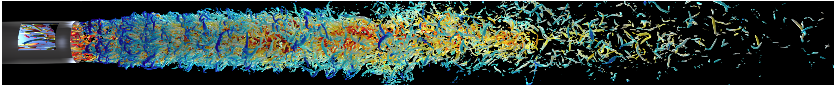
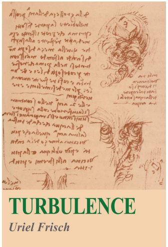
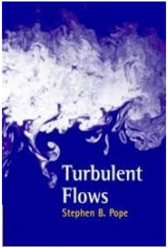
Turbulence
Equations and Conserved Quantities

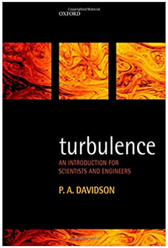


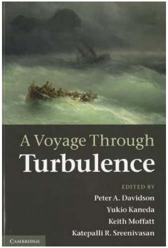
Alexandros ALEXAKIS
alexakis@phys.ens.fr
Dep. Physique ENS Ulm

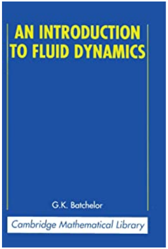
Turbulence: Books

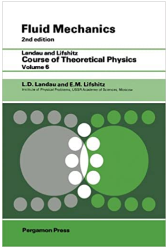


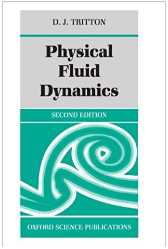


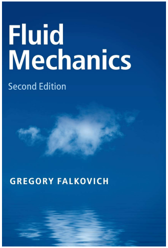






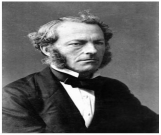






The Navier Stokes equations

Claude-Louis Navier
1785-1836



Sir George Stokes
1819-1903

Isaac Newton
1642-1727

Leonhard Euler
1707-1783

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot [\mathbf{u}(\rho \mathbf{u}))] = -\nabla P + \nabla \cdot \left(\mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - \frac{2}{3}\mu \mathbf{I}(\nabla \cdot \mathbf{u}) \right) + \mathbf{f}$$

- $\mathbf{u}(\mathbf{x}, t)$ is the velocity field
- $\rho(\mathbf{x}, t)$ is the density
- $P(\mathbf{x}, t)$ is the pressure
- μ is the *dynamic viscosity*
- $\mathbf{f}$ represents (external) forcing terms

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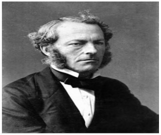

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Mass transport

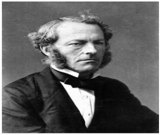
$$\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0$$

Equation of state

$$P=f(\rho,T,,\dots)$$

The Incompressible Navier Stokes equations


Claude-Louis Navier
1785-1836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$$\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0$$

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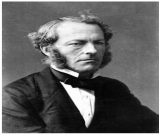

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Mass transport

$$\rho = constant$$

Incompressibility condition

$$\nabla\cdot\mathbf{u} = 0$$

The Incompressible Navier Stokes equations

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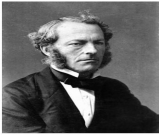

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$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{\mu}{\rho} \nabla^2 \mathbf{u} + \frac{1}{\rho} \mathbf{f}$$

Mass transport

$$\rho = constant$$

Incompressibility condition

$$\nabla \cdot \mathbf{u} = 0$$

The Incompressible Navier Stokes equations

Claude-Louis Navier
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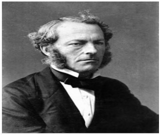

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$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

Mass transport

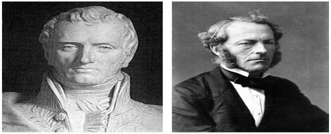
$$\rho = 1$$

Incompressibility condition

$$\nabla \cdot \mathbf{u} = 0$$

$$\nu = \mu/\rho \equiv \text{kinematic viscosity}$$

Breaking down the Navier Stokes equations



Claude-Louis Navier 1785-1836   Sir George Stokes 1819-1903

Navier Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

- time derivative
- Non-linear terms: advection and pressure
- Linear terms: viscous terms responsible for energy dissipation
- Non-homogeneous terms: Forcing - energy injection
- Divergence-free constraint

Breaking down the Navier Stokes equations

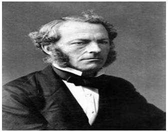
Euler Equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P$$

$$\nabla \cdot \mathbf{u} = 0$$

- time derivative
- Non-linear terms: advection and pressure
- Linear terms: viscous terms responsible for energy dissipation
- Non-homogeneous terms: Forcing - energy injection
- Divergence-free constraint

Breaking down the Navier Stokes equations



Sir George Stokes
1819-1903

Stokes Equations

$$\partial_t \mathbf{u} \qquad = \qquad + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

- time derivative
- Non-linear terms: advection and pressure
- Linear terms: viscous terms responsible for energy dissipation
- Non-homogeneous terms: Forcing - energy injection
- Divergence-free constraint

The Navier-Stokes

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

The pressure

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

Take the divergence $(\nabla \cdot)$ of the Navier Stokes equation

$$\partial_t (\nabla \cdot \mathbf{u}) + \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla^2 P + \nu \nabla^2 (\nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{f}$$

by incompressibility

$$\cancel{\partial_t (\nabla \cdot \mathbf{u}) +} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla^2 P \cancel{+ \nu \nabla^2 (\nabla \cdot \mathbf{u})} + \cancel{\nabla \cdot \mathbf{f}}$$

$$\boxed{\nabla^2 P = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})}$$

To find the pressure we have to invert a laplacian

Alternative forms of the Navier Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

using the vector identity

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = -\mathbf{A} \cdot \nabla \mathbf{A} + \frac{1}{2} \nabla (\mathbf{A} \cdot \mathbf{A})$$

$$\partial_t \mathbf{u} = \mathbf{u} \times \mathbf{w} - \nabla P' + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

where $\mathbf{w} = \nabla \times \mathbf{u}$ and $P' = P + \frac{1}{2} |\mathbf{u}|^2$

vorticity equation

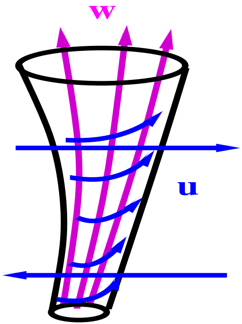
$$\boxed{\partial_t \mathbf{w} = \nabla \times (\mathbf{u} \times \mathbf{w}) + \nu \nabla^2 \mathbf{w} + \nabla \times \mathbf{f}}$$

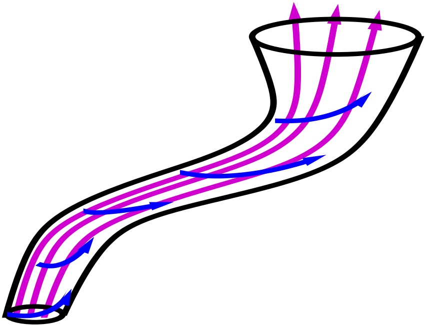
where $\nabla \times \mathbf{w} = \nabla \times \nabla \times \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}$

vorticity equation

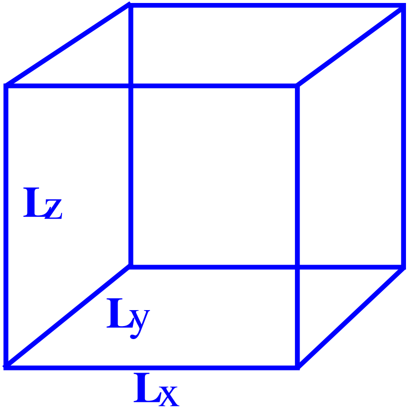
$$\partial_t \mathbf{w} = \nabla \times (\mathbf{u} \times \mathbf{w}) + \nu \nabla^2 \mathbf{w} + \nabla \times \mathbf{f}$$

Vorticity field lines move with the flow





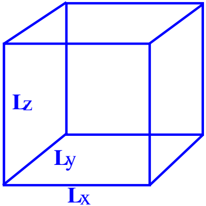
Domain $\mathcal{D}$



For 99,99% of the course $L_x = L_y = L_z = L$.

In many cases we want $L \to \infty$

Boundary Conditions on $\partial\mathcal{D}$



- No-slip

$$\mathbf{u} = 0$$

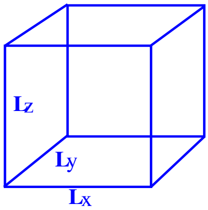
- Free slip

$$\mathbf{u} \cdot \mathbf{e}_n = 0 \quad \& \quad \mathbf{w} \times \mathbf{e}_n = 0$$

- Periodic

$$\mathbf{u}(x, y, z, t) = \mathbf{u}(x + n_x L_x, y + n_y L_y + z + n_z L_z, t)$$

Fourier Space (Finite box)



$$\mathbf{u}(\mathbf{x},t) = \sum \tilde{\mathbf{u}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \tilde{\mathbf{u}}_{\mathbf{k}}(t) = \left\langle \mathbf{u} e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle$$

where

$$\langle f(\mathbf{x}) \rangle = \frac{1}{L_x L_y L_z} \int f(\mathbf{x}) dV$$

and

$$\mathbf{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z} \right), \qquad \text{with} \quad n_x, n_y, n_z \in \mathbb{Z}$$

Fourier Space (finite and infinite space)

$$\tilde{\mathbf{u}}(\mathbf{k},t) = \frac{1}{L^3}\int \mathbf{u}e^{-i\mathbf{k}\cdot\mathbf{x}}dV, \qquad \tilde{\mathbf{u}}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\int \mathbf{u}e^{-i\mathbf{k}\cdot\mathbf{x}}dV$$

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \mathbf{u}(\mathbf{x},t) = \int \tilde{\mathbf{u}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}dk^3$$

$$\mathbf{k} \in \mathbb{N}^3 \qquad \mathbf{k} \in \mathbb{R}^3$$

Energy Spectrum:

- Finite Box

$$E(k) = \frac{L}{2}\sum_{k\le|\mathbf{k}|<k+1} |\tilde{\mathbf{u}}_{\mathbf{k}}|^2$$

- Infinite space

$$E(k) = \frac{1}{2}\int |\tilde{\mathbf{u}}|^2\delta(k-|\mathbf{k}|)dk^3$$

Navier Stokes in Fourier Space

$$\left\langle \left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle = \left\langle \left(-\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}\right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle$$

$$\partial_t \tilde{\mathbf{u}}_{\mathbf{k}} + \left\langle (\mathbf{u}\cdot\nabla\mathbf{u}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle = -\nabla\tilde{P}_{\mathbf{k}} - \nu|\mathbf{k}|^2\tilde{\mathbf{u}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}}$$

Navier Stokes in Fourier Space

$$\left\langle \left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle = \left\langle \left(-\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}\right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle$$

$$\partial_t \tilde{\mathbf{u}}_{\mathbf{k}} + \left\langle (\mathbf{u}\cdot\nabla\mathbf{u}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle = -\nabla\tilde{P}_{\mathbf{k}} - \nu|\mathbf{k}|^2\tilde{\mathbf{u}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}}$$

$$\begin{aligned}
\left\langle (\mathbf{u}\cdot\nabla\mathbf{u}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle &= \left\langle \left(\left[\sum_{\mathbf{q}} \tilde{\mathbf{u}}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\right] \cdot \nabla \left[\sum_{\mathbf{p}} \tilde{\mathbf{u}}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right] \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle \\
&= \left\langle \left(\sum_{\mathbf{p},\mathbf{q}} \tilde{\mathbf{u}}_{\mathbf{q}} \cdot (i\mathbf{p})\tilde{\mathbf{u}}_{\mathbf{p}} \right) e^{i(\mathbf{q}+\mathbf{p}-\mathbf{k})\mathbf{x}} \right\rangle \\
&= \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} i\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}}\tilde{\mathbf{u}}_{\mathbf{p}}
\end{aligned}$$

Navier Stokes in Fourier Space: Pressure term

$$\left\langle e^{-i\mathbf{k}\cdot\mathbf{x}}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\right\rangle = \left\langle -\left(\nabla^2 P\right) e^{-i\mathbf{k}\cdot\mathbf{x}}\right\rangle$$

$$i\mathbf{k}\cdot\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} i(\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})\tilde{\mathbf{u}}_{\mathbf{p}} = |\mathbf{k}|^2\tilde{P}_{\mathbf{k}}$$

$$\tilde{P}_{\mathbf{k}} = -\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \frac{(\mathbf{k}\cdot\tilde{\mathbf{u}}_{\mathbf{p}})(\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})}{|\mathbf{k}|^2}$$

Navier Stokes:

$$\boxed{\partial_t\tilde{\mathbf{u}}_{\mathbf{k}} = -\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} i\left((\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})\tilde{\mathbf{u}}_{\mathbf{p}} - \mathbf{k}\frac{(\mathbf{k}\cdot\tilde{\mathbf{u}}_{\mathbf{p}})(\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})}{|\mathbf{k}|^2}\right) - \nu|\mathbf{k}|^2\tilde{\mathbf{u}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}}}$$

and

$$\mathbf{k}\cdot\mathbf{u}_{\mathbf{k}} = 0$$

Volume averaged quantities

$$\langle f(\mathbf{x})\rangle = \frac{1}{L_x L_y L_z}\int f(\mathbf{x})dV$$

Identities

$$\langle \nabla\cdot\mathbf{a}\rangle = \frac{1}{V}\int_{\partial\mathcal{D}}(\mathbf{a}\cdot\mathbf{n})d\mathcal{S},$$

$$\langle \nabla\times\mathbf{a}\rangle = \frac{1}{V}\iint_{\partial\mathcal{A}}\nabla\times\mathbf{a}d\mathcal{A}dz = \frac{1}{V}\int\oint_{\partial\mathcal{C}}\mathbf{a}\cdot d\ell dz,$$

In periodic domains this implies

$$\langle \nabla\cdot\mathbf{a}\rangle = 0$$

$$\langle \nabla\times\mathbf{a}\rangle = 0$$

Volume averaged quantities

in periodic domains we thus have

- $\langle \partial_i f \rangle = 0$
- $\langle (\nabla f) g \rangle = \langle \nabla (fg) \rangle - \langle f(\nabla g) \rangle = -\langle f(\nabla g) \rangle$
- $\left\langle (\nabla^2 f) g \right\rangle = \langle \nabla (g \nabla f) \rangle - \langle (\nabla f)(\nabla g) \rangle = -\langle (\nabla f)(\nabla g) \rangle$
- $$\begin{aligned} \langle \mathbf{a} \cdot \nabla \mathbf{b} \rangle = \langle \nabla_i (a_i b_j) \rangle - \langle \mathbf{b} \nabla \cdot \mathbf{a} \rangle &= -\langle \mathbf{b} \nabla \cdot \mathbf{a} \rangle \\ &= 0 \quad if \quad \nabla \cdot \mathbf{a} = \mathbf{0} \end{aligned}$$
- $\langle \mathbf{a} \cdot \nabla \times \mathbf{b} \rangle = \langle \nabla \cdot (\mathbf{b} \times \mathbf{a}) \rangle + \langle \mathbf{b} \cdot \nabla \times \mathbf{a} \rangle = \langle \mathbf{b} \cdot \nabla \times \mathbf{a} \rangle$
- $\left\langle \mathbf{a} \cdot \nabla^{\mathbf{2}} \mathbf{b} \right\rangle = -\langle \mathbf{a} \cdot \nabla \times \nabla \times \mathbf{b} \rangle = -\langle (\nabla \times \mathbf{b}) \cdot (\nabla \times \mathbf{a}) \rangle$

Conservation laws: Momentum

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\begin{aligned}
\partial_t \mathbf{u} + \quad & \mathbf{u}\cdot\nabla\mathbf{u} && = -\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f} \\
\partial_t \langle\mathbf{u}\rangle + \quad & \langle\mathbf{u}\cdot\nabla\mathbf{u}\rangle && = -\langle\nabla P\rangle + \nu\left\langle\nabla^2\mathbf{u}\right\rangle + \langle\mathbf{f}\rangle \\
\partial_t \langle\mathbf{u}\rangle + \quad & \langle\nabla_i(u_i u_j)\rangle - \langle\mathbf{u}\nabla\cdot\mathbf{u}\rangle && = -\langle\nabla P\rangle + \nu\left\langle\nabla^2\mathbf{u}\right\rangle + \langle\mathbf{f}\rangle \\
\partial_t \langle\mathbf{u}\rangle + \quad & \cancel{\langle\nabla_i(u_i u_j)\rangle} - \cancel{\langle\mathbf{u}\nabla\cdot\mathbf{u}\rangle} && = -\cancel{\langle\nabla P\rangle} + \nu\cancel{\left\langle\nabla^2\mathbf{u}\right\rangle} + \cancel{\langle\mathbf{f}\rangle}
\end{aligned}$$

$$\partial_t\langle\mathbf{u}\rangle = 0$$

(here $\langle\mathbf{f}\rangle = 0$ is assumed)

Conservation laws: Energy

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\begin{aligned}
\langle \mathbf{u}\cdot\partial_t\mathbf{u}\rangle + \langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\rangle &= -\langle \mathbf{u}\cdot\nabla P\rangle + \nu\big\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\big\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle \\
\frac{1}{2}\frac{d}{dt}\big\langle|\mathbf{u}|^2\big\rangle + \frac{1}{2}\big\langle(\mathbf{u}\cdot\nabla|\mathbf{u}|^2)\big\rangle &= -\langle \mathbf{u}\cdot\nabla P\rangle + \nu\big\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\big\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle \\
\frac{1}{2}\frac{d}{dt}\big\langle|\mathbf{u}|^2\big\rangle + \frac{1}{2}\big\langle(\nabla_i u_i|\mathbf{u}|^2)\big\rangle &= -\langle \nabla\cdot(\mathbf{u}P)\rangle + \langle P\nabla\cdot\mathbf{u}\rangle \\
&\quad +\nu\langle\nabla_i(u_j\nabla_i u_j)\rangle - \nu\langle\nabla_j u_i\nabla_j u_i\rangle \\
&\quad +\langle\mathbf{u}\cdot\mathbf{f}\rangle
\end{aligned}$$

Conservation laws: Energy

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\begin{aligned}
\langle \mathbf{u}\cdot\partial_t\mathbf{u}\rangle + \langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\rangle &= -\langle \mathbf{u}\cdot\nabla P\rangle + \nu\big\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\big\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle \\
\frac{1}{2}\frac{d}{dt}\big\langle|\mathbf{u}|^2\big\rangle + \frac{1}{2}\big\langle(\mathbf{u}\cdot\nabla|\mathbf{u}|^2)\big\rangle &= -\langle \mathbf{u}\cdot\nabla P\rangle + \nu\big\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\big\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle \\
\frac{1}{2}\frac{d}{dt}\big\langle|\mathbf{u}|^2\big\rangle + \frac{1}{2}\cancel{\big\langle(\nabla_i u_i|\mathbf{u}|^2)\big\rangle} &= -\cancel{\langle\nabla\cdot(\mathbf{u}P)\rangle} + \langle P\cancel{\nabla\cdot}\mathbf{u}\rangle \\
&\quad +\nu\cancel{\langle\nabla_i(u_j\nabla_i u_j)\rangle} - \nu\langle\nabla_j u_i\nabla_j u_i\rangle \\
&\quad +\langle \mathbf{u}\cdot\mathbf{f}\rangle
\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\big\langle|\mathbf{u}|^2\big\rangle = -\nu\langle\nabla_j u_i\nabla_j u_i\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle$$

$$\boxed{\frac{d}{dt}\mathcal{E} = -\epsilon + \mathcal{I}}$$

Energy Dissipation rate ϵ

Let $\mathcal{S}_{ij} = (\partial_i u_j + \partial_j u_i)/2$ and $\Omega_{ij} = (\partial_i u_j - \partial_j u_i)/2$
The local energy dissipation rate is then given by

$$2\nu \mathcal{S}_{i,j}\mathcal{S}_{i,j}$$

In **periodic** domains we have

$$\begin{aligned}
\epsilon &= 2\nu\langle(\partial_i u_j + \partial_j u_i)(\partial_i u_j + \partial_j u_i)\rangle/4 \\
&= \nu\ \left\langle(\partial_i u_j)^2 + 2\partial_j u_i \partial_i u_j + (\partial_j u_i)^2\right\rangle/2 \\
&= \nu\ \left\langle(\partial_i u_j)^2\right\rangle + \langle\partial_j(u_i\partial_i u_j)\rangle \\
&= \nu\ \left\langle(\partial_i u_j)^2\right\rangle + \cancel{\langle\partial_j(u_i\partial_i u_j)\rangle} \\
&= \boxed{\nu\ \left\langle(\partial_i u_j)^2\right\rangle} \\
&= \nu\ \left\langle(\partial_i u_j)^2 - 2\partial_j u_i \partial_i u_j + (\partial_j u_i)^2\right\rangle/2 \\
&= \boxed{2\nu\langle\Omega_{i,j}\Omega_{i,j}\rangle = \nu\left\langle|\mathbf{w}|^2\right\rangle} \qquad (1)
\end{aligned}$$

Conservation laws: Helicity

$$\mathcal{H} = \frac{1}{2}\langle \mathbf{u} \cdot \mathbf{w} \rangle$$

$$\begin{aligned}
\partial_t \mathbf{u} &= \mathbf{u} \times \mathbf{w} - \nabla P & +\nu \nabla^2 \mathbf{u} + \mathbf{f} \\
\partial_t \mathbf{w} &= \nabla \times (\mathbf{u} \times \mathbf{w}) & +\nu \nabla^2 \mathbf{w} + \nabla \times \mathbf{f}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{u} \cdot \mathbf{w} \rangle &= \langle \mathbf{u} \cdot \partial_t \mathbf{w} \rangle + \langle \mathbf{w} \partial_t \mathbf{u} \rangle \\
&= \langle \mathbf{u} \cdot \nabla \times (\mathbf{u} \times \mathbf{w}) \rangle + \nu \left\langle \mathbf{u} \cdot \nabla^2 \mathbf{w} \right\rangle + \langle \mathbf{u} \cdot \nabla \times \mathbf{f} \rangle + \\
&\quad \langle \mathbf{w} \cdot (\mathbf{u} \times \mathbf{w}) \rangle + \langle \mathbf{w} \cdot \nabla P \rangle + \left\langle \mathbf{w} \cdot \nabla^2 \mathbf{u} \right\rangle + \langle \mathbf{w} \cdot \mathbf{f} \rangle
\end{aligned}$$

Conservation laws: Helicity

$$\mathcal{H} = \frac{1}{2}\langle \mathbf{u} \cdot \mathbf{w} \rangle$$

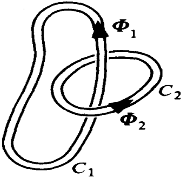
$$\begin{aligned} \partial_t \mathbf{u} &= \mathbf{u} \times \mathbf{w} - \nabla P & +\nu \nabla^2 \mathbf{u} + \mathbf{f} \\ \partial_t \mathbf{w} &= \nabla \times (\mathbf{u} \times \mathbf{w}) & +\nu \nabla^2 \mathbf{w} + \nabla \times \mathbf{f} \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\langle \mathbf{u} \cdot \mathbf{w} \rangle &= \langle \mathbf{u} \cdot \partial_t \mathbf{w} \rangle + \langle \mathbf{w} \partial_t \mathbf{u} \rangle \\ &= \cancel{\langle \mathbf{u} \cdot \nabla \times (\mathbf{u} \times \mathbf{w}) \rangle} + \nu \langle \mathbf{u} \cdot \nabla^2 \mathbf{w} \rangle + \langle \mathbf{u} \cdot \nabla \times \mathbf{f} \rangle + \\ &\quad \cancel{\langle \mathbf{w} \cdot (\mathbf{u} \times \mathbf{w}) \rangle} + \cancel{\langle \mathbf{w} \cdot \nabla P \rangle} + \nu \langle \mathbf{w} \cdot \nabla^2 \mathbf{u} \rangle + \langle \mathbf{w} \cdot \mathbf{f} \rangle \\ \frac{1}{2}\frac{d}{dt}\langle \mathbf{u} \cdot \mathbf{w} \rangle &= -\nu \langle \mathbf{w} \cdot \nabla \times \mathbf{w} \rangle + \langle \mathbf{w} \cdot \mathbf{f} \rangle \end{aligned}$$

$$\frac{d}{dt}\mathcal{H} = -\epsilon_{\mathcal{H}} + \mathcal{I}_{\mathcal{H}}$$

Conservation laws: Helicity

$\mathcal{H} = \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{w}\rangle$

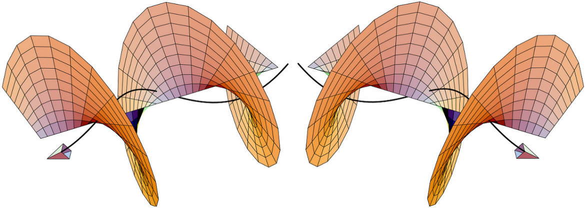


Consider two infinitesimal vorticity tubes

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\int_{V_1}\mathbf{u}\cdot\mathbf{w}dx^3 + \frac{1}{2}\int_{V_2}\mathbf{u}\cdot\mathbf{w}dx^3 \\
&= \frac{1}{2}\oint_{\mathcal{C}_\infty}\int_{S_1}\mathbf{u}\cdot\mathbf{w}dsd\ell + \frac{1}{2}\oint_{\mathcal{C}_\in}\int_{S_2}\mathbf{u}\cdot\mathbf{w}dsd\ell \\
&= \frac{1}{2}\Phi_1\oint_{\mathcal{C}_\infty}\mathbf{u}\cdot d\ell + \frac{1}{2}\Phi_2\oint_{\mathcal{C}_\in}\mathbf{u}\cdot d\ell \\
&= \frac{1}{2}\Phi_1\oint_{\mathcal{A}_\infty}\mathbf{w}\cdot d\mathbf{s} + \frac{1}{2}\Phi_2\oint_{\mathcal{A}_\in}\mathbf{w}\cdot d\mathbf{s} \\
&= \Phi_1\Phi_2
\end{aligned}$$

Helicity in Fourier Space

$$\tilde{\mathbf{u}}_{\mathbf{k}} = \frac{1}{L^3} \int e^{i\mathbf{k}\mathbf{x}} \mathbf{u} d\mathbf{x}^3, \quad \mathbf{u}(x) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} \tilde{\mathbf{u}}_{\mathbf{k}}$$

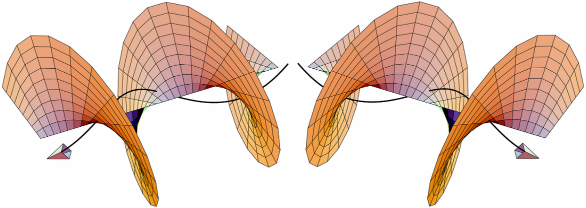


$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u^+_{\mathbf{k}} \mathbf{h}^+_{\mathbf{k}} + u^-_{\mathbf{k}} \mathbf{h}^-_{\mathbf{k}}}$$

$$\mathbf{h}^{\pm}_{\mathbf{k}} = \frac{\mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{e}})}{\sqrt{2}|\mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{e}})|} \pm i \frac{\mathbf{k} \times \hat{\mathbf{e}}}{\sqrt{2}|\mathbf{k} \times \hat{\mathbf{e}}|}$$

Helicity in Fourier Space

$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u^{+}_{\mathbf{k}}\,\mathbf{h}^{+}_{\mathbf{k}} + u^{-}_{\mathbf{k}}\,\mathbf{h}^{-}_{\mathbf{k}}}$$



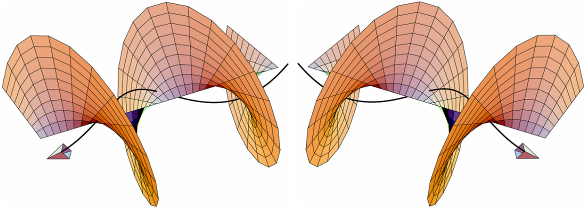
$$\mathbf{h}^{\pm}_{\mathbf{k}} = \frac{\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})}{\sqrt{2}|\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})|} \pm i\frac{\mathbf{k}\times\hat{\mathbf{e}}}{\sqrt{2}|\mathbf{k}\times\hat{\mathbf{e}}|}$$

$$i\mathbf{k}\times\mathbf{h}^{\pm}_{\mathbf{k}} = \pm k\mathbf{h}^{\pm}_{\mathbf{k}}, \quad i\mathbf{k}\cdot\mathbf{h}^{\pm}_{\mathbf{k}} = 0$$

$$\mathbf{h}^{s}_{\mathbf{k}}\cdot(\mathbf{h}^{s'}_{\mathbf{k}})^{*} = \mathbf{h}^{s}_{\mathbf{k}}\cdot\mathbf{h}^{-s'}_{\mathbf{k}} = \mathbf{h}^{s}_{\mathbf{k}}\cdot\mathbf{h}^{s'}_{-\mathbf{k}} = \delta_{s,s'}$$

Helicity in Fourier Space

$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u^{+}_{\mathbf{k}}\,\mathbf{h}^{+}_{\mathbf{k}} + u^{-}_{\mathbf{k}}\,\mathbf{h}^{-}_{\mathbf{k}}}$$



Energy:

$$\mathcal{E} = \frac{1}{2}\sum_{\mathbf{k}} \left[|u^{+}_{\mathbf{k}}|^2 + |u^{-}_{\mathbf{k}}|^2\right]$$

Helicity:

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}} k\left[|u^{+}_{\mathbf{k}}|^2 - |u^{-}_{\mathbf{k}}|^2\right]$$

Navier Stokes in helical modes

Let $s_{\mathbf{k}} = \pm 1$, $s_{\mathbf{q}} = \pm 1$, $s_{\mathbf{p}} = \pm 1$.
Then

$$\partial_t \left\langle \mathbf{h}_{-\mathbf{k}}^{s_{\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{x}} \cdot \mathbf{w} \right\rangle = \left\langle \mathbf{h}_{-\mathbf{k}}^{s_{\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{x}} \cdot \left(\nabla \times (\mathbf{u} \times \mathbf{w}) + \nu \nabla^2 \mathbf{w} + \nabla \times \mathbf{f} \right) \right\rangle$$

Navier Stokes:

$$\boxed{\partial_t u_{\mathbf{k}}^{s_{\mathbf{k}}} = \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}}, s_{\mathbf{q}}, s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}} \right) - \nu |\mathbf{k}|^2 u_{\mathbf{k}}^{s_{\mathbf{k}}} + f_{\mathbf{k}}^{s_{\mathbf{k}}}}$$

where

$$C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}}, s_{\mathbf{q}}, s_{\mathbf{p}}} u^{s_{\mathbf{q}}} u^{s_{\mathbf{p}}} = \frac{1}{2} (s_{\mathbf{q}} q - s_{\mathbf{p}} p) \left\langle \mathbf{h}_{-\mathbf{k}}^{s_{\mathbf{k}}} \cdot \mathbf{h}_{\mathbf{q}}^{s_{\mathbf{q}}} \times \mathbf{h}_{\mathbf{p}}^{s_{\mathbf{p}}} \right\rangle$$

Number of Degrees of Freedom of N wavenumber modes

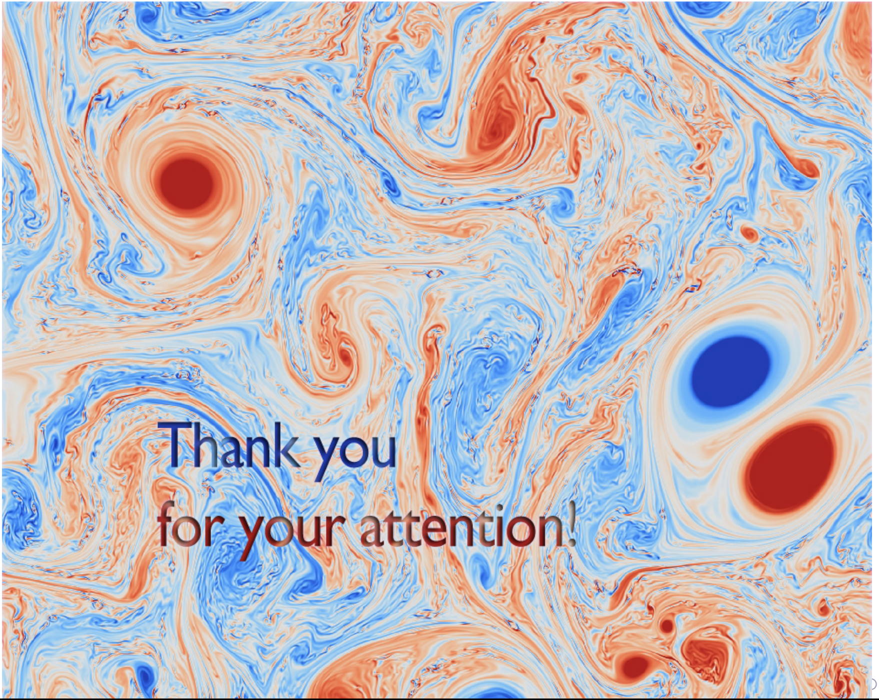
$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u^{+}_{\mathbf{k}}\,\mathbf{h}^{+}_{\mathbf{k}} + u^{-}_{\mathbf{k}}\,\mathbf{h}^{-}_{\mathbf{k}}}$$

$$\boxed{\partial_t u^{s_{\mathbf{k}}}_{\mathbf{k}} = \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}} C^{s_{\mathbf{k}}, s_{\mathbf{q}}, s_{\mathbf{p}}}_{\mathbf{k},\mathbf{q},\mathbf{p}} u^{s_{\mathbf{q}}}_{\mathbf{q}} u^{s_{\mathbf{p}}}_{\mathbf{p}} \right) - \nu |\mathbf{k}|^2 u^{s_{\mathbf{k}}}_{\mathbf{k}} + f^{s_{\mathbf{k}}}_{\mathbf{k}}}$$

If N wavenumbers are sufficient to resolve a given problem then the number of degrees of freedom N_F are

- (2) $u^{+}_{\mathbf{k}}, u^{-}_{\mathbf{k}}$ two modes
- (2) $u^{\pm}_{\mathbf{k}}$ is complex
- (1/2) $u^{\pm}_{-\mathbf{k}} = (u^{\pm}_{\mathbf{k}})^*$ realizability condition

$$N_F = 2 \times 2 \times \frac{1}{2} \times N = 2N$$


Thank you
for your attention!