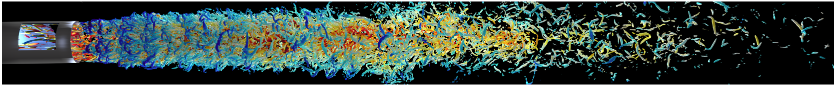
Two dimensional Turbulence



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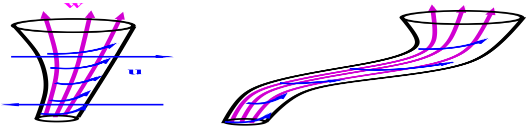
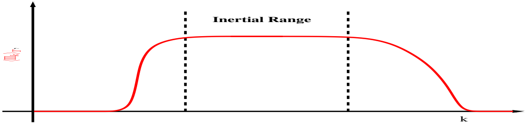
Three dimensional turbulence

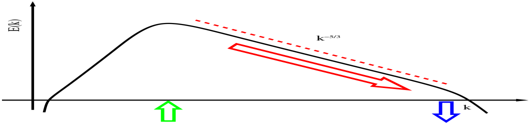
- Vorticity stretching



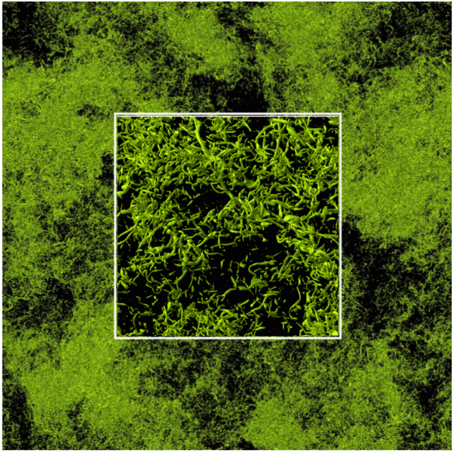
- Forward energy flux

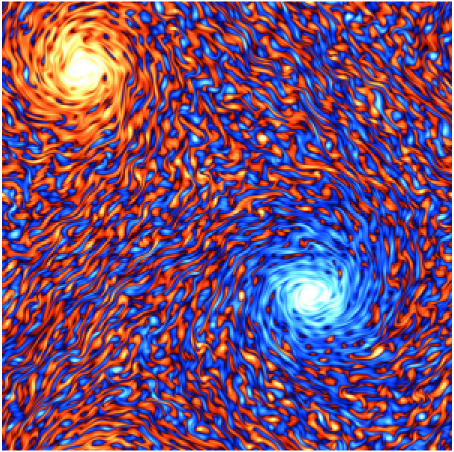


- Kolmogorov Energy Spectrum



3D Turbulence vs 2D Turbulence





2D Navier Stokes, $u_z = 0$, $\partial_z \mathbf{u} = 0$

$$\partial_t \mathbf{w} = \nabla\times(\mathbf{u}\times\mathbf{w}) + \nu\nabla^2\mathbf{w} + \nabla\times\mathbf{f}$$

$$u_z = 0, \qquad \partial_z \mathbf{u} = 0, \quad \Rightarrow w_x = w_y = 0$$

$$\partial_t w_z = \partial_x(\mathbf{u}\times\mathbf{w})_y - \partial_y(\mathbf{u}\times\mathbf{w})_x + \nu\nabla^2 w_z + (\nabla\times\mathbf{f})_z$$

$$\partial_t w_z = \partial_x(-u_x w_z) - \partial_y(u_y w_z) + \nu\nabla^2 w_z + f_w$$

$$\partial_t w_z = -u_x\partial_x w_z - w_z\partial_x u_x - u_y\partial_y w_z - w_z\partial_y u_y + \nu\nabla^2 w_z + f_w$$

$$\partial_t w_z + u_x\partial_x w_z + u_y\partial_y w_z = -w_z(\partial_x u_x + \partial_y u_y) + \nu\nabla^2 w_z + f_w$$

$$\partial_t w_z + \mathbf{u}\cdot\nabla w_z = -w_z(\cancel{\nabla\cdot\mathbf{u}}) + \nu\nabla^2 w_z + f_w$$

$$\boxed{\partial_t w_z + \mathbf{u}\cdot\nabla w_z = \nu\nabla^2 w_z + f_w}$$

2D Navier Stokes, Stream function formulation

$$\boxed{\partial_t w_z + \mathbf{u} \cdot \nabla w_z = \nu \nabla^2 w_z + f_w}$$

Set $u_x = \partial_y \psi$, $u_y = -\partial_x \psi$ so that $\nabla \cdot \mathbf{u} = \partial_x \partial_y \psi - \partial_y \partial_x \psi = 0$

$$\begin{aligned} w_z &= \partial_x u_y - \partial_y u_x \\ &= -\partial_x \partial_x \psi - \partial_y \partial_y \psi \\ &= -\nabla^2 \psi \end{aligned}$$

$$\partial_t \nabla^2 \psi - \partial_y \psi \partial_x \nabla^2 \psi + \partial_x \psi \partial_y \nabla^2 \psi = \nu \nabla^2 \nabla^2 \psi - f_w$$

$$\boxed{\partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) = \nu \nabla^2 \nabla^2 \psi - f_w}$$

Energy Conservation

$$\left\langle \psi \partial_t \nabla^2 \psi \right\rangle - \left\langle \psi \partial_y \psi \partial_x \nabla^2 \psi \right\rangle + \left\langle \psi \partial_x \psi \partial_y \nabla^2 \psi \right\rangle = \nu \left\langle \psi \nabla^2 \nabla^2 \psi \right\rangle - \langle \psi f_w \rangle$$

$$-\frac{1}{2}\frac{d}{dt}\langle \nabla \psi \cdot \nabla \psi \rangle - \frac{1}{2}\left\langle \partial_y \psi^2 \partial_x \nabla^2 \psi \right\rangle + \frac{1}{2}\left\langle \partial_x \psi^2 \partial_y \nabla^2 \psi \right\rangle$$
$$= \nu \left\langle \nabla^2 \psi \nabla^2 \psi \right\rangle - \langle \psi f_w \rangle$$

$$-\frac{1}{2}\frac{d}{dt}\left\langle |\nabla \psi|^2 \right\rangle + \frac{1}{2}\cancel{\left\langle \psi^2 \partial_y \partial_x \nabla^2 \psi \right\rangle} - \frac{1}{2}\cancel{\left\langle \psi^2 \partial_x \partial_y \nabla^2 \psi \right\rangle}$$
$$= \nu \left\langle |\nabla^2 \psi|^2 \right\rangle - \langle \psi f_w \rangle$$

$$\boxed{\frac{1}{2}\frac{d}{dt}\left\langle |\nabla \psi|^2 \right\rangle = -\nu \left\langle |\nabla^2 \psi|^2 \right\rangle + \langle \psi f_w \rangle}$$

Energy Conservation

$$\boxed{\frac{1}{2}\frac{d}{dt}\left\langle|\nabla\psi|^2\right\rangle = -\nu\left\langle|\nabla^2\psi|^2\right\rangle + \langle\psi f_w\rangle}$$

$$\boxed{\frac{d}{dt}\mathcal{E} = -\epsilon + \mathcal{I}_{\mathcal{E}}}$$

where

$$\mathcal{E} = \frac{1}{2}\left\langle|\mathbf{u}|^2\right\rangle = \frac{1}{2}\langle\psi w_z\rangle = \frac{1}{2}\left\langle|\nabla\psi|^2\right\rangle$$

$$\epsilon = \nu\left\langle|\nabla^2\psi|^2\right\rangle = \nu\left\langle w_z^2\right\rangle$$

$$\mathcal{I}_{\mathcal{E}} = \langle\psi f_w\rangle = \langle\mathbf{u}\cdot\mathbf{f}\rangle$$

Enstrophy Conservation

$$\partial_t w_z + \mathbf{u} \cdot \nabla w_z = \nu \nabla^2 w_z + f_w$$

$$\langle w_z \partial_t w_z \rangle + \langle w_z \mathbf{u} \cdot \nabla w_z \rangle = \nu \left\langle w_z \nabla^2 w_z \right\rangle + \langle w_z f_w \rangle$$

$$\boxed{\frac{1}{2} \frac{d}{dt} \left\langle w_z^2 \right\rangle = -\nu \left\langle |\nabla w_z|^2 \right\rangle + \langle w_z f_w \rangle}$$

$$\boxed{\frac{d}{dt} \Omega = -\eta + \mathcal{I}_\Omega}$$

where

$$\Omega = \frac{1}{2} \left\langle w_z^2 \right\rangle \qquad \text{Enstrophy}$$

$$\eta = \nu \left\langle |\nabla w_z|^2 \right\rangle \qquad \text{Enstrophy dissipation}$$

$$\mathcal{I}_\Omega = \langle w_z f_w \rangle \qquad \text{Enstrophy injection}$$

An infinity of invariants

In fact for any function $F(w_z)$ for which

$$F'(w) = \frac{dF}{dw} \qquad \text{we} \quad \text{have}$$

$$\big\langle F'(w_z)\partial_t w_z\big\rangle + \big\langle F'(w_z)\mathbf{u}\cdot\nabla w_z\big\rangle = \nu\big\langle F'(w_z)\nabla^2 w_z\big\rangle + \big\langle F'(w_z)f_w\big\rangle$$

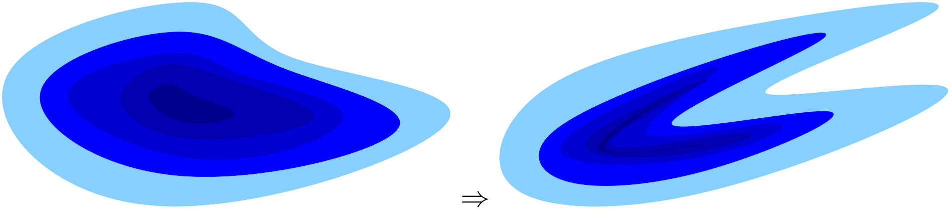
$$\frac{d}{dt}\langle F(w_z)\rangle + \cancel{\langle \mathbf{u}\cdot\nabla F(w_z)\rangle} = -\nu\big\langle \nabla F'(w_z)\cdot\nabla w_z\big\rangle + \big\langle F'(w_z)f_w\big\rangle$$

$$\frac{d}{dt}\langle F(w_z)\rangle = -\nu\big\langle F''(w_z)|\nabla w_z|^2\big\rangle + \big\langle F'(w_z)f_w\big\rangle$$

An infinity of invariants

For $\nu = 0$ $f_w = 0$

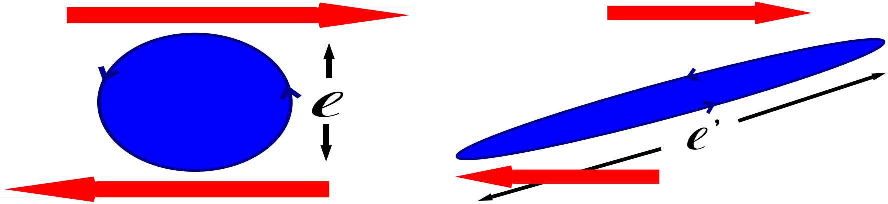
$$\frac{d}{dt}\langle F(w_z)\rangle = 0$$



- The flow will move every infinitesimal area element with out changing its area nor its vorticity.

Vortex shearing

Consider a vortex patch of size ℓ sheared by a larger scale eddy:



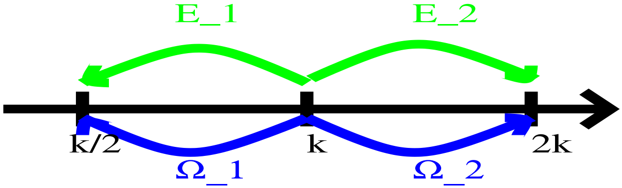
$$\ell \to \ell' \gg \ell$$

$$\int w \; d^2x = \oint \mathbf{u} d\ell \qquad \Rightarrow \qquad u_\ell \ell = u'_\ell \ell'$$

$$\boxed{u'_\ell \propto u_\ell \frac{\ell}{\ell'}}$$

Three modes

Consider the transfer of energy and enstrophy between three modes



$$\Omega_k = k^2 E_k, \qquad \Omega_{k/2} = k^2/4 E_{k/2}, \qquad \Omega_{2k} = 4k^2 E_{2k}$$

$$\Omega_k = \Omega_{k/2} + \Omega_{2k}, \qquad E_k = E_{k/2} + E_{2k}$$

$$k^2 E_k = \frac{1}{4} k^2 E_{k/2} + 4k^2 E_{2k}, \qquad E_k = E_{k/2} + E_{2k}$$

$$E_{k/2} = \frac{4}{5} E_k, \qquad E_{2k} = \frac{1}{5} E_k$$

$$\Omega_{k/2} = \frac{1}{5} \Omega_k, \qquad \Omega_{2k} = \frac{4}{5} \Omega_k$$

Implications of vorticity Conservation

No finite dissipation theorem Let

- $\mathcal{I}_{\mathcal{E}} = \langle \psi f_w \rangle$ The energy injection rate (at wavenumber k_f)
- $\mathcal{I}_{\Omega} = \langle w_z f_w \rangle$ The enstrophy injection rate
- $\epsilon = \nu \left\langle w_z^2 \right\rangle$ the energy dissipation rate
- $\eta = \nu \left\langle |\nabla w_z|^2 \right\rangle$ the estrophy dissipation rate (palinstrophy)

We have

$$\mathcal{I}_{\Omega} = \sum_{\mathbf{k}} \tilde{f}_{w\mathbf{k}}^{*} \tilde{w}_{\mathbf{k}} = \sum_{\mathbf{k}} \tilde{f}_{w\mathbf{k}}^{*} |\mathbf{k}|^2 \tilde{\psi}_{\mathbf{k}} = k_f^2 \sum_{\mathbf{k}} \tilde{f}_{w\mathbf{k}}^{*} \tilde{\psi}_{\mathbf{k}} = k_f^2 \mathcal{I}_{\mathcal{E}}$$

and from energy and enstrophy balance

$$\eta = \mathcal{I}_{\Omega} = k_f^2 \mathcal{I}_{\mathcal{E}} = k_f^2 \epsilon$$

Implications of vorticity Conservation

No finite dissipation theorem

$$\begin{aligned}
\eta &= \mathcal{I}_\Omega \\
&= k_f^2 \mathcal{I}_\mathcal{E} \\
&= k_f^2 \epsilon \\
&= k_f^2 \nu \langle w_z^2 \rangle \\
&= k_f^2 \nu \langle \nabla \times \mathbf{u} \cdot \mathbf{w} \rangle \\
&= k_f^2 \nu \langle \mathbf{u} \cdot \nabla \times \mathbf{w} \rangle \\
&\leq k_f^2 \nu \langle |\mathbf{u}|^2 \rangle^{1/2} \langle |\nabla \times \mathbf{w}|^2 \rangle^{1/2} \\
&= k_f^2 \nu^{1/2} (2\mathcal{E})^{1/2} \left(\nu \langle |\nabla \mathbf{w}|^2 \rangle \right)^{1/2} \\
&= k_f^2 \nu^{1/2} (2\mathcal{E})^{1/2} \eta^{1/2}
\end{aligned}$$

$$\eta \leq k_f^2 \nu^{1/2} (2\mathcal{E})^{1/2} \eta^{1/2}$$

Implications of vorticity Conservation

No finite dissipation theorem

$$\eta \leq k_f^2 \nu^{1/2} (2\mathcal{E})^{1/2} \eta^{1/2}$$

$$k_f^2 \epsilon = \eta \leq 2 k_f^4 \nu \mathcal{E}$$

- $\displaystyle \lim_{\nu \to 0} \frac{\epsilon}{\mathcal{E}^{3/2} k_f} \leq \lim_{\nu \to 0} \frac{2\nu k_f^4 \mathcal{E}}{\mathcal{E}^{3/2} k_f} = 0,$ No finite energy dissipation
- $\displaystyle \lim_{\nu \to 0} \frac{\eta}{\mathcal{E}^{3/2} k_f^3} = \lim_{\nu \to 0} \frac{2\nu k_f^2 \mathcal{E}}{\mathcal{E}^{3/2} k_f^3} = 0,$ No finite enstrophy dissipation

A dual cascade

$$\partial_t w + \mathbf{u} \cdot \nabla w = \nu \nabla^2 w - \alpha w + f_w$$

$$\frac{1}{2}\frac{d}{dt}\left\langle |\nabla \psi|^2 \right\rangle = -\nu \left\langle |\nabla^2 \psi|^2 \right\rangle - \alpha \left\langle |\nabla \psi|^2 \right\rangle + \langle \psi f_w \rangle$$

$$\boxed{\frac{d}{dt}\mathcal{E} = -\epsilon_\nu - \epsilon_\alpha + \mathcal{I}_\mathcal{E}}$$

and

$$\frac{1}{2}\frac{d}{dt}\left\langle w^2 \right\rangle = -\nu \left\langle |\nabla w|^2 \right\rangle - \alpha \left\langle w^2 \right\rangle + \langle w f_w \rangle$$

$$\boxed{\frac{d}{dt}\Omega = -\eta_\nu - \eta_\alpha + \mathcal{I}_\Omega}$$

(Note $\eta_\nu \neq k_f^2 \epsilon_\nu$ that was used before)

A dual cascade

$$\partial_t w + \mathbf{u} \cdot \nabla w = \nu \nabla^2 w - \alpha w + f_w$$

$$\frac{1}{2}\frac{d}{dt}\left\langle |\nabla \psi_k^<|^2 \right\rangle - \left\langle \psi_k^< \mathbf{u} \cdot \nabla w \right\rangle = -\nu \left\langle |\nabla^2 \psi_k^<|^2 \right\rangle + \alpha \left\langle |\nabla \psi_k^<|^2 \right\rangle + \left\langle \psi^< f_w \right\rangle$$

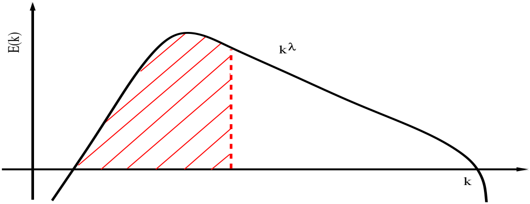
$$\boxed{\frac{d}{dt}\mathcal{E}^<(k) + \Pi_{\mathcal{E}}(k) = -\epsilon_\nu^<(k) - \epsilon_\alpha^<(k) + \mathcal{I}_{\mathcal{E}}^<(k)}$$

and

$$\frac{1}{2}\frac{d}{dt}\left\langle |w_k^<|^2 \right\rangle + \left\langle w^< \mathbf{u} \cdot \nabla w \right\rangle = -\nu \left\langle |\nabla w^<|^2 \right\rangle - \alpha \left\langle |w^<|^2 \right\rangle + \left\langle w^< f_w \right\rangle$$

$$\boxed{\frac{d}{dt}\Omega^<(k) + \Pi_{\Omega}(k) = -\eta_\nu^<(k) - \eta_\alpha^<(k) + \mathcal{I}_{\Omega}^<(k)}$$

A dual cascade



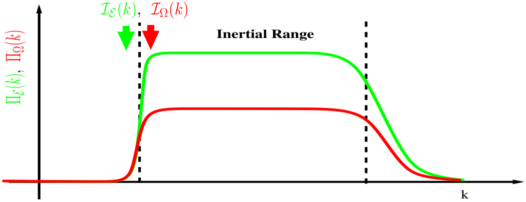
$$\epsilon_\alpha^<(k) = \alpha \int_0^k E(k')dk', \qquad \epsilon_\nu^<(k) = \nu \int_0^k E(k')k^2 dk',$$

$$\eta_\alpha^<(k) = \alpha \int_0^k E(k')k^2 dk', \qquad \eta_\nu^<(k) = \nu \int_0^k E(k')k^4 dk'$$

if $\lambda < \ \ 0$ then $\epsilon_\alpha^<(k)$ is dominated by small k
if $\lambda > -3$ then $\epsilon_\nu^<(k)$ and $\eta_\alpha^<(k)$ are dominated by large k
if $\lambda < -1$ then $\eta_\nu^<(k)$ is dominated by large k

A dual cascade

Suppose $\mathcal{E}, \Omega$ both cascade forward, with ϵ_ν and η_ν finite in the limit $\nu \to 0$.



For k in the inertial range we have

$$\begin{aligned} \Pi_\mathcal{E}(k) &= \epsilon_\nu \\ \Pi_\Omega(k) &= \eta_\nu \end{aligned}$$

A dual cascade

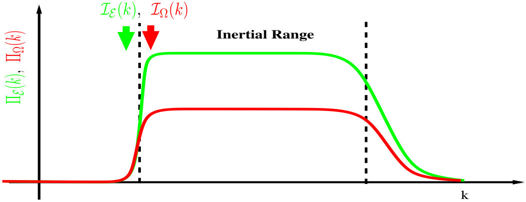
For k in the inertial range we have

$$
\begin{aligned}
\Pi_{\mathcal{E}}(k) &= \epsilon_\nu \\
&= \nu \sum_{q=0}^{\infty} q^2 E(q) \\
&\simeq \nu \sum_{q=k}^{\infty} q^2 E(q) \\
&\leq \nu k^{-2} \sum_{q=k}^{\infty} q^4 E(q) \\
&= k^{-2} \eta_\nu
\end{aligned}
$$

$$\Pi_{\mathcal{E}}(k) \leq k^{-2} \eta_\nu$$

A dual cascade

Suppose $\mathcal{E}, \Omega$ both cascade forward, with ϵ_ν and η_ν finite in the limit $\nu \to 0$.



For k in the inertial range we have

$$\Pi_\mathcal{E}(k) \leq k^{-2}\eta_\nu$$

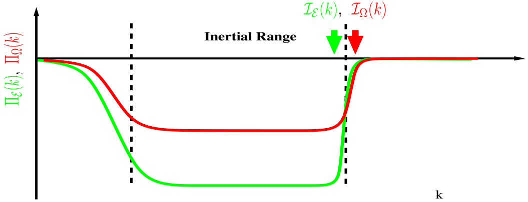
When we take the limit

$$\lim_{k\to\infty}\lim_{\nu\to 0}\Pi_\mathcal{E}(k) = 0$$

We can not have a forward Energy and Enstrophy cascade!

A dual cascade

Suppose $\mathcal{E}, \Omega$ both cascade Inversly, with ϵ_α and η_α finite in the limit $\alpha \to 0$.



For k in the inertial range we have

$$\begin{aligned} \Pi_\mathcal{E}(k) &= -\epsilon_\nu \\ \Pi_\Omega(k) &= -\eta_\nu \end{aligned}$$

A dual cascade

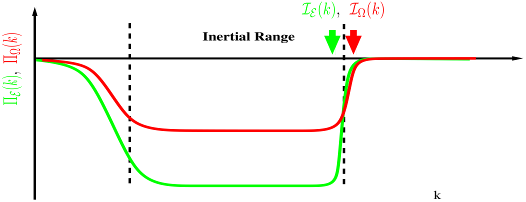
For k in the inertial range we have

$$\begin{aligned} |\Pi_\Omega(k)| &= \eta_\alpha \\ &= \alpha \sum_{q=0}^{\infty} q^2 E(q) \\ &\simeq \alpha \sum_{q=0}^{k} q^2 E(q) \\ &\leq \alpha k^2 \sum_{q=k}^{\infty} E(q) \\ &= k^2 \epsilon_\alpha \end{aligned}$$

$$|\Pi_\Omega(k)| \leq k^2 \epsilon_\alpha$$

A dual cascade

Suppose $\mathcal{E}, \Omega$ both cascade Inversly, with ϵ_α and η_α finite in the limit $\alpha \to 0$.



For k in the inertial range we have

$$|\Pi_\Omega(k)| \leq \nu k^2 \epsilon_\alpha$$

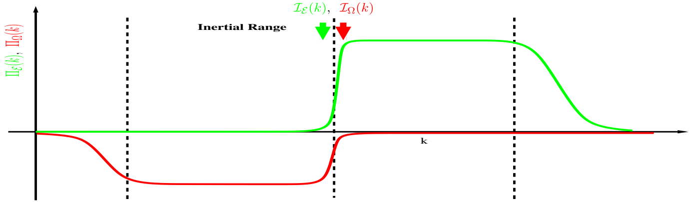
When we take the limit

$$\lim_{k\to 0} \lim_{\alpha \to 0} \Pi_\Omega(k) = 0$$

We can not have an inverse Energy and Enstrophy cascade!

A dual cascade

Suppose $\mathcal{E}$ cascades forward, with ϵ_ν finite in the limit $\nu \to 0$ and Ω cascades inversly, with η_α finite in the limit $\alpha \to 0$ and



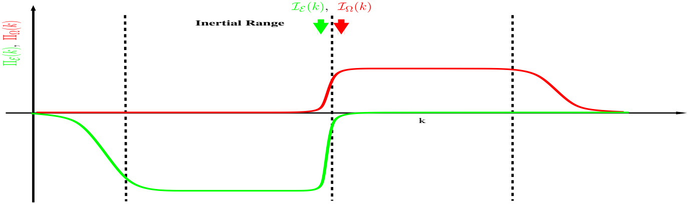
For k in the inverse inertial range $\eta_\alpha \leq k^{-2}\epsilon_\alpha$

For k in the forward inertial range $\epsilon_\nu \leq k^2\eta_\nu$

We can not have inverse Enstrophy and forward Energy cascade!

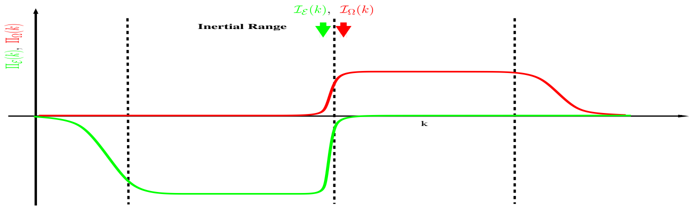
A dual cascade

Forward Enstrophy and Inverse Energy cascade is the only option!



$$
\begin{aligned}
\Pi_{\Omega}(k) &= \lim_{\nu \to 0} \eta_{\nu}, \qquad \text{Finite} \\
\Pi_{\mathcal{E}}(k) &= \lim_{\nu \to 0} \epsilon_{\nu} = 0 \\
\Pi_{\Omega}(k) &= \lim_{\alpha \to 0} \eta_{\alpha} = 0 \\
\Pi_{\mathcal{E}}(k) &= \lim_{\alpha \to 0} \epsilon_{\alpha}, \qquad \text{Finite}
\end{aligned}
$$

Forward Enstrophy cascade

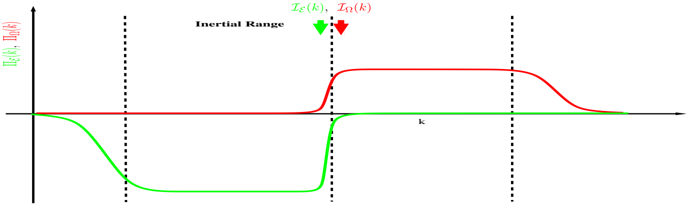


$$\Pi_\Omega(k) \quad = \quad \lim_{\nu\to 0} \eta_\nu, \qquad \text{Finite}$$

$$\Pi_\Omega(k) = \left\langle w_k^< \mathbf{u}\cdot\nabla w \right\rangle \propto \frac{u_\ell w_\ell^2}{\ell} \propto \frac{u_\ell^3}{\ell^3} = \eta_\nu > 0$$

$$u_\ell \propto \eta_\nu^{1/3}\ell, \qquad E(k) \propto \frac{u_\ell^2}{k} \propto \eta_\nu^{2/3} k^{-3}$$

Forward Enstrophy cascade



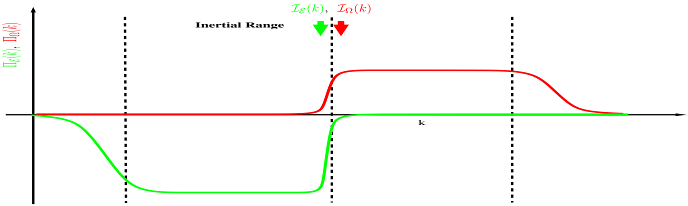
$$\Pi_\Omega(k) = \lim_{\nu \to 0} \eta_\nu, \qquad \text{Finite}$$

More precisely

- $\left\langle \delta u_\parallel^3 \right\rangle = \frac{1}{8}\eta_\nu r^3$
- $E(k) \propto \eta_\nu^{2/3} k^{-3} \log^{-1/3}(k/k^*)$

See: R.H. Kraichnan, J. Fluid Mech. 47 (3) (1971) 525-535. D. Bernard, Phys. Rev. E 60 (5) (1999) 6184. E. Lindborg, J. Fluid Mech. 388 (1999) 259-288. V. Yakhot, Phys. Rev. E 60 (5) (1999) 5544.

Forward Enstrophy cascade



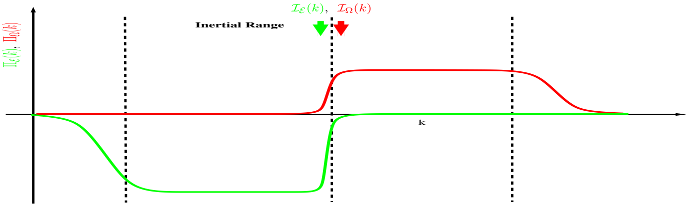
$$\Pi_\Omega(k) = \lim_{\nu\to 0} \eta_\nu, \qquad \text{Finite}$$

Vorticity dissipation scale ℓ_ν

$$\nu \frac{u_\ell^2}{\ell_\nu^4} \propto \eta_\nu \qquad \Rightarrow \qquad \nu \frac{\eta_\nu^{2/3} \ell_\nu^2}{\ell_\nu^4} \propto \eta_\nu$$

$$\boxed{\ell_\nu \propto \nu^{1/2} \eta_\nu^{-1/6}} \qquad \boxed{k_\nu \propto \nu^{-1/2} \eta_\nu^{1/6}}$$

Inverse Energy Cascade



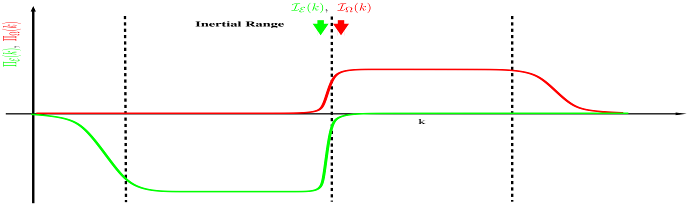
$$\Pi_{\mathcal{E}}(k) \quad = \quad \lim_{\alpha \to 0} \epsilon_\alpha, \qquad \text{Finite}$$

$$\Pi_{\mathcal{E}}(k) = \left\langle \psi_k^{<} \mathbf{u} \cdot \nabla w \right\rangle \propto \frac{u_\ell^3}{\ell} = \eta_\nu > 0$$

$$u_\ell \propto \epsilon_\alpha^{1/3} \ell^{1/3}, \qquad \boxed{E(k) \propto \frac{u_\ell^2}{k} \propto \epsilon_\alpha^{2/3} k^{-5/3}}$$

$$\boxed{\left\langle \delta u_\parallel^3 \right\rangle = \frac{2}{3} \epsilon_\alpha r}$$

Inverse Energy cascade



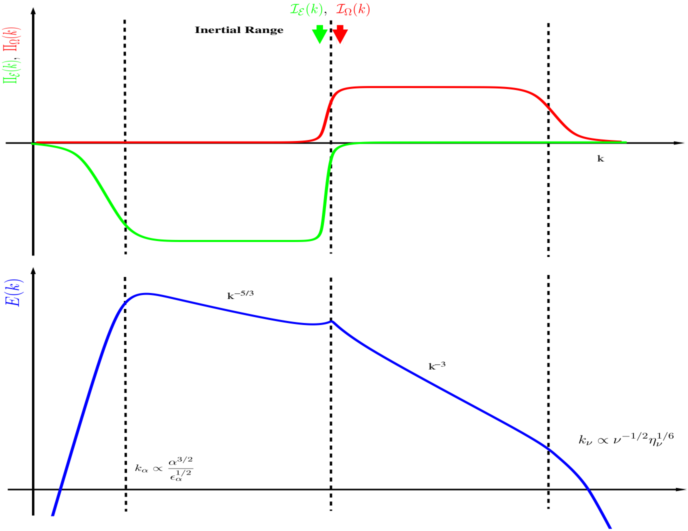
$$\Pi_{\mathcal{E}}(k) = \lim_{\alpha \to 0} \epsilon_\alpha, \qquad \text{Finite}$$

Drag energy dissipation scale ℓ_α

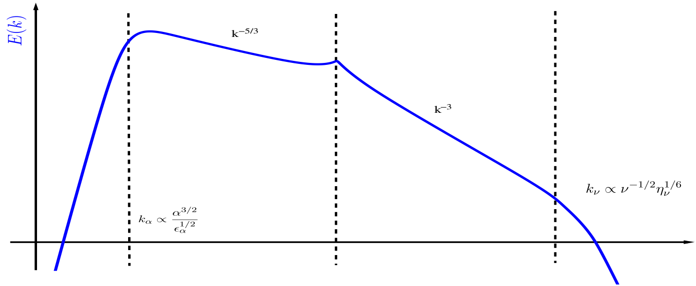
$$\alpha u_\ell^2 \propto \epsilon_\alpha \qquad \Rightarrow \qquad \alpha \left(\epsilon_\alpha^{1/3} \ell_\alpha^{1/3}\right)^2 \propto \epsilon_\alpha$$

$$\boxed{\ell_\alpha \propto \frac{\epsilon_\alpha^{1/2}}{\alpha^{3/2}}} \qquad \boxed{k_\alpha \propto \frac{\alpha^{3/2}}{\epsilon_\alpha^{1/2}}}$$

Dual energy cascade in 2D

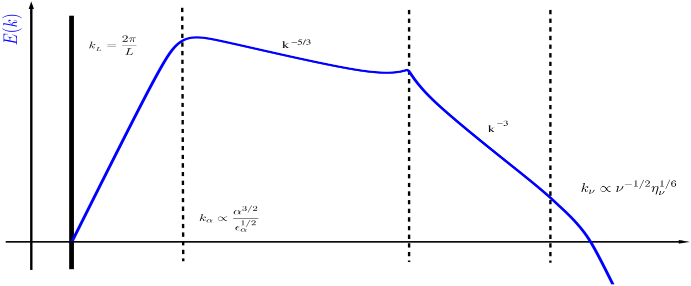


Dual energy cascade in 2D

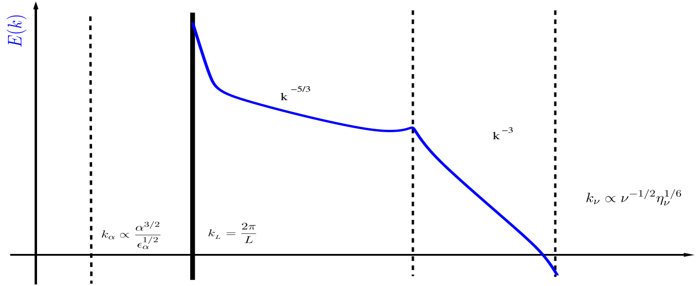


- $\eta_\nu = \mathcal{O}(1)$
- $\epsilon_\alpha = \mathcal{O}(1)$
- $\epsilon_\nu \propto \nu u_\ell^2/\ell_\nu^2 \propto \eta_\nu \ell_\nu^2 = \nu\eta_\nu^{2/3} \ll \mathcal{O}(1)$
- $\eta_\alpha \propto \alpha u_\ell^2/\ell_\alpha^2 \propto \epsilon_\alpha/\ell_\alpha^2 = \alpha^3 \ll \mathcal{O}(1)$

What happens when $\ell_\alpha > L$?



What happens when $\ell_\alpha > L$?

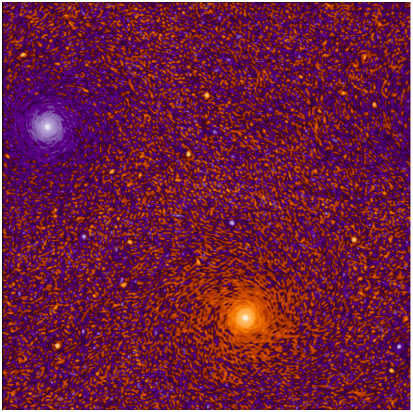


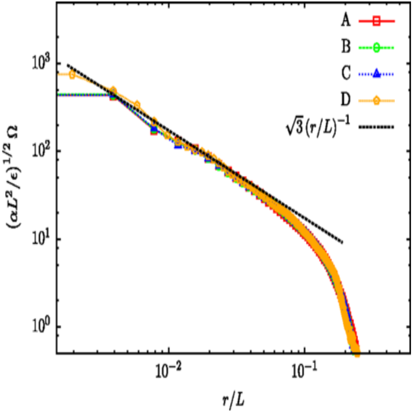
A spectral condensate! if $\alpha = 0$

$$\mathcal{I}_\mathcal{E} = \nu \frac{U^2}{L^2} \qquad \Rightarrow \qquad U^2 \propto \frac{\mathcal{I}_\mathcal{E} L^2}{\nu}$$

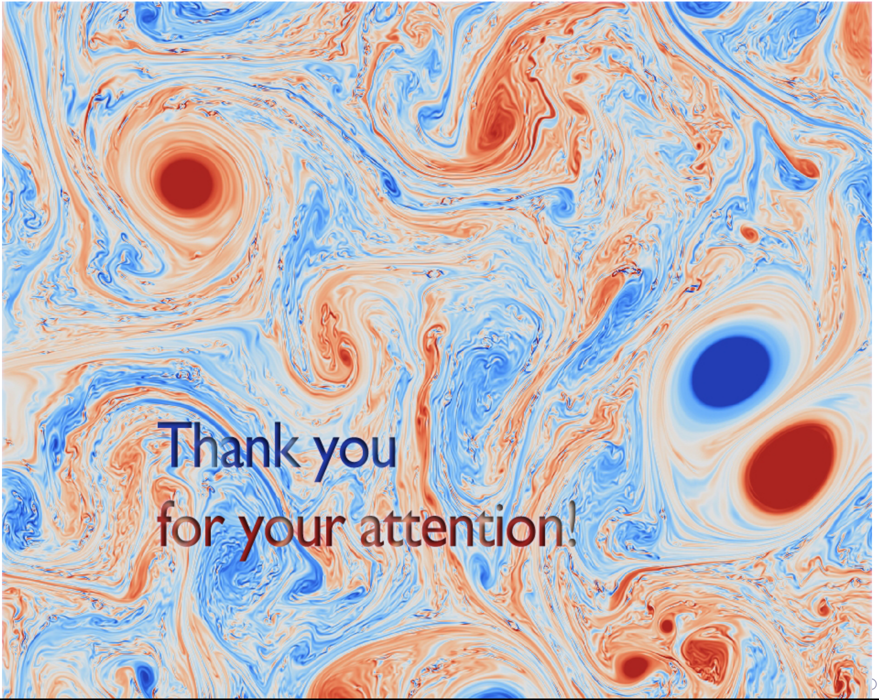
$$\eta_\nu^< = \nu \frac{U^2}{L^4} = \frac{\mathcal{I}_\mathcal{E}}{L^2} = \mathcal{I}_\Omega \frac{\ell_f^2}{L^2} = \mathcal{O}(1)$$

Condensate





$$\overline{w} \propto 1/r$$


Thank you
for your attention!