

Exercises 01

1. Consider the flow in a rectangular box of size $(L \times L \times H)$ with periodic boundary conditions along the horizontal directions (x, y) and free slip boundary conditions along the vertical direction (z) :

$$\mathbf{u}(x + n_x L, y + n_y L, z) = \mathbf{u}(x, y, z), \quad \text{and} \quad \partial_z u_x = \partial_z u_y = u_z = 0 \quad \text{at} \quad z = 0 \text{ \& } z = H$$

Are energy and helicity conserved for the incompressible Euler equations in this domain?

2. Consider the incompressible Navier-Stokes equations in infinite space in a rotating reference frame given by:

$$\partial_t \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}' + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u}' + \mathbf{f}$$

where $\boldsymbol{\Omega}$ is a constant vector indicating the direction and amplitude of the rotation.

Which of the aforementioned symmetries remain

- Space translations: $x' = x + \ell$
- Time translations: $t' = t + T$
- Galilean transformations: $\mathbf{x}' = \mathbf{x} - \mathbf{c}t$, $t' = t$, $\mathbf{u}' = \mathbf{u} + \mathbf{c}$
- Rotations $\mathbf{u}' = \mathcal{R}\mathbf{u}$, $\mathbf{x}' = \mathcal{R}^{-1}\mathbf{x}$
- Parity (reflections): $\mathbf{u}' = -\mathbf{u}$, $\mathbf{x}' = -\mathbf{x}$
- Scaling $\mathbf{x}' = \lambda\mathbf{x}$, $t' \rightarrow \lambda^\alpha t$, $\mathbf{u}' = \lambda^\beta \mathbf{u}$

3. Consider the equation

$$\partial_t \mathbf{a} + \mathbf{b} \times \mathbf{a} = -\nabla P' + \nu \nabla^2 \mathbf{a} \tag{1}$$

where $\nabla \cdot \mathbf{a} = 0$.

\mathbf{b} is related to \mathbf{a} as $\mathbf{b} = (\nabla \times)^n \mathbf{a}$ for some $n \in \mathbb{N}$ so that

$$(\nabla \times)^1 \mathbf{a} = \nabla \times \mathbf{a},$$

$$(\nabla \times)^2 \mathbf{a} = \nabla \times \nabla \times \mathbf{a},$$

$$(\nabla \times)^3 \mathbf{a} = \nabla \times \nabla \times \nabla \times \mathbf{a} \text{ and so on}$$

For $n = 1$ the system reduces to the Navier Stokes with $\mathbf{a} = \mathbf{u}$ and $\mathbf{b} = \mathbf{w}$.

For $n \neq 1$:

Are Energy $\langle \frac{1}{2} |\mathbf{a}|^2 \rangle$ & Helicity $\langle \mathbf{a} \cdot \mathbf{b} \rangle$ conserved for $\nu = 0$ for these systems (for smooth \mathbf{a}, \mathbf{b})?

What are the scaling symmetries they have for $\nu = 0$ and which one of these survives for $\nu \neq 0$?