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Crack dynamics in elastic media

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ABSTRACT

The classical theory of fracture mechanics states that a crack propagating in an unbounded body should smoothly accelerate until it reaches the Rayleigh wave speed. We introduce here a general approach for solving the equation of motion of the crack tip. We show that the loading conditions and the geometry of the configuration do not produce inertial effects. The equation of motion of a propagating crack is always a first-order differential equation.

§1. INTRODUCTION

Dynamic fracture experiments (Fineberg *et al.* 1992, Gross *et al.* 1993, Gross 1995, Boudet *et al.* 1996, Sharon *et al.* 1996) have shown many phenomena which are now admitted to be related to fundamental physical processes. Concerning the instabilities of dynamic fractures, it has been observed that, when the crack velocity v exceeds a critical speed V_c, the acoustic emission from the crack increases (Gross *et al.* 1993, Boudet *et al.* 1996), the velocity oscillations are amplified and a pattern correlated with the velocity oscillations appears on the fracture surface (Fineberg *et al.* 1992, Boudet *et al.* 1996). At that point, the initially flat broken surface of the material becomes *rough*. At higher velocities $V_B(>V_c)$ a *macroscopic branching* instability occurs; the crack tip splits, or deviates from its original direction (Lawn 1993). However, recent experiments (Sharon *et al.* 1996) have also distinguished another transition region of fine-scale fracturing. Indeed, above a velocity threshold $V_b(<V_c)$, dynamical instability occurs; the straight crack branches locally and its velocity oscillates at the frequency of the appearance of these microfractures.

Recently, we developed a formalism in the framework of elastodynamic fracture mechanics (Adda-Bedia *et al.* 1996) that could account for the roughening and the branching mechanisms. In this approach, the crack is viewed as a surface of discontinuity of stresses and deformations. The local analysis in the neighbourhood of the crack tip shows that the stress tensor field exhibits a *universal* square-root singularity and an angular variation in each stress component, which depends on the motion of the crack tip only through *the instantaneous crack tip speed* (Freund 1990). Any information about loading and configuration is embedded in scalar multipliers called the dynamic stress intensity factors, characterizing the dominant divergent contribution to the stress near the crack tip.

Using this simple formulation, we studied the largest principal stress near the tip, combined with symmetry considerations on the broken surface. We correlated the behaviour of this quantity after and before cracking and studied the curves of constant largest principal stress. Then, we have shown that at 'low' crack velocities, the path of crack extension is that of a pure opening mode (mode I). However, this property disappears when the crack speed exceeds V_c and reappears again beyond a speed V_B, but in a direction different from that of 'low' velocities. These variations have been interpreted (Adda-Bedia *et al.* 1996) as corresponding to the roughening and the branching instabilities respectively. The thresholds V_c and V_B are below the well known Yoffe (1951) critical velocity and are within the range of the experimental measurements.

The major remaining theoretical challenge is the determination of the dynamics of the crack (Gross 1995), and the origin of the dynamic instability at $v = V_b$. There have been a number of suggestions to explain the motion of the crack tip. It has been argued (Marder and Gross 1995) that conventional continuum theories are inherently inadequate to describe the crack properties. On the other hand, studies based on a continuum approach to the crack problem have suggested that the crack instabilities can be due to three-dimensional effects (Lund 1996) or have emphasized that complete dynamical models of deformation and decohesion at crack tips (Cheng *et al.* 1996) are necessary in order to understand the experimental observations.

According to the classical theory of fracture mechanics (Freund 1990), the crack tip should smoothly accelerate until it reaches the Rayleigh wave speed V_R, the speed at which elastic waves travel across a flat surface. Experiments, however, seldom show crack speeds exceeding half this speed. It can be argued that this discrepancy arises because the theoretical analysis has been done for a crack propagating in an unbounded body. That is, reflections of elastic waves on the remote boundaries could explain the lower terminal velocity of the crack tip. In order to take into account these experimental restrictions, we introduce here a different approach to the equation of motion of the crack tip. We show that *a finite geometry does not produce* inertial effects at the crack tip. The equation of motion of a propagating crack is always a first-order differential equation, independently of the loading conditions and the geometry of the configuration. The theory based on two-dimensional, purely elastic assumptions with energy considerations at the crack tip is not able to explain fully the mechanisms that govern the dynamics of brittle cracks.

§2. The crack tip equation of motion

In the study of the propagation of a crack through a solid within the context of continuum mechanics, the field equations can be solved once the motion of the crack edge is specified, together with the configuration of the body and the details of the loading. However, when developing a theoretical model of a dynamic crack growth process, the motion of the crack edge should not be specified *a priori* but, instead, should follow from the analysis. Unless the constitutive equations for bulk response already include the possibility of material separation, a mathematical statement of a *crack growth criterion* must be added. Such a criterion must be related to some physical parameter defined as a function of the crack edge mechanical fields, separately from theorems governing deformation and material response at any point in the continuous medium. In an energy approach, such a physical quantity is given by the *dynamic energy release rate* $\mathcal{G}$ (Kostrov and Nitkin 1970, Freund 1990).

$\mathcal{G}$ is the total mechanical energy release rate and is not only a rate of release of elastic energy. For example, there can be an exchange between elastic and kinetic energy of a body through wave reflection at a remote boundary; such a variation in elastic energy has no immediate influence on the energy release rate. $\mathcal{G}$ is determined by the mechanical fields near the crack surface. In the case of a purely elastodynamic model, the contribution comes only from the near-crack-tip distribution of the mechanical fields. Consequently, one obtains a relationship between the energy release rate and the stress intensity factors. For a pure mode I propagation, the result is (Kostrov and Nitkin 1970, Freund 1990)

$$\mathcal{G} = \frac{1}{2\mu} A(\dot{l}) K_I^2, \tag{1}$$

where μ is the Lamé coefficient and K_I is the mode I stress intensity factor. The function $A(\dot{l})$ is universal, in the sense that it does not depend on the detail of the applied loading or the configuration of the body being analysed. It depends on the instantaneous crack speed $\dot{l}$ and on the properties of the material.

In the study of crack growth process in materials which fail in a purely brittle manner, the most commonly used crack growth criteria are the generalization of the Griffith critical energy release rate criterion. According to the generalized Griffith (1921) criterion, the crack must grow in such a way that $\mathcal{G}$ is always equal to a newly defined quantity: the dynamic fracture energy Γ of the material. The energy release rate is a property of the local mechanical fields. The dynamic fracture energy, on the other hand, represents the resistance of the material to crack advance; it is assumed to be a property of the material and its value can be determined only through laboratory measurements, or eventually by pure microscopic models. The growth criterion is

$$\mathcal{G}(t, l, \dot{l}, \ldots) = \Gamma(\dot{l}). \tag{2}$$

This relation is called an *equation of motion* for the crack tip because it has the form of an ordinary differential equation for the crack tip position $l(t)$ as a function of time. In general, the rate Γ of energy dissipation may depend on the instantaneous crack tip speed $\dot{l}$.

Although the dynamic fracture problem has become self-consistent with the introduction of an equation of motion, there are very few cases for which the crack dynamics can be explicitly computed by solving equation (2). For an elasto-dynamic crack growth, it has been shown (Freund 1990) that, independently of the imposed loading conditions, a mode I crack tip in an unbounded body should accelerate smoothly until it reaches the terminal velocity V_R. We shall show in the following that the equation of motion of the crack tip is always a first-order differential equation, even in the case of a bounded body.

§3. The inertia of the crack tip

Equations (1) and (2) when combined together give

$$K_I(t) = \left(\frac{2\mu\Gamma(\dot{l})}{A(\dot{l})} \right)^{1/2}. \tag{3}$$

The right-hand side of equation (3) depends on the instantaneous speed of the crack tip only. Thus, the dependence on acceleration $\ddot{l}$ of equation (3) must be embedded in the stress intensity factor K_I. However, the stress intensity factor is explicitly found when the momentum equations in the bulk with appropriate boundary conditions are completely solved. This can be done analytically for special cases only (Freund 1990).

Consider two different configurations of the crack tip motion in the same finite material with the same loading configuration (figure 1). The first configuration is the actual configuration and is characterized by $l(t)$, $\dot{l}(t)$, $\ddot{l}(t), \ldots$. The corresponding

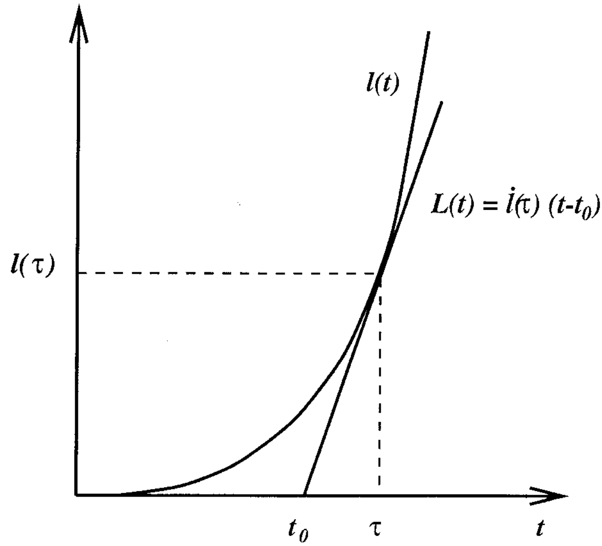

Figure 1. Scheme of the 'real'-time-dependent extension $l(t)$ of a crack tip and a 'virtual' extension $L(t)$ which satisfies $L(\tau) = l(\tau)$ and $\dot{L}(\tau) = \dot{l}(\tau)$.

time-dependent stress intensity factor is labelled K_I^R. The second configuration is a virtual configuration characterized by $L(t)$, $\dot{L}(t)$, $\ddot{L}(t)$, and by a stress intensity factor K_I. Suppose that, at a given time τ, the two cracks are at the same position $l(\tau) = L(\tau)$, with the same instantaneous tip speed $\dot{l}(\tau) = \dot{L}(\tau)$. From equation (3) one must have

$$K_I^R(\tau, l(\tau), \dot{l}(\tau), \ddot{l}(\tau), \ldots) \equiv K_I(\tau, l(\tau), \dot{l}(\tau), \ddot{L}(\tau), \ldots). \tag{4}$$

Remember that equation (3) is an instantaneous energy balance. Therefore the stress intensity factors must be equal *independently* of the instantaneous distribution of the stress field in the bulk and on the history of the crack propagation corresponding to these two motions.

Suppose now that a particular motion of the crack tip $L(t)$, is given by (figure 1)

$$L(t) = \dot{l}(\tau)(t - t_0)H(t - t_0), \tag{5}$$

where $H(\ )$ is the Heaviside function and

$$t_0 = \tau - \frac{l(\tau)}{\dot{l}(\tau)}.$$

By construction, the conditions $L(\tau) = l(\tau)$ and $\dot{L}(t) = \dot{l}(\tau)$ are satisfied. Moreover, one has

$$\dot{L}(t) = \dot{l}(\tau)H(t - t_0). \tag{6}$$

For this virtual motion, the crack propagation occurs at a constant speed and is due to a general space- and time-dependent loading. Finding the stress intensity factor of such growth is, at least in principle, a tractable problem. A method of resolution (figure 2) consists in solving first the stationary crack problem of a body subjected to given applied tractions at its boundaries. This allows the determination of the

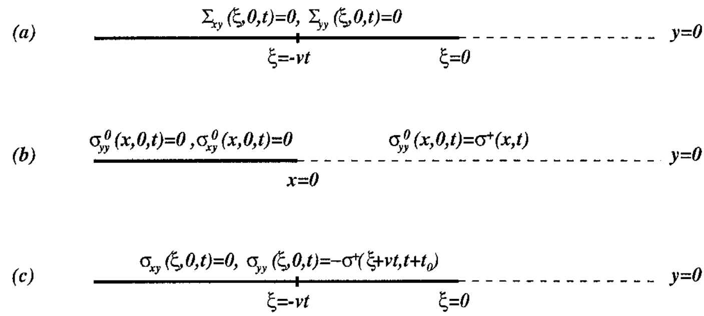

Figure 2. An example of resolution of a mode I crack propagation (*a*) by solving a stationary crack problem (*b*) of a body subjected to given applied tractions at its boundaries and a dynamic problem (*c*) of a crack that started to grow at time $t = t_0$, with a constant speed $\dot{l}(\tau) = v$ and with tractions σ_{ij} on $y = 0$.

corresponding stress field $\sigma^0_{ij}(x, y, t)$. The second step consists in solving the problem of a crack that started to grow, in the *x*-direction, at time $t = t_0$, with a constant speed $\dot{l}(\tau) = v$. For this problem, the material is free from tractions at its boundaries and on $y = 0$, the stress field σ_{ij} must satisfy

$$\sigma_{xy}(\xi, 0, t_1) = 0, \tag{7}$$

$$\sigma_{yy}(\xi, 0, t_1) = -\sigma^0_{yy}(\xi + vt_1, 0, t_1 + t_0)H(-\xi) + \sigma_{yy}(\xi, 0, t)H(\xi). \tag{8}$$

Here, $\xi = x - vt_1$ and $t_1 = t - t_0$. The instantaneous stress intensity factor K_I is given by the solution of the second problem. It is clear that K_I depends on v, t_1 and t_0 : $K_I \equiv K_I(v, t_1, t_0)$. On the other hand, according to equation (4), the stress intensity factor found by solving this problem is exactly equal to that of the real motion of the crack at time $t = \tau$. Thus,

$$K^R_I(\tau) = K_I\left(\dot{l}(\tau), \frac{l(\tau)}{\dot{l}(\tau)}, \tau - \frac{l(\tau)}{\dot{l}(\tau)}\right). \tag{9}$$

The procedure of finding $K^R_I(\tau)$ can be repeated for any time *t*. So, the stress intensity factor $K^R_I(t)$ is always given by equation (9) and, consequently, it depends on $t, l(t)$ and $\dot{l}(t)$ only. On the other hand, the stress intensity factor scales as $\sigma l_0^{1/2}$, where the stress scale magnitude σ and the length scale l_0 are fixed by the applied tractions and the initial conditions. Taking $\Gamma(0) = \Gamma_0$, the equation of motion can be formally written in a non-dimensioned form as

$$\frac{\dot{l}(t)}{V_R} = F\left(\frac{l(t)}{l_0}, \frac{V_R t}{l_0}, \frac{\mu \Gamma_0}{\sigma^2}\right). \tag{10}$$

The equation of motion of a crack tip in a brittle material is always a first-order differential equation. The initial boundary condition given by the value of $l(0)$ is sufficient to solve such an equation. The material geometry and loading configuration do not produce inertial effects at the crack tip. In contrast with dislocations (Lund 1996), one cannot define an effective mass for the crack tip. In fact, the crack tip singularity for the elastic field is rather the equivalent of the photon for the electromagnetic field. If one pushes this analogy further, one can expect that the selection of a terminal velocity is independent of the loading conditions and the geometry of the configuration. Thus, according to the result in an infinite medium, the terminal velocity of a crack tip should always be the Rayleigh wave speed.

§4. Conclusion

We have introduced a new approach to compute the equation of motion of the crack tip without being aware of radiation effects that result from reflections of elastic waves on the remote boundaries. This approach is different from the analysis of Freund (1990), where the crack tip motion $l(t)$ has been approximated by a polygonal curve $L(i)$ which have vertices lying on the original curve and where the crack tip is supposed to stop suddenly. We have shown that a finite geometry does not produce inertial effects at the crack tip. The equation of motion of a propagating crack is always a first-order differential equation with one initial boundary condition. The inertia of the crack tip cannot exist without an explicit dependence of the fracture energy Γ on $\ddot{l}$.

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