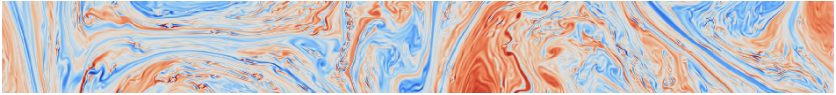
Turbulence
Equilibrium Dynamics



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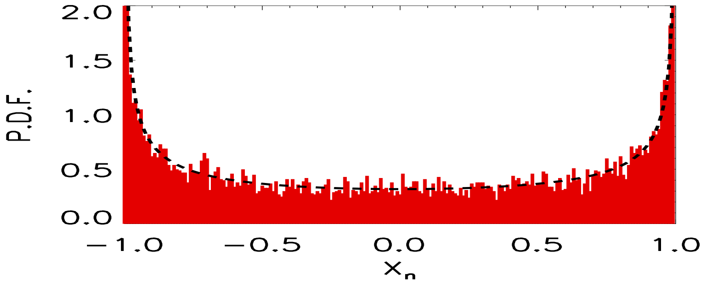
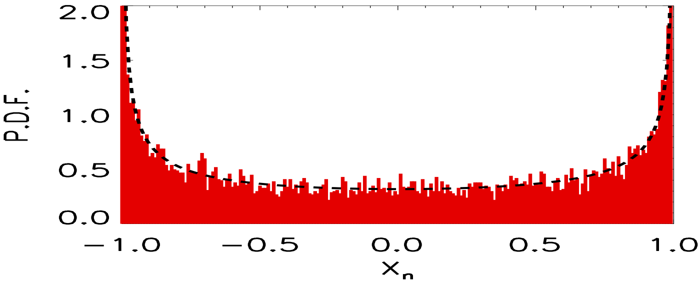
Statistical steady state



- Flow reaches a random state
- We look for a $\mathcal{P}[\mathbf{u}]$
- Respects symmetries of the Navier-Stokes (Euler?)
- Nonlinearities conserve Energy and Helicity
- There is energy/helicity injection and dissipation

Statistical steady state



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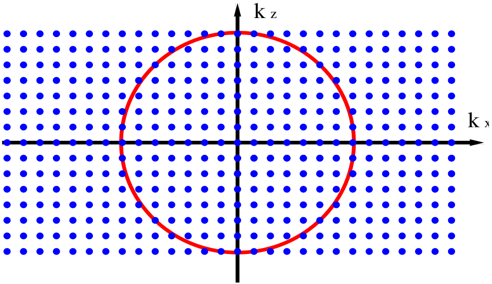
What if we forget dissipation?

Euler-equations

If $\nu = 0$ dissipation can still occur by the formation of singularities.

The Galerkin Truncated Euler-equations

$$\frac{d}{dt}u_{\mathbf{k}}^{s_{\mathbf{k}}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}}^{|\mathbf{k}|,|\mathbf{q}|,|\mathbf{p}|\leq k_{\max}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}},s_{\mathbf{q}},s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}}$$



Equilibrium Dynamics

(Lee 1952; Hopf 1952; Kraichnan 1967, 1973; Orszag 1977)

$$\frac{d}{dt} u_{\mathbf{k}}^{s_{\mathbf{k}}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}}^{|\mathbf{k}|,|\mathbf{q}|,|\mathbf{p}| \leq k_{\max}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}}, s_{\mathbf{q}}, s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}}$$

Liouville's condition

$$\frac{\partial}{\partial u_{\mathbf{k}}^{s_{\mathbf{k}}}} \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}}^{|\mathbf{k}|,|\mathbf{q}|,|\mathbf{p}| \leq k_{\max}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}}, s_{\mathbf{q}}, s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}} \right) = 0$$

Density of states constant along trajectories

+Ergodicity assumption $\Rightarrow$

$\mathcal{P}[\mathbf{u}]$ **is determined by the invariants of the system** $(\mathcal{E}, \mathcal{H})$

Equilibrium Dynamics

Micro-Canonical Ensemble

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]\delta\left[\mathcal{H} - \frac{1}{2}\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} s_{\mathbf{k}} k |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]$$

Canonical Ensemble

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\exp\left[\frac{1}{2}\beta\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2 + \frac{1}{2}\gamma\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} s_{\mathbf{k}} k |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]$$

Micro-Canonical Ensamble (Neglecting Helicity)

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_\mathbf{k},|\mathbf{k}|\leq k_{\max}} |u_\mathbf{k}^{s_\mathbf{k}}|^2\right]$$

$$\begin{aligned}
\left\langle |u_\mathbf{k}^{s_\mathbf{k}}|^2 \right\rangle &= \frac{1}{Z}\int |u_\mathbf{k}^{s_\mathbf{k}}|^2 \delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_\mathbf{q},|\mathbf{q}|\leq k_{\max}} |u_\mathbf{q}^{s_\mathbf{q}}|^2\right]\prod_\mathbf{q} du_\mathbf{q}^{s_\mathbf{q}} \\
&= \frac{1}{NZ}\int \left(\sum_{\mathbf{k}|,s_\mathbf{k}} |u_\mathbf{k}^{s_\mathbf{k}}|^2\right)\delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_\mathbf{k},|\mathbf{k}|\leq k_{\max}} |u_\mathbf{k}^{s_\mathbf{k}}|^2\right]\prod_{\mathbf{q},s_\mathbf{q}} du_\mathbf{q}^{s_\mathbf{q}} \\
&= \frac{2\mathcal{E}}{NZ}\int \delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_\mathbf{k},|\mathbf{k}|\leq k_{\max}} |u_\mathbf{k}^{s_\mathbf{k}}|^2\right]\prod_\mathbf{q} du_\mathbf{q}^{s_\mathbf{q}} = 2\mathcal{E}/N
\end{aligned}$$

$$E(k) = \frac{1}{2}\sum_{k\leq|\mathbf{q}|<k+1} \left\langle |u_\mathbf{k}^{s_\mathbf{k}}|^2 \right\rangle \propto k^2$$

Canonical Ensamble

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\exp\left[\frac{1}{2}\left[\sum_{s_{\mathbf{k}},|\mathbf{k}|\le k_{\max}}(\beta + s_{\mathbf{k}}k\gamma)|u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]\right]$$

$$\left\langle |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2 \right\rangle = \frac{1}{Z}\int |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2 \mathcal{P}[\mathbf{u}] \prod_{\mathbf{q}} du_{\mathbf{q}}^{s_{\mathbf{q}}}$$

$$= \frac{1}{(\beta + s_{\mathbf{k}}\gamma k)}, \qquad \left(\int_{-\infty}^{+\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}\right)$$

$$\left\langle u_{-\mathbf{k}}^{s_{\mathbf{k}}} \cdot w_{\mathbf{k}}^{s_{\mathbf{k}}} \right\rangle = \frac{s_{\mathbf{k}}k}{(\beta + s_{\mathbf{k}}\gamma k)}, \qquad \left(\int_{-\infty}^{+\infty} x^2 e^{-ax^2}dx = \sqrt{\frac{\pi}{4a^3}}\right)$$

$$\langle e_k \rangle = \frac{1}{2}\left\langle |u_{\mathbf{k}}^{+}|^2 + |u_{\mathbf{k}}^{-}|^2 \right\rangle = \frac{\beta}{(\beta^2 - \gamma k^2)},$$

$$\langle h_k \rangle = \frac{k}{2}\left\langle |u_{\mathbf{k}}^{+}|^2 - |u_{\mathbf{k}}^{-}|^2 \right\rangle = \frac{\gamma k^2}{(\beta^2 - \gamma^2 k^2)}$$

Canonical Ensamble

$$\langle e_k \rangle = \frac{1}{2}\langle |u^+_{\mathbf{k}}|^2 + |u^-_{\mathbf{k}}|^2 \rangle = \frac{\beta}{(\beta^2 - \gamma k^2)},$$

$$\langle h_k \rangle = \frac{k}{2}\langle |u^+_{\mathbf{k}}|^2 - |u^-_{\mathbf{k}}|^2 \rangle = \frac{\gamma k^2}{(\beta^2 - \gamma^2 k^2)}$$

$$E(k) = \frac{4\pi k^2}{(\beta^2 - \gamma^2 k^2)}, \quad H(k) = \frac{4\pi \gamma k^4}{(\beta^2 - \gamma^2 k^2)}$$

$$\mathcal{E} = \sum_{\mathbf{k}} \frac{\beta}{(\beta^2 - \gamma k^2)}, \qquad \mathcal{H} = \sum_{\mathbf{k}} \frac{\gamma k^2}{(\beta^2 - \gamma k^2)}$$

Canonical Ensamble

$$\langle e_k \rangle = \frac{1}{2}\langle |u_{\mathbf{k}}^{+}|^2 + |u_{\mathbf{k}}^{-}|^2 \rangle = \frac{\beta}{(\beta^2 - \gamma k^2)},$$

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$$k_{\max} < |\beta/\gamma|$$

Canonical Ensamble

$$E(k) = \frac{4\pi k^2}{(\beta^2 - \gamma k^2)}, \quad H(k) = \frac{4\pi\gamma k^4}{(\beta^2 - \gamma k^2)}$$

Thermal equilibrium of large scales

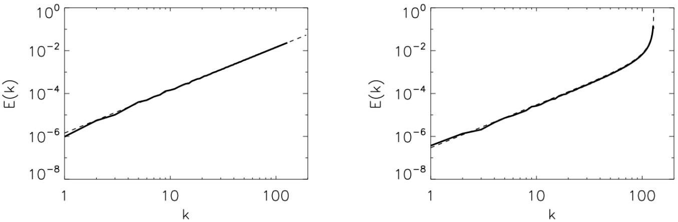
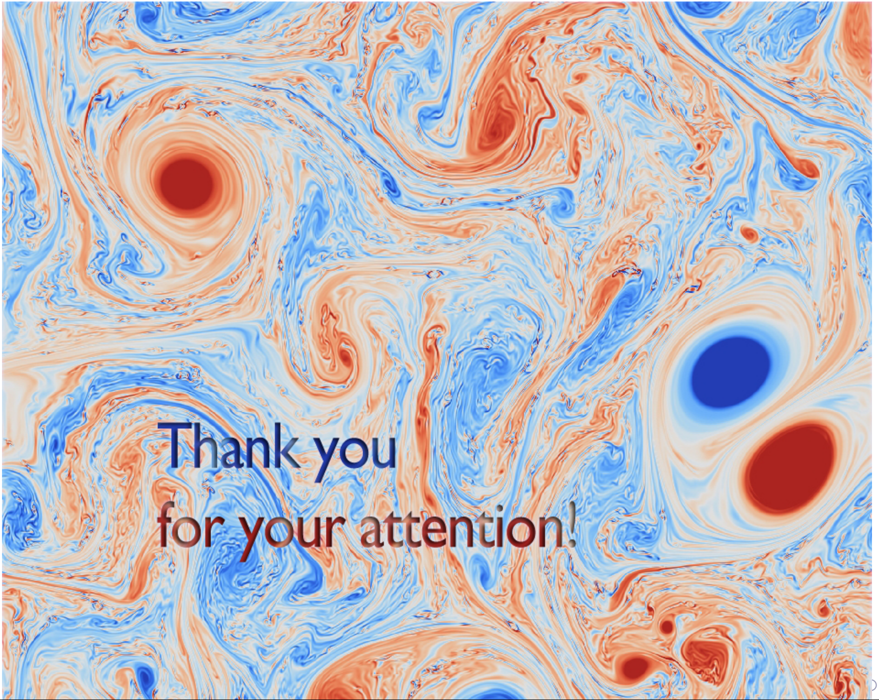


Figure 1: The energy spectra from two simulations of the truncated Euler equations with $k_{max} = 128$ and zero helicity (left) and $\mathcal{H}/\mathcal{E}k_{max} = 0.82$ (right). The dashed lines show the theoretical predictions given in eq. 1.2.


Thank you
for your attention!