 \mp i \right). \quad (A11)$$

The contours $C_\pm$ encircle the branch points $\xi_1^\pm$ and $\xi_2^\pm$ respectively, with

$$\xi_1^\pm = -\frac{s v}{(c_\alpha \gamma)^2} \pm iq_0; \quad q_0 = \frac{1}{\alpha} \sqrt{1 - \frac{v^2}{(c_\alpha \gamma)^2}}, \quad (A12)$$

$$\xi_2^\pm = -\frac{s v}{(c_\beta \gamma)^2} \pm iq_0; \quad q_0 = \frac{1}{\beta} \sqrt{1 - \frac{v^2}{(c_\beta \gamma)^2}}, \quad (A13)$$

$$\xi_3^\pm = -\frac{s v}{(c_\gamma)^2} \pm iq_0; \quad q_0 = \frac{1}{\gamma} \sqrt{1 - \frac{v^2}{(c_\gamma)^2}}, \quad (A14)$$

with $\alpha^2 = 1 - v^2/c_\alpha^2$, $\beta^2 = 1 - v^2/c_\beta^2$, $\gamma^2 = 1 - v^2/c_\gamma^2$. The kinematic functions $\Theta_3(v)$ and $\omega_{31}(v)$ are defined in the main text.

Finally, using the homogeneity of the functions $U$ and $P^{(1)}$ and the time-independance of the zeroth order stresses, one can show that the function $T_{11}$ can be written in Fourier space as

$$T_{11}(\xi_2,\omega) = K_{II}^{(0)} \sqrt{\frac{\xi_2}{L}} \mathcal{T}_{11}(\omega/|\xi_2|,|\xi_2| L) \overline{\psi}(\xi_2,\omega), \quad (A15)$$

where $\mathcal{T}_{11}$ is a complex function of degree zero and $L$ is the geometrical length scale introduced in the main text. Here we used the fact that to leading order in $\epsilon$ one has $\psi(X,x_2,t) = \psi(X + vt,x_2)$.

II. CALCULATIONS

For a real $s$ and $\sqrt{s^2 + v^2} < c_R$, Eqs. (A6)-(A7) reduce to

$$\Re[\mathcal{F}_{11}(s)] = -q_c + \frac{1}{2} q_b + \left( \frac{a_+ a_- - b_+ b_-}{a_+ a_- + b_+ b_-} \right) + \Re[F_+(s)], \quad (A16)$$

$$\Im[\mathcal{F}_{11}(s)] = \frac{s v}{(c_\gamma)^2} \ln(1 + \frac{s v}{(c_\gamma)^2} + \Im[F_+(s)] \quad (A17)$$

$$\Re[\mathcal{F}_{13}(s)] = q_c + \frac{1}{2} q_a + \Re[F_+(s)], \quad (A18)$$

$$\Im[\mathcal{F}_{13}(s)] = \frac{s v}{(c_\gamma)^2} \ln(1 + \frac{s v}{(c_\gamma)^2} + \Im[F_+(s)] \quad (A19)$$

where

$$F_+(s) = \int_{C_-} \frac{d\xi_1}{2\pi} \ln(T(s,\xi_1,1)), \quad (A20)$$

The functions $a_+ a_-$ and $b_+ b_-$ are given by

$$b_+ b_- = 1 + \sqrt{1 - H^2/c_s^2}, \quad (A21)$$

$$a_+ a_- = \frac{c_s^2 (H(v)(1 + \sqrt{1 - H^2/c_s^2)^2} \exp[G_+(s) + G_-(s)], \quad (A22)$$

with

$$H^2 = s^2 + v^2, \quad (A23)$$
Let us introduce the following change of variables in the above defined integrals:

$$G_{\pm}(s) = - \int_{c_\pm} \frac{d\xi}{2\pi i} \ln(T(s, \xi, 1)).$$  \hspace{1cm} (A24)

The main calculations consist in computing the integrals $F_+(s)$ and $G_{\pm}(s)$. The algebraic manipulation of $F_+(s)$ was performed in [S3, S4] and is given by

$$F_+(s) = \frac{1}{2\pi} \int_{c_\pm} dJ \frac{2v^2 J - H^2(J + v^2)}{\sqrt{J - H^2}(J - v^2)^2} \tan^{-1} \left[ \frac{4\sqrt{1 - J/c_R^2} \sqrt{J/c_R^2} - 1}{(2 - J/c_R^2)^2} \right].$$

It is now established that the energy balance of the in-plane crack front dynamics leads to non-dispersive waves with real velocity $s_c$ for all crack front propagation speeds [S4]. Using Eq. (11) of the main text, one then has

$$\Re \left[ \mathcal{G}_{33}(s_c) \right] = 0, \hspace{1cm} (A26)$$

and

$$\Im \left[ 2\mathcal{G}_{33}(s_c) - i s_c \partial_v \left[ \ln(\alpha(1 - \beta^2)/R(v)) \right] \right] = 0. \hspace{1cm} (A27)$$

Eq. (A25) shows that $\Im[F_+]$ depends linearly on $s$. Consequently, the crack front wave solution $s_c$ is a solution of Eq. (A26) while Eq. (A27) is an identity which is satisfied for every real $s$ and $v$. This result can be checked through direct computation of the integral in (A27). Moreover, one has

$$\Im \left[ \mathcal{G}_{11}(s) \right] - \Im \left[ \mathcal{G}_{33}(s) \right] = \frac{1}{2} \frac{s v}{(c_d \alpha)^2} - \frac{1}{2} \frac{s v}{(c_s \beta)^2}. \hspace{1cm} (A28)$$

Combining this result with Eq. (A27) one gets

$$\Im \left[ \mathcal{G}_{11}(s) \right] = \frac{s}{2} \partial_v \left[ \ln(\beta(1 - \beta^2)/R(v)) \right]. \hspace{1cm} (A29)$$

In the following, we present the computation of $G_{\pm}(s)$ following the same steps as for the computation of $F_+$ [S3]. Let us introduce the following change of variables in the above defined integrals:

$$J = \left( \frac{s - v \xi}{\xi_\perp} \right)^2 = \frac{(s - v \xi)^2}{1 + \xi^2}. \hspace{1cm} (A30)$$

We then have

$$T(J) = \frac{s^2 v^2 [(2 - J/c_s^2)^2 - 4\sqrt{1 - J/c_R^2} \sqrt{1 - J/c_s^2}]}{R(v)J(J/c_R^2 - 1)}, \hspace{1cm} (A31)$$

and

$$\xi_{\pm}(J) = \frac{-s v \pm i \sqrt{J(J - H^2)}}{J - v^2}, \hspace{1cm} (A32)$$

leading to

$$J(\xi_{\pm}^0) = c_s^2, \hspace{1cm} J(\xi_{\pm}^\perp) = c_d^2, \hspace{1cm} (A33)$$

and

$$\xi_{\pm}(J) = \frac{d\xi_{\pm}}{dJ} = \frac{s v}{(J - v^2)^2} \pm \frac{i}{2} \frac{[H^2(J + v^2) - 2v^2J]}{\sqrt{J(J - H^2)(J - v^2)^2}}. \hspace{1cm} (A34)$$

Note that we search for solutions for which $v^2 < H^2 < c_R^2$. Thus, the branch cut runs from $c_s$ to $c_d$ and the contribution of $\ln T(J)$ to the contour integral does not come from the denominator of $T(J)$ which is positive. Then, one has

$$G_{\pm}(s) = \frac{1}{\pi} \int_{c_\pm} dJ \frac{\xi_{\pm}(J)}{\xi_{\perp}(J) + i} \tan^{-1} \left[ \frac{4\sqrt{1 - J/c_D^2} \sqrt{J/c_s^2} - 1}{(2 - J/c_D^2)^2} \right]. \hspace{1cm} (A35)$$
and thus

$$G_+(s) + G_-(s) = \frac{2}{\pi} \int_{c_2^+}^{c_2^-} dJ \Re \left[ \frac{\xi'_-(J)}{\xi_-(J) - i} \tan^{-1} \left( \frac{4\sqrt{1 - J/c_2^2} \sqrt{J/c_2^2} - 1}{(2 - J/c_2^2)^2} \right) \right], \tag{A36}$$

with

$$2 \Re \left[ \frac{\xi'_-(J)}{\xi_-(J) - i} \right] = -\frac{[2v^2(J + s^2) - H^2(J + v^2)]/(J - v^2)}{(sv)^2 + (\sqrt{J(H - s^2)} + J - v^2)^2} - \frac{[2v^2J - H^2(J + v^2)]/\sqrt{J(H^2 - s^2)}}{(sv)^2 + (\sqrt{J(H - s^2)} + J - v^2)^2}. \tag{A37}$$

With these algebraic manipulations and simplifications at hand, the numerical computation of $\Re[\xi_{33}(s)]$ and $\Re[\xi_{11}(s)]$ is straightforward.

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