Nonlinearity 7 (1994) 765–776. Printed in the UK

Investigations on the dendrite problem at zero surface tension in 2D and 3D geometries

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Received 7 June 1993
Recommended by Y Couder

Abstract. Steady state dendritic growth at zero surface tension is analysed both in two-dimensional and three-dimensional geometries. It is demonstrated that the solutions found by Ivantsov and later extended by Horvay and Cahn are obtained under very general assumptions. In two-dimensional growth, Ivantsov's ansatz is equivalent to searching for solutions in the space of conformal transformations. We show that the parabolae are the unique solutions in this space. For the three-dimensional dendrite problem, we rederive, by a new method, the previously found solutions and prove that they are the only ones allowed by Ivantsov's ansatz. Finally, the linearization around Ivantsov's paraboloid is given in a more convenient form.

PACS numbers: 0340, 6870

1. Introduction

Usually, for solving a growth problem, the most appropriate method is to start by finding theoretically exact solutions in the vanishing surface tension limit. Indeed, the global problem without this step is often intractable analytically and even numerically, since iterative calculations require a good starting zero-order approximation. The models with zero surface tension are admittedly not physical, but once they are solved, they allow great improvements in the understanding of the dynamics of the corresponding growth and its selection mechanism [1–3]. In the case of the growth of a new phase, the only known solutions are those pointed out in the paper of Horvay and Cahn [2]. These solutions concern two different models of growth: a self-similar growth process where all lengths are proportional to the square root of time, and the steady growth process with a constant velocity in one favoured direction: the dendritic growth. In his famous paper [3], Ivantsov showed that the dendrite problem admits as a solution a paraboloid of revolution (a parabola for a dendrite platelet), and by using the Ivantsov method Horvay and Cahn solved the problem of the paraboloid with elliptical cross section.

In crystal growth experiments, other shapes have been observed either in two-dimensional geometries such as the dendrite in a channel [4a] and the free dendrite with a groove at its tip [4b], or in three-dimensional ones such as the free dendrite with four fold symmetry [4c]. One may wonder if other exact solutions for the dendrite problem can be found at zero surface tension. In this paper, we try to answer this question and show in which way the investigations must be focused if these solutions exist. We will adopt, at least in the beginning, the same approach as Ivantsov. We use the Ivantsov ansatz which appears to be a very clever method to solve free boundary problems. Unfortunately it turns out that, inside this ansatz, the already known solutions are the only ones. Moreover, even when one modifies this approach by making a more general ansatz, it becomes difficult to see that new shapes may be found.

2. The dendrite problem

In the following, we assume for convenience that the crystal growth is limited by the diffusion of latent heat in its pure melt, although the theory can be applied to any diffusive process with arbitrary diffusion coefficients either in the solid or in the liquid. We adopt a representation of the dendrite model which is independent of the space dimension. When surface tension effects are neglected, the Gibbs–Thomson law indicates that the liquid–solid interface is an isothermal surface. So, the temperature is constant throughout the entire crystal and is equal to the melting temperature T_{m}. With these assumptions, the equations of the growing crystal reduce to the one of diffusion in the liquid phase:

$$\Delta U = \frac{\partial U}{\partial \tau} \tag{1}$$

where U is a dimensionless temperature field that measures the departure from the temperature at infinity T_∞:

$$U = \frac{c(T - T_\infty)}{L}$$

where c denotes the heat capacity and L the latent heat. The stationary space coordinates X_i ($i = 1, \ldots, d$; d is the space dimension) are scaled by a length parameter R, which is defined by a complete solution of the growth problem when it is not fixed by the geometry. The dimensionless time τ is then given by $\tau = Dt/R^2$, where D is the diffusion coefficient in the liquid phase.

The boundary condition at the interface is given by the Stefan law which relates the jump in the temperature gradient across the interface to the normal velocity. Ivantsov transformed this condition using the fact that the interface is an isotherm of dimensionless temperature U_{f} equal to the supercooling $c(T_m - T_\infty)/L$. Thus the Stefan law may be rewritten as (see [2] or [3] for details):

$$|\nabla U|^2 = \frac{\partial U}{\partial \tau} \qquad \text{at } U = U_{\mathrm{f}} \tag{2}$$

Ivantsov's method consists of generalizing the boundary condition (2) to the whole liquid phase by introducing an arbitrary smooth function of temperature $F(U)$ in such a way that:

$$|\nabla U|^2 = F(U)\frac{\partial U}{\partial \tau} \tag{3}$$

and the boundary condition (2) becomes:

$$F(U_t) = 1. \tag{4}$$

So Ivantsov's treatment consists of replacing the free-boundary problem of growth into the solution of a system of two-coupled partial differential equations assumed to be valid everywhere in the liquid phase. Note that the choice of F as a function of U is clearly restrictive, because the dependence on the position and time coordinates is supposed to be implicit. For example, equation (3) is not satisfied for the Saffman–Taylor viscous fingering. One simple way to remove these restrictions, is to consider a most general smooth function F which depends explicitly on X_i and τ. One recovers Ivantsov's ansatz when $F(X_i, \tau)$ is an appropriate combination of X_i and τ such that it becomes a function of U only.

Now, let us suppose that we look for steady state growing crystals in the X_d direction. Then, we can define new dimensionless coordinates x_i moving forward at the velocity V of the freezing front as follows:

$$x_i = X_i \qquad i \neq d \tag{5a}$$

$$x_d = X_d - 2P\tau \tag{5b}$$

where $P = RV/2D$ is the Péclet number. For this stationary problem, F must depend on time only through x_d. In this case, the term $\partial U/\partial \tau$ is replaced by $-2P\partial U/\partial x_d$, or under a vectorial form by $-2P\nabla U \cdot \nabla x_d$. By combining (1) and (3), we can write the diffusion equation in a more convenient form. Then, in the new frame of reference, the equations to be solved may be written as follows:

$$F(x_i)\,\Delta U = |\nabla U|^2 \tag{6a}$$

$$\nabla U \cdot (\nabla U + 2PF(x_i)\,\nabla x_d) = 0 \tag{6b}$$

with the same boundary condition:

$$F(x_i) = 1 \text{ at the interface.} \tag{6c}$$

This covariant writing of the equations is helpful, because it brings out scalar products, which are invariant under coordinate transformations. Until now, we did not really impose any drastic restrictions on the space of solutions of equations (1)–(3). The only hypothesis we made is that the function F must be smooth and single-valued at each point of the liquid phase. In fact, the assumption above requires only that the temperature field $U(x_i)$ is well defined and regular. In consequence, ∇U must be a non-vanishing vector field at every point of the space, which is the case of the temperature fields of most known dendritic growths.

Equation (6*b*) states that at each point of the liquid phase and of the interface, one can construct two orthogonal vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ given by:

$$\boldsymbol{u} = \nabla U \qquad \boldsymbol{v} = \nabla U + 2PF(x_i)\,\nabla x_d. \tag{7}$$

This construction leads to a pair of vector fields ($\boldsymbol{u}$, $\boldsymbol{v}$), where $\boldsymbol{u}$ is orthogonal to the isothermal surfaces, and $\boldsymbol{v}$ belongs to the tangent space of these surfaces. Seeing that these fields are well defined (especially $\boldsymbol{u}$, which has a physical meaning), we can conclude that it is always possible to build an orthogonal coordinate system where $\boldsymbol{u}$ is parallel to one of the directions of these new axis. In consequence, $\boldsymbol{v}$ will automatically belong to the orthogonal space to this direction. The important result which follows from this coordinate transformation is that the temperature field becomes a function of only one variable. Effectively, since ∇U (or $\boldsymbol{u}$) has an invariant direction in the new frame, U must depend only on the corresponding component of this direction. In view of this, solving the system of equations (6) reduces essentially to finding an appropriate change of coordinates $x_i \to \eta_i$, such that equation (6b) gives:

$$U = U(\eta_d) \tag{8a}$$

$$v_d(\eta_i) \equiv U'(\eta_d) + 2PF(\eta_i)\frac{\partial x_d(\eta_i)}{\partial \eta_d} = 0 \tag{8b}$$

where we choose U as a function of the variable η_d. $v_d(\eta_i)$ is the component of $\boldsymbol{v}$ in this direction. From the equations (8), one can immediately see that the trivial solutions of the dendrite problem, a line in the 2D growth and a plane in the 3D one, are found when one chooses the identity coordinate transformation $x_i \to \eta_i = x_i$.

All the propositions made above do not depend on the space dimension. Moreover, they are often insensitive to supplementary boundary conditions if these do not induce forbidden singularities to the temperature field. Indeed, when treating one kind of a dendrite problem (free boundary one, a dendrite in a channel, a dendrite with a groove at the tip, etc.) one requires only particular additional boundary conditions, which comes down to imposing the equivalent ones to the chosen $F(x_i)$. Since the Stefan growth problem seems to correspond to the search of a suitable curvilinear coordinates, we begin our investigations with the 2D case by trying to find suitable mappings in order to go beyond Ivantsov's ansatz.

3. 2D dendritic growth

3.1. Conformal mappings

The simplest group of coordinate transformations one can think of is the conformal mapping. It is indeed a huge one since *any analytic function* $\phi(z) = \xi + \mathrm{i}\eta$, $z = x + \mathrm{i}y$, *can be represented geometrically as a transformation of two-dimensional coordinates.* One can imagine two complex planes, one to represent the chosen values of z and the other to represent the resulting values of ϕ whose real part is related to the temperature field. Any line drawn on the z plane has a resulting line on the ϕ plane. Of course many other pairs of functions ξ, η of x and y can be used to define such a general coordinate transformation. The transformations represented by the real and imaginary parts of analytic functions, however, have several unique and useful characteristics.

So, let us suppose the existence of a conformal mapping from x ($\equiv x_1$), y ($\equiv x_2$) to ξ ($\equiv \eta_1$), η ($\equiv \eta_2$), such that equations (8) are satisfied. The complex function $\phi = \xi + \mathrm{i}\eta$ (resp. the complex conjugate $\phi^* = \xi - \mathrm{i}\eta$) must be an analytic function of the complex variable $z = x + \mathrm{i}y$ (resp. the complex conjugate $z^* = x - \mathrm{i}y$). By using the differentials $\partial/\partial z = (\mathrm{d}\phi/\mathrm{d}z)\partial/\partial\phi$ and $\partial/\partial z^* = (\mathrm{d}\phi^*/\mathrm{d}z^*)\partial/\partial\phi^*$, we see that the equation (6*a*) keeps the same form in these new coordinates:

$$F(\xi, \eta)\,\Delta U = |\nabla U|^2 \tag{9}$$

where the Laplacian and the gradient operators are now written with the new variables ξ and η. But from (8*a*), U does not depend on ξ. One immediately finds from (9) that F depends only on η. Therefore, the chosen F must be, at least formally, an explicit function of U only:

$$F(\xi, \eta) \equiv F(\eta) \equiv F(U) = \frac{U'^2(\eta)}{U''(\eta)}. \tag{10}$$

This result is very important, because it states that: *among all the functions F, the only possible ones in this case have the form of Ivantsov's ansatz.* This means that the choice of a coordinate system deduced by a conformal mapping implies automatically Ivantsov's ansatz. Moreover, we now prove that this condition gives automatically a unique solution: Ivantsov's parabola for the free boundary dendrite problem.

By replacing (10) in (8*b*), one deduces the condition that must be satisfied by the coordinate transformation:

$$\frac{\partial y}{\partial \eta} = -\frac{1}{2P}\frac{U''(\eta)}{U'(\eta)} \tag{11}$$

By integrating (11), one finds that $y(\xi, \eta)$ is a separable function of ξ and η. The analytic function $z = z(\phi)$ is easily found by using the Cauchy–Riemann conditions and the fact that $y(\xi, \eta)$ is a harmonic function. After simple algebraic manipulations, we obtain the coordinate transformation under the following form:

$$x(\xi, \eta) = \eta\xi \tag{12a}$$

$$y(\xi, \eta) = \tfrac{1}{2}(\eta^2 - \xi^2) \tag{12b}$$

This is simply the parabolic change of coordinates. The isotherm surfaces are the parabolae indexed by $\eta =$ constant, and the liquid–solid interface is at $\eta = 1$ (here, R is the tip radius of curvature of this parabola). Thus, from equations (10) and (11) and the boundary condition (6*c*), one can deduce the external temperature distribution $U(\eta)$, the ansatz $F(\eta)$ and the relation between the supercooling and the Péclet number:

$$U(\eta) = U_f \frac{\displaystyle\int_\eta^\infty \mathrm{d}x\, \exp(-Px^2)}{\displaystyle\int_1^\infty \mathrm{d}x\, \exp(-Px^2)}$$

$$F(\eta) = \exp(P)\frac{\exp(-P\eta^2)}{\eta}$$

with

$$U_t = 2P\exp(P)\int_1^\infty \mathrm{d}x\, \exp(-Px^2).$$

This result has been obtained by Ivantsov a long time ago by a completely different method. Although the space of conformal mapping is infinite, it provides only one $2D$ solution to the Stefan problem: the well-known $2D$ dendrite.

3.2. Arbitrary frame transformation

When the coordinate transformation from (x, y) to (ξ, η) is not necessarily conformal, one has to write the orthogonality conditions under the form:

$$\frac{\partial x}{\partial \xi} = \lambda(\xi, \eta)\frac{\partial y}{\partial \eta} \tag{13a}$$

$$\frac{\partial x}{\partial \eta} = -\frac{1}{\lambda(\xi, \eta)}\frac{\partial y}{\partial \xi} \tag{13b}$$

where $\lambda(\xi, \eta)$ is any arbitrary function of ξ and η. The particular case of conformal mappings corresponds to $\lambda = 1$. Note that the scale factors $h_1(\xi, \eta)$ and $h_2(\xi, \eta)$ defined by: $h_1 = [(\partial x/\partial \xi)^2 + (\partial y/\partial \xi)^2]^{1/2}$ and $h_2 = [(\partial x/\partial \eta)^2 + (\partial y/\partial \eta)^2]^{1/2}$ are related to λ by: $\lambda = h_1/h_2$. In the new coordinates, the diffusion equation: $\Delta U + 2P\, \partial U/\partial y = 0$, is transformed to:

$$\frac{\partial \log \lambda(\xi, \eta)}{\partial \eta} + \frac{U''(\eta)}{U'(\eta)} + 2P\frac{\partial y(\xi, \eta)}{\partial \eta} = 0$$

where we have chosen the direction of ∇U in the η one. The integration of this equation fixes λ in terms of ξ, η and the function $y(\xi, \eta)$:

$$\lambda^{-1}(\xi, \eta) = a(\xi)U'(\eta)\exp(2Py(\xi, \eta)). \tag{14}$$

The unknown functions $a(\xi)$ and $U(\eta)$ can be determined from the behaviour of the isotherm surfaces far away from the liquid–solid interface. For example, in the case of a dendrite in a channel [5], it is obvious that the coordinate transformation tends to that of the identity transformation in the vicinity of $U = 0$ ($\eta \to \infty$). So, one deduces that $a(\xi)$ is a constant and $U(\eta)$ behaves as $\exp(-2P\eta)$. In other hand, the integrability conditions of the coordinate transformation (14) give, when combined with equation (14), the equation that satisfies $\lambda(\xi, \eta)$:

$$\left[\lambda_\xi^{-1} - \frac{a'}{a}\lambda^{-1}\right]_\xi = \left[\lambda_\eta + \frac{U''}{U'}\lambda\right]_\eta. \tag{15}$$

This is highly nonlinear partial differential equation, which has no obvious solutions even when one treats a specific dendrite problem and takes into account the corresponding values of $a(\xi)$ and $U(\eta)$. Note that when λ is taken as the product of two separable functions of η and ξ, one still recovers Ivantsov's solution.

Then, from the previous remarks, one notices that it is rather difficult to find analytically solutions of the Stefan problem other than the needle parabola of Ivantsov. This result may be surprising, since Ivantsov's ansatz can appear rather physical. When one tries to adopt another point of view, one is faced with a complicated problem. We know growth solutions without surface tension in the Laplace approximation ($P = 0$), such as dendrites with a groove or viscous fingering solutions either in a linear or a sector geometry [1]. They have been found by the conformal mapping technique. Unfortunately, it appears quite impossible to extend these solutions to a diffusion field since, as shown in the last part of this paper, they do not obey the Ivantsov ansatz. Let us now examine the 3D case.

4. 3D dendritic growth

For 3D changes of coordinates we do not have the analogues of the 2D conformal transformations. So, let us suppose from the beginning that the function F satisfies Ivantsov's ansatz. Because $F(U)$ is assumed smooth and single-valued, it is always possible to define [2] a function $\omega(U)$ related to $F(U)$ such that:

$$P\frac{d\omega(U)}{dU} = -\frac{1}{F(U)} \tag{16}$$

The new function $\omega(U)$ is a one to one mapping; so, the inverse function $U(\omega)$ exists. The isotherms already given by constant values of U are now also indexed by the correspondant constant values of ω. Now, let us make the coordinate transformation from x ($\equiv x_1$), y ($\equiv x_2$), z ($\equiv x_3$) to η_1, η_2, η_3, such that equations (8) are satisfied. When using the variable change (16) and integrating (8*b*), we obtain the condition that must be satisfied by the change of coordinates:

$$\omega(\eta_3) - 2z(\eta_1, \eta_2, \eta_3) = a(\eta_1, \eta_2) \tag{17}$$

where $a(\eta_1, \eta_2)$ is an unknown function independent on the third variable η_3. One can see that the parabolic coordinates and the paraboloidal ones satisfy the condition (17), but are they the only ones? Instead of searching the coordinates systems which are involved by this condition, we will use another approach to answer this question.

Let us suppose that one can extract, at least in a formal way, from the two equations $x = x(\eta_1, \eta_2, \eta_3)$ and $y = y(\eta_1, \eta_2, \eta_3)$ the functions $\eta_1(\eta_3, x, y)$ and $\eta_2(\eta_3, x, y)$. By using the fact that $\eta_3 = \eta_3(\psi)$ (because $\omega(\eta_3)$ is a one to one function), the two coordinates η_1, η_2 can be formally given in terms of x, y and ω. So one can state that the isotherms in the cartesian coordinates are given by:

$$\omega - 2z = \phi(x, y, \omega) \tag{18}$$

where ϕ is a function independent of z to be determined. This condition means that: If the isotherm surfaces satisfy a certain general equation $S(x, y, z, \omega) = 0$, then this equation must be separable under the form (18). Instead of taking in equations (6) the temperature field as a function of the coordinates x, y and z, it is more suitable to use the general parametrization $S(x, y, z, \omega) = 0$. Thus, one can easily transform equations (6) and derive the conditions that satisfy the functions S and $U(\omega)$:

$$\Delta S = -2Q(\omega)S_z - 2S_{z\omega} \tag{19a}$$

$$|\nabla S|^2 + 2S_z S_\omega = 0 \tag{19b}$$

$$U(1) = U_f \text{ and } U'(1) = -P \tag{19c}$$

where S_z (resp. S_ω) is the partial derivative of S with respect to z (resp. ω). $Q(\omega)$ is an unknown function of ω related to the variable change $U(\omega)$ by:

$$Q(\omega) = P + \frac{U''(\omega)}{U'(\omega)} \tag{20}$$

The condition (18) simplifies greatly equations (19*a*) and (19*b*), because it reduces the space dimension variables from 3 to 2. Therefore, the 3D Laplacian and gradient operators become the 2D ones for the function $\phi(x, y, \omega)$. So, the equations satisfied by ϕ can be written under the form where the derivatives are taken with respect to the complex variable $z = x + \mathrm{i}y$ and its complex conjugate $z^* = x - \mathrm{i}y$:

$$\frac{\partial^2 \phi}{\partial z\, \partial z^*} = -Q(\omega) \tag{21a}$$

$$\frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial z^*} + \phi_\omega = 0. \tag{21b}$$

The integration of equation (21*a*) over z and z^* gives ϕ as the sum of an anharmonic function and a harmonic one such that:

$$\phi(\zeta, \zeta^*) = -Q(\omega)\zeta\zeta^* + \psi(\zeta, \omega) + \psi^*(\zeta^*, \omega) \tag{22}$$

where $\psi(\zeta, \omega)$ is any analytic function and $\psi^*(\zeta^*, \omega)$ its analytic complex conjugate one. Consequently, equation (21*b*) becomes:

$$2\,\mathrm{Re}(-Q\zeta\psi_\zeta + \psi_\omega) + \psi_\zeta\psi^*_{\zeta^*} + (Q^2 - Q')\zeta\zeta^* = 0 \tag{23}$$

with Re denoting the real part of the function in parenthesis. By differentiating equation (23) to ζ and ζ^*, one can deduce the form of the only permitted function $\psi(\zeta, \omega)$:

$$\psi(\zeta, \omega) = \tfrac{1}{2}(Q'(\omega) - Q^2(\omega))^{1/2}\zeta^2. \tag{24}$$

The problem remaining is to determine $Q(\omega)$. This is done when one replaces $\psi(\zeta, \omega)$ by its value from (24). One finds a differential equation for Q, and verifies that it admits as a solution:

$$Q(\omega) = -\tfrac{1}{2}\left(\frac{1}{\omega} + \frac{1}{\omega + B}\right) \tag{25}$$

where B is a non-determined constant. Therefore, the equation of isotherm surfaces is given, in cylindrical coordinates (r, θ, z), by:

$$\omega - 2z = \frac{r^2}{2}\left(\left(\frac{1}{\omega} + \frac{1}{\omega + B}\right) + \left(\frac{1}{\omega} - \frac{1}{\omega + B}\right)\cos 2\theta\right). \tag{26}$$

They are the paraboloids with elliptic cross section already found by Horvay and Cahn (the Ivantsov paraboloids correspond to the particular case $B = 0$). The variable change $U(\omega)$ and the undercooling U_f are then those shown in [2]. So one can state that: *in 3D dendritic growth, the paraboloidal and the paraboloid shapes for a free boundary problem are the only solutions allowed by Ivantsov's ansatz*. This is the equivalent result of the 2D case.

The linearization around the Ivantsov paraboloid made by Kessler *et al* and Langer *et al* [6] for finding solutions which have higher order symmetries (for example a four-fold one), appears automatically from equation (23). If one assumes that the non-linear term $\psi_\zeta\psi^*_{\zeta^*}$ can be neglected, the function $Q(\omega)$ will have the same form as in Ivantsov case ($Q(\omega) = -1/\omega$) and $\psi(\zeta, \omega)$ will satisfy the simple differential equation:

$$\zeta\psi_\zeta + \omega\psi_\omega = 0 \tag{27}$$

which admits as a general solution the function:

$$\psi(\zeta, \omega) \equiv \psi(\zeta/\omega) \tag{28}$$

where $\psi(\zeta/\omega)$ is any analytic function of ζ. So, with this reasoning, we find a more general linearization method than in [6], where the function ψ was only defined on the interface. Moreover, in our case one can easily extract all the terms of any linearized solution which satisfies any selected symmetry. For the linearized solutions with four fold symmetry, we find that they are expanded in series of homogeneous functions $\alpha_n\,(r/\omega)^{4n}\cos(4n\theta)$, where α_n is a free parameter and n is a positive integer, because one excludes the terms which diverge when $\omega \to \infty$ (which is the equivalent condition to $U \to 0$). Unfortunately, this linearization is not completely valid, at least when $F = F(U)$, because the nonlinear term cannot be neglected. In consequence, when one restricts to Ivantsov's ansatz, any approximate solution constructed by this linearization is invalid due to the fact that the global problem has not other solutions than those already found. Anyway, these linearized degrees of feedom of the diffusion field are an essential ingredient in the theory of the 3D dendritic selection as shown recently [7].

5. Laplacian growth process and the Ivantsov ansatz

For Laplacian field in 2D, steady state solutions can be (or have been found) by conformal mapping techniques. This technique consists in mapping the $z = x + iy$ plane to the complex potential one: $\Psi = \phi + i\psi$, where ϕ is the velocity potential and ψ the stream-function. Known and explicit solutions exist such as the 2D needle-crystal and the Saffman–Taylor solutions. Extending these families, either to the 3D geometries or to diffusive fields is of special interest in studies on growth phenomena.

5.1. The dendritic growth

As it has been shown in [8], the conformal mapping technique allows one to treat the 2D needle-crystal growth when the Laplace equation is assumed for the temperature field.

$$iz = -\frac{1}{4P^2}(\Psi - P)^2 + \tfrac{1}{4}. \tag{29}$$

Here, 2R is the length unit, so the interface which corresponds to $\phi = 0$ is given by: $y = -x^2$. Using ϕ and ψ as frame coordinates transforms the Stefan law into:

$$\left.\frac{\partial y}{\partial \phi}\right|_{\phi=0} = -\frac{1}{2P} \tag{30}$$

and the Ivantsov ansatz, which is verified here, into:

$$\frac{\partial y}{\partial \phi} = \frac{1}{F(\phi)} \qquad \text{with } F^{-1}(\phi) = \frac{\phi - P}{2P^2}. \tag{31}$$

Another interesting dendritic solution with a central groove is easily found:

$$\mathrm{i}z = -\frac{1}{4P^2}(\Psi - P)^2 + \tfrac{1}{4} - \frac{2x_0}{\pi}\log\Psi. \tag{32}$$

Since we have not found this solution in the literature, let us give some details concerning the isotherms. Taking the real and imaginary parts, one easily finds:

$$x = -\frac{1}{P^2}(\phi - P)\psi - \frac{2x_0}{\pi}A\tan\left(\frac{\psi}{\phi}\right) \tag{33a}$$

$$y = \frac{1}{4P^2}((\phi - P)^2 - \psi^2) - \tfrac{1}{4} + \frac{x_0}{\pi}\log(\phi^2 + \psi^2) \tag{33b}$$

and the interface equation is:

$$y = -(|x| - x_0)^2 + \frac{x_0}{\pi}\log(|x| - x_0)^2. \tag{34}$$

So $2x_0$ represents the relative width of the groove. Note that this family presents a double degree of freedom: the length scale R and x_0. Here, the Stefan condition (on the interface given by: $\phi = 0$) is obviously verified but the Ivantsov ansatz is not since:

$$\frac{\partial y}{\partial \phi} = \frac{\phi - P}{2P^2} + \frac{2x_0}{\pi}\frac{\phi}{\phi^2 + \psi^2}. \tag{35}$$

All our attempts to extend this new family of dendritic solutions to more realistic diffusion fields were not successful. The fact that the Ivantsov ansatz is not valid did not help.

5.2. The Saffman–Taylor solution and its extension to dendritic growth in a channel

It is often interesting to study the effect of confinement in order to make the dendritic growth theories closer to the experiments. The Saffman–Taylor viscous fingering instability in a linear cell, which is now very well understood, is a good example that physicists would like to extend to a diffusion field [1]. The solutions for symmetric fingers have been discovered a long time ago by Saffman and Taylor: if λ is the relative finger width compared to the cell width, the family of solutions is given by:

$$-\mathrm{i}z = \frac{\Psi}{\lambda U} + \frac{2}{\pi}(1 - \lambda)\log\tfrac{1}{2}\left(1 + \exp - \frac{\pi\Psi}{\lambda U}\right). \tag{36}$$

Here, half the cell-width a is chosen in length units and U is the finger velocity. We give also the y coordinate in terms of (ϕ, ψ):

$$y = \frac{\phi}{\lambda U} + \frac{1}{\pi}(1 - \lambda)\log\tfrac{1}{4}\left(1 + \exp - \frac{2\pi\phi}{\lambda U} + 2\exp - \frac{\pi\phi}{\lambda U}\cos\frac{\pi\psi}{\lambda U}\right). \tag{37}$$

The Stefan condition is verified but not the Ivantsov ansatz.

6. Conclusion

In this paper, we have shown that the Ivantsov approach does not give more solutions than those already found. For the 2D dendritic growth, we found that the Ivantsov parabolae are the unique allowed patterns in the large space of conformal representations. Beyond Ivantsov's ansatz, the equations in the orthogonal coordinates of the temperature field become nonlinear and hard to solve. For the 3D problem, we showed that, in a subspace where the ansatz F is an explicit function of the temperature, the Ivantsov and Horvay–Cahn solutions are the only permitted ones. We gave the linearization around Ivantsov's paraboloid in a simple way and we saw that, even for the linearized problem, one has to treat it beyond Ivantsov's ansatz.

Acknowledgments

The authors acknowledge E Brener, V Hakim and Y Pomeau for their helpful and enlightening discussions.

Appendix. Derivation of equations (19)

When doing the variable transformation (16), equations (6) can be rewritten under the form:

$$\Delta\omega = -2Q(\omega)\frac{\partial\omega}{\partial z} \tag{A1}$$

$$|\nabla\omega|^2 - 2\frac{\partial\omega}{\partial z} = 0 \tag{A2}$$

where $Q(\omega)$ is the function defined in (20). The isotherms $\omega = \text{constant}$ are defined by a function S of x, y, z and ω such that:

$$S(x, y, z, \omega) = 0 \tag{A3}$$

so when differentiating S to all the variables, one obtains the equalities:

$$\nabla S + S_\omega \nabla\omega = 0 \tag{A4}$$

$$S_z + S_\omega \omega_z = 0. \tag{A5}$$

Therefore, equation (A2) can be rewritten as follows:

$$|\nabla S|^2 + 2S_z S_\omega = 0 \tag{A6}$$

which is equation (19b). Applying the gradient operator to equation (A4) yields:

$$S_\omega \Delta\omega = -\Delta S - 2(\nabla S)_\omega \cdot \nabla\omega - S_{\omega\omega}|\nabla\omega|^2 \tag{A7}$$

or by replacing $\nabla\omega$ by its value from (A4), one finds:

$$S_\omega \Delta\omega = -\Delta S + \left(\frac{|\nabla S|^2}{S_\omega}\right)_\omega . \tag{A8}$$

Finally, we recover equation (19*a*) by using the identities (A5), (A6) and (A7) in (A1).

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