Exercises 02

1. Consider again the equation

$$\partial_t \mathbf{b} + \mathbf{v} \times \mathbf{b} = -\nabla P' + (-1)^m \nu_m \nabla^{2m+2} \mathbf{b} + \mathbf{F} \tag{1}$$

$$\nabla \cdot \mathbf{b} = 0 \tag{2}$$

in a periodic box of size L where m is an integer and $\mathbf{F}$ is a forcing that injects energy at scale L at a rate $\mathcal{I}$.

$\mathbf{v}$ is related to $\mathbf{b}$ as $\mathbf{v} = (\nabla\times)^n \mathbf{b}$ for some $n \in \mathbb{N}$ so that
$(\nabla\times)^1 \mathbf{b} = \nabla \times \mathbf{b}$,
$(\nabla\times)^2 \mathbf{b} = \nabla \times \nabla \times \mathbf{b}$,
$(\nabla\times)^3 \mathbf{b} = \nabla \times \nabla \times \nabla \times \mathbf{b}$ and so on
n, m are integers. For $n = 1$, $m = 0$ the system reduces to the Navier Stokes with $\mathbf{b} = \mathbf{u}$.
For any n Energy $\mathcal{E} = \langle \frac{1}{2} |\mathbf{b}|^2 \rangle$ is conserved for $\nu_m = 0$ and $\alpha = 0$ (see last homework).
Assuming :

- energy cascades to smaller scales
- similar size eddies dominate the cascade

show the following:

(a) Write an expression for the energy dissipation and the energy balance relation
(b) Predict the energy spectrum of $\mathbf{b}$ based on the assumptions above.
(c) Predict the lengthscale ℓ_ν that dissipation becomes effective and dissipates the energy.
(d) For which value of m viscosity will not be sufficient to dissipate the injected energy as $\nu_m \to 0$?.

2. Consider incompressible the NS for the velocity $\mathbf{u}$ along with the advection diffusion equation for the scalar field ϕ in a cubic periodic box of size L:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{F} \tag{3}$$

$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi = +\kappa \nabla^2 \phi + S \tag{4}$$

where $\mathbf{F}$ and S is a forcing and a source injecting energy $\frac{1}{2}\langle |\mathbf{u}|^{\mathbf{2}} \rangle$ and scalar variance $\frac{1}{2}\langle \phi^2 \rangle$ at scale L (with $\langle F \rangle = 0$ and $\langle S \rangle = 0$).

Using *à la Kolmogorov* arguments predict the spectrum on ϕ. Distinguish between the $\kappa \ll \nu$ and the $\nu \ll \kappa$ cases.