Dynamic Instability of Brittle Fracture

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Using Eshelby's energy-momentum tensor, it is shown that the elastic configurational force acting on a moving crack tip does not necessarily point in the direction of crack propagation. A generalization of Griffith's approach that takes into account this fact is proposed to describe dynamic crack propagation in two dimensions. The model leads to a critical velocity below which motion proceeds in a pure opening mode, while above it, it does not. The possible relevance of this instability to recent experimental observations is discussed. [S0031-9007(99)08705-0]

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When an elastic medium that contains a crack is subject to an external load, the energy stored in the elastic field is focused into the region around the crack tip. When the applied load exceeds a threshold value, the crack propagates, thereby creating new surface. According to the present theory of fracture mechanics in two dimensions [1], the crack tip should smoothly accelerate until it reaches the Rayleigh wave speed V_R, the speed at which elastic waves travel along a flat surface. Experiments in isotropic media, however, seldom show crack speeds exceeding half this value, with a trajectory that is far from smooth: a dynamic instability occurs above a well-defined critical velocity where microcracks appear, the fractured surface becomes increasingly rough, and acoustic emissions become markedly stronger [2,3]. Although there have been a number of attempts [4–7] at a theoretical understanding of these facts, it seems fair to say that no coherent explanation has been achieved yet. The purpose of this paper is to propose a model for elastodynamic crack propagation that leads to a dynamic instability.

A basic tenet of current macroscopic fracture theory [1,8] is that crack advance is governed by the fact that the change in energy per unit crack advance (also called the energy release rate G) must be equal to a material parameter, Γ, the specific fracture energy:

$$G \equiv \Gamma\,. \tag{1}$$

The latter parameter includes the energy associated with the creation of new crack surface, as well as with the energy associated with whatever nonlinear processes take place on a microscopic scale very near the crack tip. In thermodynamical terms, G is the generalized force conjugate to the extension of the crack, and there are two ways to compute it: the first [9,10] is through a global dissipation analysis that takes into account the fact that the fracture of a material sample is thermodynamically irreversible, while the local mechanical behavior of the bulk material may be fully elastic. The second one [11,12] directly involves the computation of the generalized or configurational force, of a *non-Newtonian* type, which acts at the tip of a crack which is considered as a defect. This is the point of view of the theory of defects and material forces on singularities introduced by Eshelby in 1951 [11].

An energy argument, being scalar, is insufficient to completely describe a crack trajectory that is allowed to deviate from a straight line. In order to complete the description of the crack motion, additional criteria, such as the principle of local symmetry [13,14], have been introduced. In this paper, we wish to explore the consequences of taking into account all components of the configurational force acting at the tip of a moving crack. In this framework, we develop a model of force balance instead of energy balance, that under minimal assumptions leads to a critical crack velocity. Below this critical velocity, the crack propagates in a direction that keeps a pure opening mode at its tip, and above it this ceases to hold, leading to a dynamic instability at the crack tip.

We consider an effectively two-dimensional elastic medium, with a crack tip in motion, close to which the elastic fields become singular. The equations of dynamic elasticity in the absence of body force can be written as Euler-Lagrange equations

$$\frac{\partial}{\partial X_\nu}\left(\frac{\partial \mathcal{L}}{\partial u_{i,\nu}}\right) = 0\,, \tag{2}$$

with Lagrangian

$$\mathcal{L} = \frac{1}{2}\rho \dot{u}_i \dot{u}_i - \frac{1}{2} c_{ijkl} u_{i,j} u_{k,l}\,, \tag{3}$$

where ρ is the material density, c_{ijkl} the elastic constants, $u_i(X_\nu)$ the displacement field, $X_0 = t$, X_i the space coordinates, a comma means partial differentiation [15], and an overdot means partial differentiation with respect to time. For a homogeneous and stationary medium, they imply the conservation laws

$$\frac{\partial T_{\mu\nu}}{\partial X_\nu} = 0\,, \tag{4}$$

with $T_{\mu\nu}$ the elastodynamic energy-momentum tensor [12]:

$$T_{\mu\nu} = -\mathcal{L}\delta_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial u_{i,\nu}} u_{i,\mu}\,, \tag{5}$$

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with T_{00} the energy density, T_{0i} the energy density flux, $-T_{i0}$ the field momentum density, and $-T_{ij}$ the field momentum density flux. Note that the field momentum $-T_{i0}$, also called quasimomentum or pseudomomentum [12], is dimensionally a linear momentum but it is not the physical one. Indeed, the physical linear momentum is defined by $\rho \dot{u}_i$, and the quantity $\rho \dot{u}_i + T_{i0}$ is the canonical momentum. Thus, the field momentum is the difference between the linear momentum and the canonical momentum [12].

Consider now a crack tip that is enclosed by a curve C that starts on one lip of the crack, ends on the other, and moves with it. The energy flow $\mathcal{F}(t)$ and the flow of field momentum F_i through the curve C into the crack tip are [16]:

$$\mathcal{F}(t) = -\lim_{C\to 0} \int_C dC[T_{0j}N_j - V_j T_{00} N_j], \qquad (6)$$

$$F_i(t) = \lim_{C\to 0} \int_C dC[T_{ij}N_j - V_j T_{i0} N_j], \qquad (7)$$

with $\vec{N}$ the unit normal at a given point of the curve C. $F_i(t)$ can be identified as a configurational force acting on the crack tip. These flows of energy and field momentum are independent of the shape of the curve C, as long as it lies close to the crack tip [16]. The configurational force $\vec{F}$ is related to the energy flow rate $\mathcal{F}$ through

$$\mathcal{F}(t) = F_i(t) V_i(t) . \qquad (8)$$

This means that the work done by the configurational force $\vec{F}$ for an infinitesimal advance of the crack tip, $\vec{F} \cdot d\vec{R}$, is equal to the energy entering into the crack tip region during that time, $\mathcal{F} dt$, with $\vec{V} = d\vec{R}/dt$.

We define a local frame $\vec{e}_i$ such that $\vec{e}_1$ is in the direction of crack motion and $\vec{e}_2$ is perpendicular to it. We suppose a smooth motion of the crack front with relatively small curvature and smooth velocity $V(t)$. Substitution of the universal expressions for stress and displacement velocity near the moving crack tip [1] into Eqs. (3)–(7) gives

$$\frac{1}{V} \mathcal{F}(t) \equiv G(t) = F_1(t) , \qquad (9)$$

$$F_1(t) = \frac{1}{2\mu} [A_{\rm I}(V) K_{\rm I}^2 + A_{\rm II}(V) K_{\rm II}^2] , \qquad (10)$$

$$F_2(t) = -\frac{1}{2\mu} B(V) K_{\rm I} K_{\rm II} , \qquad (11)$$

$K_{\rm I}$ and $K_{\rm II}$ are the stress intensity factors; $A_{\rm I}(V)$, $A_{\rm II}(V)$, and $B(V)$ are universal functions of $V \equiv \vec{V} \cdot \vec{e}_1$:

$$A_{\rm I}(V) = \frac{a(1 - b^2)}{D(V)} , \qquad (12)$$

$$A_{\rm II}(V) = \frac{b(1 - b^2)}{D(V)} , \qquad (13)$$

$$B(V) = \frac{4ab(1 - b^4)(a - b)}{D(V)^2} , \qquad (14)$$

with $a(V) \equiv \sqrt{1 - V^2/c_d^2}$, $b(V) \equiv \sqrt{1 - V^2/c_s^2}$; c_d and c_s the dilatational and shear sound velocities respectively, and $D(V) \equiv 4ab - (1 + b^2)^2$, with $D(V_R) = 0$. Expressions (9) and (10) are standard expressions in dynamic fracture. Expression (11) does not appear to have received much attention.

In Eq. (9), we recover the anticipated result (8). Indeed, F_1 is the configurational force in the direction of motion that does work, and F_2 is perpendicular to it and does no work. The possible relevance of such forces to the understanding of crack dynamics was pointed out in Ref. [17]. As evidenced in Eq. (11), the elastic configurational force on the crack tip ceases to be in the direction of motion when in addition to mode-I loading, we have a component of mode-II loading ($K_{\rm II} \neq 0$).

From Eqs. (9) and (10), we see that the energy release rate G is equivalent in the Eshelbian approach to the configurational force per unit length of the crack front. Thus, Eq. (1) can be reinterpreted as a balance between the elastic configurational force F_1 and a resistance force to crack advance, that is Γ (all per unit length of the crack front). Usually, in addition to Eq. (1), it is assumed that cracks proceed in a purely local opening mode at the tip. This is the *principle of local symmetry* [13,14]:

$$K_{\rm II} = 0 \Longleftrightarrow \text{smooth crack propagation.} \qquad (15)$$

We wish to write down an equation of motion for two dimensional dynamic crack propagation without assuming *a priori* a principle of local symmetry. In order to do so, two equations are needed. So far, we have determined the two components of the elastic configurational forces acting on the crack tip [Eqs. (10) and (11)]. In order to write down an equation of motion for it, we will assume that it involves the crack velocity only, i.e., that the crack tip does not have inertia [1]. The elastic configurational forces must be balanced by some configurational forces acting at the crack tip region level. These forces are of a dissipative nature and should represent the resistance of the material to crack advance. Their origin should be the adjustment and breaking of bonds at an atomic level. According to our presentation, Griffith's resistance force for mode I loading is

$$\vec{F}_d = -\Gamma \vec{e}_1 . \qquad (16)$$

As a generalization of this resistance force in the presence of mode II loading, we propose the following form for $\vec{F}_d$:

$$\vec{F}_d = -\Gamma(\cos\phi_d \vec{e}_1 + \sin\phi_d \vec{e}_2) , \qquad (17)$$

with ϕ_d an angle to be modeled. Thus, we state that for $K_{\rm II} \neq 0$ the resistance force is not necessarily in the direction of motion, but that its magnitude will remain as Γ. The asymmetry introduced at the tip by a mode II loading, we think justifies the fact that the resistance force will not point in the direction of motion. Furthermore, we think the resistance force should be weakened in the direction of motion by the shearing produced by the mode II loading, as it is reflected in Eq. (17).

We model the angle ϕ_d as a function of K_{II} (that acts as the forcing mechanism here) and the velocity of the crack tip, V [16]. In order to be nondimensioned, ϕ_d should depend on q, a parameter proportional to K_{II}/K_I,

$$q \equiv (b/a)(K_{II}/K_I).$$

Also, it should be an odd function of q in order to respect mode II symmetry. Therefore, one can always write

$$\tan\phi_d = -2\alpha(V)q\psi(q,V), \tag{18}$$

with $\alpha(V)$ an unknown material parameter, and $\psi(q,V) = 1 + \beta(V)q^2 + \dots$ a function that represents higher order corrections in q.

The energy balance arguments at the crack tip imply local force balance in the direction of motion. Then, it is natural to assume that this balance also holds perpendicularly to the direction of motion. Having a form for the elastic configurational force and for the resistance force, we write our equation of motion by assuming vectorial balance of forces, $\vec{F} + \vec{F}_d = 0$, thus in the following way

$$\tan\phi_d = \frac{F_2}{F_1}, \tag{19}$$

$$\Gamma\cos\phi_d = F_1. \tag{20}$$

Substitution of Eqs. (10), (11), and (18) into Eq. (19) leads to

$$2\alpha(V)q\psi(q,V) - 2C(V)\frac{q}{1+\frac{a}{b}q^2} = 0, \tag{21}$$

with $C(V) \equiv 2a(1+b^2)(a-b)/D(V)$. This equation can be solved for q for a given V, independently of the specific loading conditions and geometry. The function $\psi(q,V)$ for $q \ll 1$ is approximately $\psi(q,V) \simeq 1 + \beta(V)q^2$, and we assume $\beta(V) \geq 0$, i.e., the angle of the force of resistance grows with K_{II}. We will also assume that the material parameter $\alpha(V)$ is a slowly varying function of V compared with the variation of $C(V)$. Under these assumptions on $\psi(q,V)$ and $\alpha(V)$, the solutions of Eq. (21) can be determined: for $\alpha(V) \geq 1$, there is a critical velocity V_c, given by the condition $C(V_c) = \alpha(V_c)$, such that

$$V < V_c \Longleftrightarrow K_{II} = 0, \tag{22}$$

$$V > V_c \Longleftrightarrow K_{II} = 0,\ \pm g(V)K_I. \tag{23}$$

Therefore, according to the velocity of the crack tip, there exists either one solution $K_{II} = 0$, or three solutions $K_{II} = 0$ and $K_{II} = \pm g(V)K_I$ for Eq. (21). Notice that the approximation of $\psi(q,V)$ to order q^2 allows the calculation of $g(V)$ from Eq. (21). Indeed, $g(V) \simeq h(V_c)\sqrt{V - V_c}$ for V greater but close to V_c. Thus, the velocity V acts as a bifurcation parameter at $V = V_c$ for finding the solutions of Eq. (21) as a function of q, or K_{II}/K_I.

The new solutions with $K_{II} \neq 0$ correspond to a continuously growing K_{II} as V grows over V_c, from $K_{II} = 0$ at $V = V_c$. If $\alpha(V) < 1$, there would always exist three solutions to our equation (21), since $C(V) \geq 1$. We shall not consider this case since it does not appear to be related to experimental results in glass and plexiglass [2,3]. Notice also that the critical velocity V_c always satisfies $V_c < V_R$, since $C(V) \to \infty$ as $V \to V_R$.

Having found different solutions for *smooth* crack propagation for $V > V_c$, one needs to define a selection mechanism to decide which one will correspond to the path taken by the moving crack tip. Consider first a configuration of *smooth* crack propagation at $V(t) > V_c$, with $K_I \neq 0$ and $K_{II} = 0$. From Eqs. (9) and (20), the rate of energy flow needed for the propagation of this crack is $\mathcal{F}' = V\Gamma$. Consider a second configuration of *smooth* crack propagation with the same instantaneous velocity $V(t)$, but with $K_I \neq 0$ and $K_{II} = \pm g(V)K_I$. From Eqs. (9) and (20), the rate of energy flow needed for the propagation of this crack is $\mathcal{F}'' = V\Gamma\cos\phi_d$. Clearly, $\mathcal{F}'' \leq \mathcal{F}'$. This is so because the material response to external loading provides a smaller restoring force for the second configuration than for the first one. Thus, for a given velocity V above the critical velocity V_c, the crack needs more energy to advance in a configuration with $K_{II} = 0$ than in a configuration with $K_{II} \neq 0$. Therefore, for $V > V_c$ the selected solutions are $K_{II} = \pm g(V)K_I$, instead of the solution $K_{II} = 0$, and consequently the principle of local symmetry no longer holds for $V > V_c$.

The following scenario based on our model tries to explain some features of the experimental results in fast fracture of glass and plexiglass plates under tension [2,3]. These experiments have shown an instability appearing when the crack tip velocity surpasses a certain critical velocity. This instability is associated with the roughening of the crack surfaces and the appearance of microcracks on them. As our scenario, we propose to identify the critical velocity V_c of our model with this experimental critical velocity. For PMMA, the critical velocity has been found at $V_c \approx 0.36V_R$ [2,3]. Using $c_d \approx \sqrt{3}\,c_s$, and from the condition $\alpha(V_c) = C(V_c)$, one finds $\alpha(V_c) \approx 1.073$, which is a reasonable value for the model. Indeed, a simple estimate based on an analogy with the branching process at low velocities [14] suggests $\alpha \approx 1$ [16].

For velocities below V_c, the only possible solution that was obtained corresponds to $K_{II} = 0$. From Eq. (20), we recover the well-known equation of motion [1]:

$$\frac{1}{2\mu}A_I(V)K_I^2 = \Gamma, \tag{24}$$

which allows one to determine the crack tip velocity. The result $K_{II} = 0$ means that the crack will propagate following a smooth path, with a pure opening mode at its tip. This is the statement of the principle of local symmetry (15). Our approach can be viewed as a derivation of this principle and as an extension of it to velocities $V < V_c$. Considering the experimental results of [2,3], this solution corresponds to the mirror region, where the crack propagation follows a straight line.

As the velocity of the crack surpasses V_c, according to our selection mechanism the propagation satisfying $K_{\mathrm{II}} = 0$ at the crack tip should become unstable. The crack now propagates in one of the two new directions satisfying $K_{\mathrm{II}} \neq 0$, specifically $K_{\mathrm{II}} = \pm g(V) K_{\mathrm{I}}$. Notice that the allowed values of $K_{\mathrm{II}}/K_{\mathrm{I}}$ grow continuously with V from 0 at $V = V_c$, and that these new solutions correspond to *smooth* crack propagation.

Experiments [2,3] have shown that the roughness is associated with the presence of bumps on the surfaces, together with microcracks. The solution $K_{\mathrm{II}} = \pm g(V) K_{\mathrm{I}}$ means that the trajectory of the crack tip will deviate from a straight line, which in experiments [2,3] corresponds to the solution $K_{\mathrm{II}} = 0$. The appearance of microcracks at this stage can be explained as follows: on the crack faces the stress components σ_{22} and σ_{12} vanish identically. However, in the presence of a shear mode at the crack tip ($K_{\mathrm{II}} \neq 0$), the asymptotic elastic stress field σ_{11} near the moving crack tip is singular on the crack faces [1]:

$$\sigma_{11}(r, \pm \pi) \sim \mp K_{\mathrm{II}}/\sqrt{r}\,. \qquad (25)$$

This means that there is a high tensile stress near the tip that will tend to open microcracks on one of the crack faces, in a direction initially perpendicular to the direction of motion of the main crack. The microcrack will initiate in a weak place; it starts perpendicularly and later deviates into a direction closer to the direction of motion of the main crack, since it will avoid the unloaded region which is left behind the crack tip.

Summarizing, we have developed an approach based on the balances of energy and field momentum [11,12] for a moving crack tip in two dimensions. We have derived the energy flow rate into the crack tip and the configurational forces acting on it. The components of the material force at the crack tip have been computed in the framework of the linear isotropic elastodynamic model. Within a Griffith-like approach, we have defined a generalized dissipative force at the crack front. By assuming forces balance at the crack front, we derived a vectorial equation of motion for it. Under minimal assumptions, we have shown that below a critical crack speed, the crack propagates in a direction that keeps a pure opening mode at its tip. Moreover, a second order dynamic instability has been predicted: above this critical velocity, the crack growth with a pure opening mode at the tip becomes unstable with respect to two new possible solutions.

Our approach is universal, in the sense that the instability mechanism is local at the crack tip as well as independent of the specific loading configuration and the geometry of the experiment. Throughout the analysis, it has not been specified that the fracture energy should be velocity independent, or that the configuration of pure opening mode of the crack tip has to be a straight line. Such a configuration could be a curved path, but an instability would still occur.

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