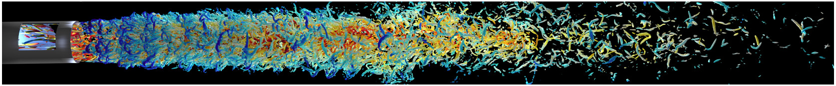
Turbulence
Navier Stokes and Symmetries



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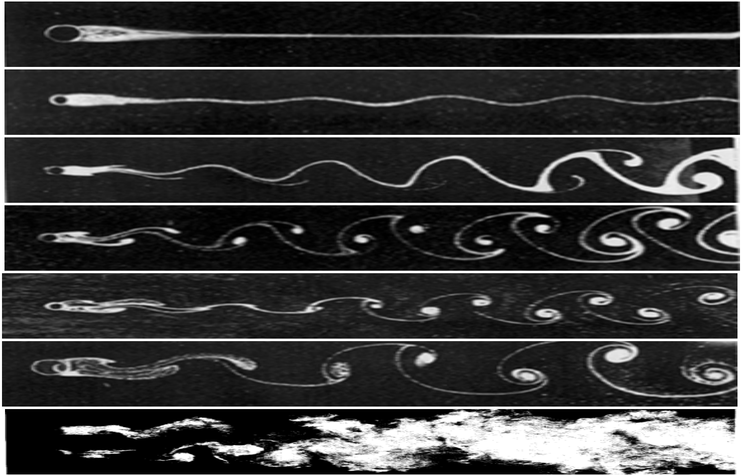
- Symmetry in fields and in equations
- Symmetries of the Navier Stokes
- Breaking of symmetries

Flow behind a cylinder



Re= 32

Re= 55

Re= 60

Re= 73

Re=102

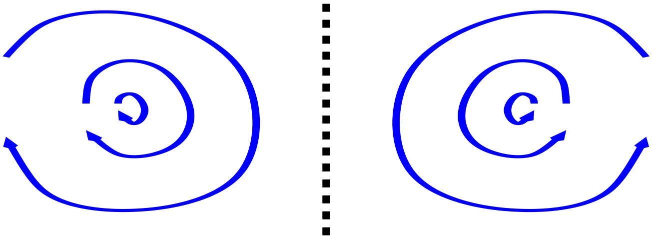
Re=161

Re=1800

Symmetries of Fields

We will say that a field $\mathbf{u}(\mathbf{x}, t)$ is invariant under a transformation $\mathcal{T}$ or that it has a $\mathcal{T}$-symmetry if under the act of the transformation it remains the same:

$$\mathcal{T}[\mathbf{u}] = \mathbf{u}$$

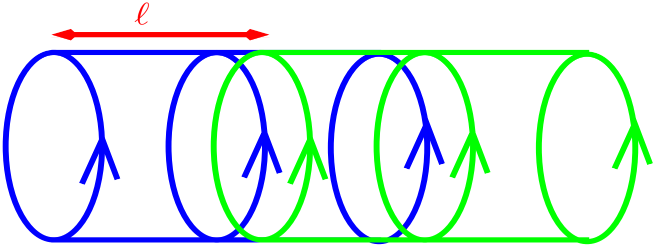


Space translations

$$\mathcal{T}[\mathbf{u}(\mathbf{x},t)] = \mathbf{u}(\mathbf{x}+\hat{\mathbf{e}}_i\ell, t)$$

Example:

- $\mathbf{u}(\mathbf{x},t) = (0, \sin(z), \cos(y))$



- continuous symmetry in x : $\quad \mathbf{u}(x,y,z) = \mathbf{u}(x+\ell_x, y, z, t)$
- discrete symmetry in $y, z,$: $\mathbf{u}(x,y,z) = \mathbf{u}(x, y+2n_y\pi, z+2n_z\pi, t)$

Time translations

$$\mathcal{T}[\mathbf{u}(\mathbf{x},t)] = \mathbf{u}(\mathbf{x},t+T)$$

Example:

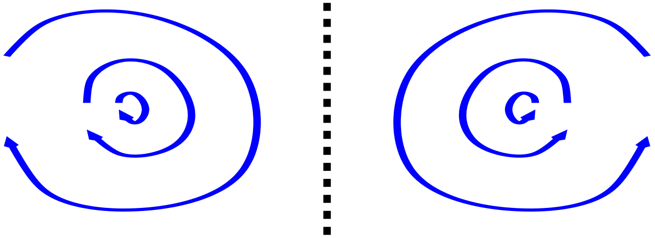
- Time periodic flows have discrete symmetries
- Constant in time flows have continuous symmetries

Reflections

$$\mathcal{T}[(u_x(x,y,z,t), u_y(x,y,z,t), u_z(x,y,z,t))] =$$
$$(-u_x(-x,y,z,t), u_y(-x,y,z,t), u_z(-x,y,z,t))$$

Example:

- $\mathbf{u}(\mathbf{x},t) = (\sin(x)\cos(y), -\cos(x)\sin(y), 0)$

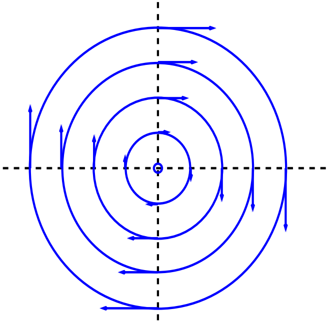


Rotations

$$\mathcal{T}[\mathbf{u}(\mathbf{x},t)] = \mathcal{R}\mathbf{u}(\mathcal{R}^{-1}\mathbf{x},t) + \mathbf{c}$$

Where $\mathcal{R}$ is the rotation matrix **Example:**

- $\mathbf{u}(\mathbf{x},t) = (y,-x,0)$



Galilean Transformations

$$\mathcal{T}[\mathbf{u}(\mathbf{x},t)] = \mathbf{u}(\mathbf{x}+\mathbf{c}t,t)+\mathbf{c}$$

Example:

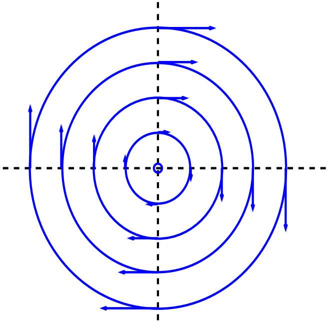
- $\mathbf{u}(\mathbf{x},t) = -x/t$

Scalling Transformations

$$\mathcal{T}[\mathbf{u}(\mathbf{x},t)] = \lambda^{\alpha}\mathbf{u}(\lambda\mathbf{x},t)$$

Example:

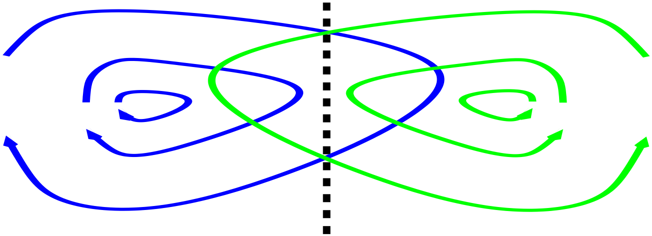
- $\mathbf{u}(\mathbf{x},t) = (y,-x,0)(x^2+y^2)^{\alpha/2-1}$



Symmetries of equations

We will say that an equation (e.g. the Navier-Stokes) is invariant under a transformation $\mathcal{T}$ or that it has a $\mathcal{T}$-symmetry if for any solution $\mathbf{u}(\mathbf{x}, t)$ of this equation $\mathcal{T}[\mathbf{u}(\mathbf{x}, t)]$ is also a solution.

Note that $\mathbf{u}(\mathbf{x}, t)$ does not have to be a symmetric field.



Symmetries of equations

Lemma: If the initial conditions $\mathbf{u}(\mathbf{x},0)$ has one of the spatial symmetries of the Navier-Stokes

$$\mathcal{T}[\mathbf{u}(\mathbf{x},0)] = \mathbf{u}(\mathbf{x},0),$$

then if $\mathbf{u}(\mathbf{x},t)$ remains smooth it retains this symmetry for all times.

$$\mathcal{T}[\mathbf{u}(\mathbf{x},t)] = \mathbf{u}(\mathbf{x},t)$$

Proof: If $\mathbf{u}(\mathbf{x},t)$ is a solution then (since $\mathcal{T}$ is one of the symmetries of the Navier-Stokes) $\mathcal{T}[\mathbf{u}(\mathbf{x},t)]$ is also a solution that has the same intitial conditions as $\mathbf{u}(\mathbf{x},t)$ ($\mathcal{T}[\mathbf{u}(\mathbf{x},0)] = \mathbf{u}(\mathbf{x},0)$). Thus either there is non-uniqueness of solutions or

$$\mathcal{T}[\mathbf{u}(\mathbf{x},t)] = \mathbf{u}(\mathbf{x},t).$$

(Non-uniqueness occurs only for non-smooth $\mathbf{u}(\mathbf{x},t)$).

Symmetries of the Navier-Stokes

- Space translations: $x' = x + \ell$
  - "if $\mathbf{u}(\mathbf{x}, t)$ is a solution $\mathbf{u}(\mathbf{x} + \ell, t)$ is also a solution."

Symmetries of the Navier-Stokes

- Space translations: $x' = x + \ell$
  - "if $\mathbf{u}(\mathbf{x}, t)$ is a solution $\mathbf{u}(\mathbf{x} + \ell, t)$ is also a solution."
  - $\partial_x \mathbf{u}(\mathbf{x} + \ell, t) = \frac{\partial x'}{\partial x} \partial_{x'} \mathbf{u}(\mathbf{x}', t) = \partial_{x'} \mathbf{u}(\mathbf{x}', t)$

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$$\partial_t \mathbf{u}(x+\ell, t) + \mathbf{u}(x+\ell, t) \cdot \nabla \mathbf{u}(x+\ell, t) = -\nabla P + \nu \nabla^2 \mathbf{u}(x+\ell, t) + \mathbf{f}(\mathbf{x}, t)$$
$$\partial_t \mathbf{u}(x', t) + \mathbf{u}(x', t) \cdot \nabla' \mathbf{u}(x', t) = -\nabla' P + \nu {\nabla'}^2 \mathbf{u}(x', t) + \mathbf{f}(\mathbf{x}' - \ell, t)$$

which is the original Navier Stokes (if $\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x} + \ell, t)$)

$ie,$ $\mathbf{u}(x + \ell, t)$ satisfies the same equations as $\mathbf{u}(x, t)$

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- Space translations: $x' = x + \ell$
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  - "if $\mathbf{u}(\mathbf{x}, t)$ is a solution $\mathbf{u}(\mathbf{x} - \mathbf{c}t, t) + \mathbf{c}$ is also a solution."

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    - "if $\mathbf{u}(\mathbf{x}, t)$ is a solution $\mathbf{u}(\mathbf{x} - \mathbf{c}t, t) + \mathbf{c}$ is also a solution."
    - $\partial_t(\mathbf{u}(\mathbf{x} - \mathbf{c}t, t) + \mathbf{c}) = \left(\frac{\partial t'}{\partial t}\right) \partial_{t'} \mathbf{u}(\mathbf{x}', t') + \left(\frac{\partial \mathbf{x}'}{\partial t}\right) \nabla_{\mathbf{x}'} \mathbf{u}(\mathbf{x}', t')$
    - $\partial_t \mathbf{u}(\mathbf{x}', t') = \partial_{t'} \mathbf{u}(\mathbf{x}', t') - \mathbf{c}\nabla' \mathbf{u}(\mathbf{x}', t')$

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    - $\partial_t \mathbf{u}(\mathbf{x}', t') = \partial_{t'} \mathbf{u}(\mathbf{x}', t') - \mathbf{c}\nabla' \mathbf{u}(\mathbf{x}', t')$
    - $(\mathbf{u} + \mathbf{c}) \cdot \nabla(\mathbf{u} + \mathbf{c}) = \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{c} \cdot \nabla \mathbf{u}$

$$\partial_t \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}' = -\nabla P + \nu \nabla^2 \mathbf{u}' + \mathbf{f}$$

$$\partial_t' \mathbf{u}' - \cancel{\mathbf{c} \cdot \nabla' \mathbf{u}} + \mathbf{u} \cdot \nabla' \mathbf{u} + \cancel{\mathbf{c} \cdot \nabla' \mathbf{u}} = -\nabla' P + \nu {\nabla'}^2 \mathbf{u} + \mathbf{f}$$

if $\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x} - \mathbf{c}t, t)$

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- Parity (reflections): $\mathbf{u}' = -\mathbf{u}$, $\mathbf{x}' = -\mathbf{x}$
  - "if $\mathbf{u}(\mathbf{x}, t)$ is a solution $-\mathbf{u}(-\mathbf{x}, t)$ is also a solution."

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- Scaling $\mathbf{x}' = \mathbf{x}/\lambda$, $t' = t/\lambda^\alpha$, $\mathbf{u}' = \lambda^\beta \mathbf{u}$
  - "if $\mathbf{u}(\mathbf{x}, t)$ is a solution $\lambda\mathbf{u}(\mathbf{x}/\lambda, t/\lambda^\alpha)$ is also a solution."

$$\partial_t \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}' = -\nabla P + \nu \nabla^2 \mathbf{u}' + \mathbf{f}$$

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$$\lambda^{\beta-\alpha}\partial_t \mathbf{u} + \lambda^{2\beta-1}\mathbf{u} \cdot \nabla' \mathbf{u} = -\lambda^{2\beta-1}\nabla P + \nu\lambda^{\beta-2}\nabla^2 \mathbf{u} + \mathbf{f}$$

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Is a solution if $\beta - \alpha = 2\beta - 1$ and $2\beta - 1 = \beta - 2$

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Is a solution if $\beta - \alpha = 2\beta - 1$ and $2\beta - 1 = \beta - 2$
$\beta = -1$ and $\alpha = 2$

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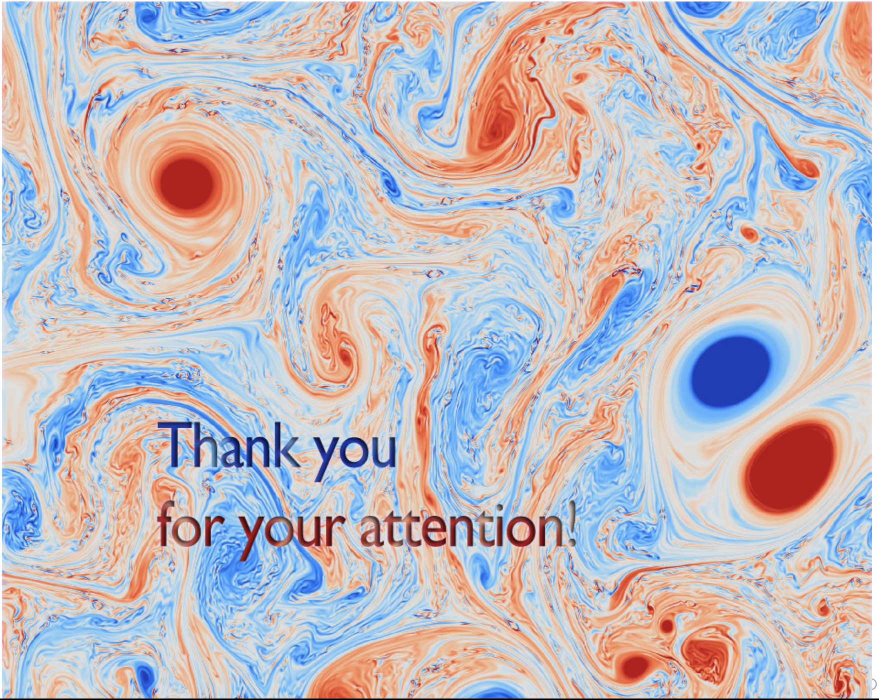
Is a solution if $\beta - \alpha = 2\beta - 1$ and $2\beta - 1 = \beta - 2$
$\beta = -1$ and $\alpha = 2$
In the absence of viscosity there is a scaling symmetry for any β and $\alpha = 1 - \beta$

Symmetries of the Navier-Stokes

- If $\mathbf{u}(\mathbf{x}, 0)$ has one of the spatial symmetries of the Navier Stokes equations then $\mathbf{u}(\mathbf{x}, t)$ will retain this symmetry at all times
- Most of the symmetries of the Navier Stokes equations break down when non-constant forcing is considered
- Most of the symmetries of the Navier Stokes equations break down when non-infinite domain sizes are considered


Thank you
for your attention!