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Energy and enstrophy dissipation in steady state 2d turbulence

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Abstract

Upper bounds on the bulk energy dissipation rate ϵ and enstrophy dissipation rate χ are derived for the statistical steady state of body forced two-dimensional (2d) turbulence in a periodic domain. For a broad class of externally imposed body forces it is shown that $\epsilon \leqslant k_f U^3 Re^{-1/2}(C_1 + C_2 Re^{-1})^{1/2}$ and $\chi \leqslant k_f^3 U^3 (C_1 + C_2 Re^{-1})$ where U is the root-mean-square velocity, k_f is a wavenumber (inverse length scale) related with the forcing function, and $Re = U/\nu k_f$. The positive coefficients C_1 and C_2 are uniform in the kinematic viscosity ν, the amplitude of the driving force, and the system size. We compare these results with previously obtained bounds for body forces involving only a single length scale, or for velocity dependent constant-energy-flux forces acting at finite wavenumbers. Implications of our results are discussed.

1. Introduction

The study of two-dimensional (2d) turbulence was originally justified as a simplified version of 3d turbulence, but it has come to be regarded as an interesting research field in its own right with deep connections to geophysical and astrophysical problems such as strongly rotating stratified flows [1]. A large number of experimental methods have been devised to constrain flows in two dimensions (e.g., soap films) allowing some theories to be tested in the lab [2]. Direct numerical simulations are far easier than the 3d case, and this has enabled researchers to investigate 2d turbulence computationally at much higher Reynolds numbers [3–8]. As a result, there are more simulation data to test theories of 2d turbulence. Nevertheless many fundamental questions remain open; see [1] for a recent review.

The inviscid conservation of enstrophy as well as energy in two dimensions results in two cascading quadratic invariants that make the phenomenology of 2d turbulence somewhat more complex than 3d turbulence and not derivable from simple dimensional arguments. Theoretical studies of turbulence usually employ a statistical description and often involve assumptions about homogeneity isotropy and the nature of the interactions. Based on physical arguments, Kraichnan [9], Leith [10] and Batchelor [11] conjectured that there is a dual cascade in 2d turbulence: energy flows to the larger scales while enstrophy moves to the small scales (when the system is driven at some intermediate scale). Kraichnan–Leith–Batchelor (KLB) theory assumes isotropy and homogeneity in the limit of infinite domain size, and in the zero viscosity limit predicts a $k^{-5/3}$ energy spectrum for the large scales and a k^{-3} energy spectrum for the small scales. The assumptions of the KLB theory, as well as the scaling laws and the universality of the two energy spectra, have been questioned in the literature [12–18].

In this Letter we derive some simple rigorous bounds for the long time averaged bulk energy and enstrophy dissipation rates for 2d statistically stationary flows sustained by a variety of driving forces. The study of physically relevant rigorous bounds on the energy dissipation rate, i.e., the power consumption of turbulence, for a class of boundary-driven flows can be traced back to the seminal work of Howard [19]. In more recent years an alternative approach [20] renewed interest in those kinds of problems, providing direct connections to experiments in some cases. Bounds for the energy dissipation of steady body

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forced flows—more convenient for theoretical and numerical investigations—in three dimensions have been derived by Foias [21] and others [22–25]. Not unexpectedly, Foias et al. [26] also derived a bound for the enstrophy dissipation rate in the statistically stationary states of 2d turbulence driven by a restricted class of forces. Bounds for the energy and enstrophy dissipation in 2d flows driven by a monochromatic forcing were derived in [27] and [16]. The case of temporally white noise forcing was studied by Eyink [28]. More recently Tran and Dritschel [29,30] derived bounds of the enstrophy dissipation for freely decaying 2d turbulence in terms of the initial ideal invariants of the flow. (The same problem has also been addressed in terms of the inviscid Euler equations [31].) Finally, we mention that dissipation rate estimates have been used to derive bounds on the dimension of the attractor for the 2d Navier–Stokes equations [32–37].

The results for the energy and enstrophy dissipation of forced flows derived in this Letter apply to a more general type of forcing than the single scale forcing [16,27]. We also consider forces that are smoothly varying in time, unlike temporally white noise forcing [28], and we are particularly interested in the behavior of the long times averaged dissipation rates in the vanishing viscosity limit.

Because viscosity is a dimensional quantity we must specify what we mean by "small" viscosity. To be precise, we measure the magnitude of the viscosity by the Reynolds number in the statistically steady state,

$$Re = \frac{U}{k_f \nu} \tag{1}$$

where U is the root mean square velocity and k_f is a natural wavenumber (inverse length scale) in the driving force. The dissipation rates are also dimensional quantities, so we measure them in terms of the inviscid scales determined by U and k_f. That is, we estimate

$$\beta = \frac{\epsilon}{k_f U^3} \quad \text{and} \quad \gamma = \frac{\chi}{k_f^3 U^3} \tag{2}$$

in terms of Re and focus on the $Re \to \infty$ limit with other parameters (such as the functional form of the forcing and, in the most general case, the large scale eddy turnover time) fixed. For a broad class of external driving we find that

$$\beta \lesssim Re^{-1/2} \quad \text{and} \quad \gamma \lesssim Re^0, \tag{3}$$

consistent with an enstrophy cascade of sorts.

However, for special cases of forcing such as "ultra narrow band" monochromatic (i.e., involving on a single length scale, albeit with a broad range of time dependence) forcing, or for a fixed energy flux forcing popular for direct numerical simulations, a stronger bound holds:

$$\beta \lesssim Re^{-1} \quad \text{and} \quad \gamma \lesssim Re^{-1}. \tag{4}$$

This kind of Re^{-1} scaling suggests "laminar" flows where the energy is concentrated at or above the smallest length scale of the forcing. This kind of scaling was previously derived for monochromatic forcing [16,27] and for white noise in time forcing [28].

In every case the bounds derived here are strictly less than those available—or expected—for 3d turbulence. The upper bounds (3) on the energy and enstrophy dissipation for 2d flows derived here are in a sense a consequence of combining previous approaches [16,26,28] applied to a class of forcing functions concentrated in a finite range of length scales. Even though some steps in our analysis have been taken before, in order to make this Letter self-contained the complete (but nevertheless short) proofs will be presented.

The rest of this Letter is organized as follows. In Section 2 we introduce the problem and basic definitions, and perform the analysis leading to (3) for the simplest case of time-independent body forces. Section 3 generalizes the analysis to include a broad class of time-dependent forces. In Section 4 we briefly review the results for time-dependent but single-length scale forcing and "fixed-flux" forces in order to establish the stronger results in (4). The concluding Section 5 contains a brief discussion of the results and their implications.

2. Time-independent forcing

Consider a two-dimensional periodic domain $[0, L]^2$, i.e., $\mathbb{T}_L^2$, filled with an incompressible fluid of unit density evolving according to the Navier–Stokes equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = +\nu \nabla^2 \mathbf{u} + \mathbf{f}, \tag{5}$$

where $\mathbf{u} = \hat{\mathbf{i}} u_x(x, y, t) + \hat{\mathbf{j}} u_y(x, y, t)$ is the incompressible (divergence-free) velocity field, $p(x, y, t)$ is the pressure, ν is the viscosity, and $\mathbf{f}(x, y) = \hat{\mathbf{i}} f_x(x, y) + \hat{\mathbf{j}} f_y(x, y)$ is a smooth, mean zero, divergence-free body force with characteristic length scale $\sim k_f$ (defined precisely below). The scalar vorticity $\omega = \partial_x u_y - \partial_y u_x$ satisfies

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + \phi, \tag{6}$$

where $\phi = \hat{\mathbf{k}} \cdot \nabla \times \mathbf{f} = \partial_x f_y - \partial_y f_x$.

The Reynolds number is defined in (1) where

$$U \equiv \langle |\mathbf{u}|^2 \rangle^{1/2} \tag{7}$$

is the root-mean-square velocity with $\langle \cdot \rangle$ representing the space–time average

$$\langle \mathbf{g} \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\frac{1}{L^2} \int_{\mathbb{T}_L^2} \mathbf{g}(x, y, t)\, d^2x \right) dt. \tag{8}$$

(The limit in the time average is assumed to exist for all the quantities of interest.) The forcing length scale associated with the wavenumber k_f is defined by

$$k_f^2 \equiv \frac{\|\nabla^2 \mathbf{f}\|}{\|\mathbf{f}\|}, \tag{9}$$

where $\| \cdot \|$ is the L_2 norm on $\mathbb{T}_L^2$. It is apparent that we are restricting ourselves to sufficiently smooth forcing functions.

The time and space averaged energy dissipation rate is

$$\epsilon \equiv \nu \langle |\nabla \mathbf{u}|^2 \rangle = \nu \langle \omega^2 \rangle, \tag{10}$$

the second expression resulting from integrations by parts utilizing incompressibility. The bulk averaged enstrophy dissipation rate is

$$\chi \equiv \nu\langle|\nabla\omega|^2\rangle = \nu\langle|\nabla^2\mathbf{u}|^2\rangle. \tag{11}$$

We think of $\beta = \epsilon/k_f U^3$ and $\gamma = \chi/k_f^3 U^3$ as functions of Re and the functional form or "shape" of the forcing, but not explicitly on its amplitude

$$F = \frac{\|\mathbf{f}\|}{L} \tag{12}$$

except indirectly through its influence on U.

We are considering the Reynolds number to be the "control parameter" even though it is defined in terms of the emergent quantity U. Strictly speaking the flow is determined by the structure and amplitude of the body force (and possibly the initial data) so the Grashof number such as $Gr = F/k_f^3\nu^2$ should naturally be used as the relevant dimensionless control parameter indicating the intensity of the driving and the resulting flow. Indeed, while we can always realize any given value of Gr, it is not at all evident that every particular value of Re can be achieved. Nevertheless, in order to express the results in terms of quantities familiar to the theory of homogeneous isotropic turbulence we will (without loss of rigor) express the bounds in terms of Re.

Poincare's inequality applied to (10) and (11) immediately yield the lower estimates

$$\epsilon \geqslant \nu\frac{4\pi^2}{L^2}U^2 \quad \text{and} \quad \chi \geqslant \nu\frac{16\pi^4}{L^4}U^2 \tag{13}$$

so that

$$\beta \geqslant 4\pi^2\alpha^2 Re^{-1} \quad \text{and} \quad \gamma \geqslant 16\pi^4\alpha^4 Re^{-1}, \tag{14}$$

where $\alpha = (k_f L)^{-1} \leqslant (2\pi)^{-1}$ is the ratio of the forcing to domain length scales. If β and γ scale both as $\sim Re^{-1}$ then we say that the flow exhibits laminar behavior because the energy is then necessarily concentrated at relatively long length scales determined by the prefactor, rather than over a broad range of scales that increases as $Re \to \infty$.

On the other hand if $\beta \sim Re^0$, the scaling expected in 3d turbulence, the flow exhibits finite (residual) dissipation in the limit of zero viscosity indicating the presence of an active and effective energy cascade to small scales. It was recently shown [22,24] that $\beta \leqslant cRe^0$ for the vanishing viscosity limit of three-dimensional versions of the systems under consideration here and in Section 3, where the coefficient c is uniform in ν, L, and F. There is, however, no known *a priori* enstrophy dissipation rate bound for the 3d turbulence; this is related to the outstanding open question of the regularity of solution for the 3d Navier–Stokes equations [38]. As the results of this Letter suggest quantitatively, the dissipation rates of 2d turbulence falls somewhere between laminar scalings and the rates for 3d turbulence.

To prove the 2d bounds we first take the inner product of the vorticity equation (6) with ω and average to obtain the enstrophy production-dissipation balance

$$\chi = \langle\omega\phi\rangle, \tag{15}$$

where the time derivative term drops out when we take the long time average. Integrating by parts to move the $\hat{\mathbf{k}}\cdot\nabla\times$ from ω onto ϕ and utilizing the Cauchy–Schwarz inequality, we easily obtain

$$\chi \leqslant k_f^2 U F. \tag{16}$$

For the second step, consider a smooth incompressible vector field $\mathbf{v}(x,y)$. Take the inner product of $\mathbf{v}$ with the Navier–Stokes equation, integrate by parts and average to obtain

$$\frac{1}{L^2}\int_{\mathbb{T}_L^2} \mathbf{v}\cdot\mathbf{f}\,d^2x = -\langle\mathbf{u}\cdot(\nabla\mathbf{v})\cdot\mathbf{u} + \nu\mathbf{u}\cdot\nabla^2\mathbf{v}\rangle. \tag{17}$$

Using the Cauchy–Schwarz and Hölder inequalities (as in [22]) we deduce

$$F \times \frac{1}{L^2\|\mathbf{f}\|}\int_{\mathbb{T}_L^2} \mathbf{v}\cdot\mathbf{f}\,d^2x \leqslant U^2\|\nabla\mathbf{v}\|_\infty + \nu\frac{U}{L}\|\nabla^2\mathbf{v}\|, \tag{18}$$

where $\|\cdot\|_\infty$ is the L_∞ norm on $\mathbb{T}_L^2$. In order for the inequality to be non-trivial we need to restrict $\mathbf{v}$ so that $\int_{\mathbb{T}_L^2}\mathbf{v}\cdot\mathbf{f}\,d^2x > 0$. This is easy to arrange. For example the choice $\mathbf{v} = \mathbf{f}/F$ will satisfy this condition if $\mathbf{f}$ is sufficiently smooth that the right hand side of (18) is finite. If it is not so smooth, then for instance we can take $\mathbf{v} \sim K * \mathbf{f}$ where $K(x,y,x',y')$ is a (positive) smoothing kernel. In any case we can choose $\mathbf{v}$ appropriately and use (18) to eliminate F in (16) so that

$$\chi \leqslant U^3 k_f^3\left(C_1 + \frac{C_2}{Re}\right) \quad \Rightarrow \quad \gamma \leqslant \left(C_1 + \frac{C_2}{Re}\right), \tag{19}$$

where the dimensionless coefficients C_1 and C_2 are independent of k_f and L, depending only on the functional "shape" of $\mathbf{v}$ (and thus also on the shape of $\mathbf{f}$) but not on its amplitude F or the viscosity ν. Explicitly,

$$C_1 = \frac{\|\nabla_l\mathbf{v}\|_\infty}{\langle\mathbf{v}\cdot\mathbf{f}/F\rangle} \quad \text{and} \quad C_2 = \frac{\langle|\nabla_l^2\mathbf{v}|\rangle^{1/2}}{\langle\mathbf{v}\cdot\mathbf{f}/F\rangle}, \tag{20}$$

where ∇_l is the gradient with respect to the non-dimensional coordinate $k_f\mathbf{x}$. An upper bound for the enstrophy dissipation rate like that in (19) was first derived in [26]. Note that for strictly band-limited forces, i.e., if the Fourier transform of the force is supported on wavenumbers $|\mathbf{k}| \in (k_{\min}, k_{\max})$ with $0 < k_{\min} < k_{\max} < \infty$, then the coefficients C_1 and C_2 are bounded by pure numbers depending only on $k_{\max}/k_{\min}$.

For the final step of the proof we use integrations by parts and the Cauchy–Schwarz inequality to see that

$$\langle\omega^2\rangle^2 = \langle\mathbf{u}\cdot\nabla\times(\hat{\mathbf{k}}\omega)\rangle^2 \leqslant \langle|\mathbf{u}|^2\rangle\langle|\nabla\omega|^2\rangle. \tag{21}$$

Combining (21) with (19) we deduce

$$\langle\omega^2\rangle^2 \leqslant \frac{k_f^3U^5}{\nu}\left(C_1 + \frac{C_2}{Re}\right), \tag{22}$$

and in terms of the energy dissipation rate this is the announced result

$$\beta \leqslant Re^{-1/2}\left(C_1 + \frac{C_2}{Re}\right)^{1/2}. \tag{23}$$

3. Time-dependent forces

Now consider the Navier–Stokes equation (5) where the time-dependent body force $\mathbf{f}(x, y, t)$ is smooth and incompressible with characteristic length scale $\sim k_f^{-1}$ given by

$$k_f^4 \equiv \frac{\langle |\nabla^2 \mathbf{f}|^2 \rangle}{\langle |\mathbf{f}|^2 \rangle}, \tag{24}$$

and time scale Ω_f^{-1} defined by

$$\Omega_f^2 \equiv \frac{\langle |\partial_t \mathbf{f}|^2 \rangle}{\langle |\mathbf{f}|^2 \rangle}. \tag{25}$$

We define

$$\tau = \frac{\Omega_f}{k_f U}, \tag{26}$$

the ratio of the "eddy turnover" time $(k_f U)^{-1}$ to the forcing time scale Ω_f^{-1}. In this time-dependent setting the amplitude F of the force is

$$F = \langle |\mathbf{f}|^2 \rangle^{1/2}. \tag{27}$$

As before, the space and time average of ω times the vorticity equation (6) yields the enstrophy balance equation (15), and integration by parts and the Cauchy–Schwarz inequality implies

$$\chi \leqslant k_f^2 U F. \tag{28}$$

For the second step here we introduce a smooth incompressible vector field $\mathbf{v}(x, y, t)$ and take space and time average of the inner product of with the Navier–Stokes equation to obtain

$$\langle \mathbf{v} \cdot \mathbf{f} \rangle = -\langle \mathbf{u} \cdot \partial_t \mathbf{v} + \mathbf{u} \cdot (\nabla v) \cdot \mathbf{u} + \nu \mathbf{u} \cdot \nabla^2 \mathbf{v} \rangle. \tag{29}$$

Cauchy–Schwarz and Hölder's inequalities then imply

$$F \frac{\langle \mathbf{v} \cdot \mathbf{f} \rangle}{\langle |\mathbf{f}|^2 \rangle^{1/2}} \leqslant U \langle |\partial_t \mathbf{v}|^2 \rangle^{1/2} + U^2 \sup_t \|\nabla \mathbf{v}\|_\infty + \nu U \langle |\nabla^2 \mathbf{v}|^2 \rangle^{1/2}. \tag{30}$$

Now we need to be able to choose $\mathbf{v}$ satisfying $\langle \mathbf{v} \cdot \mathbf{f} \rangle > 0$ such that all the coefficients on the right-hand side are all finite. Our ability to do this depends on details of $\mathbf{f}(x, y, t)$.

For example if $\mathbf{f}$ is sufficiently smooth in space and appropriately uniformly bounded in time then we can choose $\mathbf{v} \sim \mathbf{f}$. We could also choose $\mathbf{v}$ as an appropriately filtered version of $\mathbf{f}$ to cover more general cases. For the purposes of this study and to display the results in the clearest (if not the sharpest or most general form) we will simply presume that $\mathbf{f}$ is sufficiently regular that we can take $\mathbf{v} = \mathbf{f}$. In that case (30) becomes

$$F \leqslant \Omega_f U + U^2 \frac{\sup_t \|\nabla \mathbf{f}\|_\infty}{F} + \nu k_f^2 U. \tag{31}$$

Then using this to eliminate F from (28) we have

$$\chi \leqslant k_f^3 U^3 \left(\tau + C_3 + \frac{1}{Re} \right) \quad \Rightarrow \quad \gamma \leqslant \left(\tau + C_3 + \frac{1}{Re} \right), \tag{32}$$

where the coefficient C_3 is

$$C_3 = \frac{\sup_t \|\nabla_l \mathbf{f}\|_\infty}{F} \tag{33}$$

with ∇_l denoting the gradient with respect to the non-dimensional coordinate $k_f \mathbf{x}$. The dimensionless number C_3 is independent of the scales of F, k_f, L, etc., depending only on the "shape" of $\mathbf{f}$. For example if $\mathbf{f}$ is quasi-periodic with N frequencies involving only wavenumbers $\mathbf{k}$ with $0 < k_{\min} < |\mathbf{k}| < k_{\max} < \infty$, then C_3 is a pure number bounded by $\sqrt{N}$ times a function of $k_{\max}/k_{\min}$.

The final step again uses the inequality

$$\epsilon^2 = \nu^2 \langle \omega^2 \rangle^2 = \nu^2 \langle \mathbf{u} \cdot \nabla \times (\hat{\mathbf{k}} \omega) \rangle^2 \leqslant \nu U^2 \chi \tag{34}$$

and it then follows immediately from (32) that

$$\beta \leqslant Re^{-1/2} \left(\tau + C_3 + \frac{1}{Re} \right)^{1/2}. \tag{35}$$

Note in this case τ depends on U and features of the forcing through k_f and Ω_f, but *not* on ν.

4. Monochromatic and constant flux forces

An even sharper scaling bound on the energy and enstrophy dissipation rates can be derived when the driving is monochromatic in space, whether it is steady or time dependent [16,27]. Suppose the body force involves only a single length scale, i.e.,

$$-\nabla^2 \mathbf{f} = k_f^2 \mathbf{f}. \tag{36}$$

This does not preclude complex time-dependence for $\mathbf{f}(x, y, t)$, just that it involves only spatial modes with wavenumbers $\mathbf{k}$ with $|\mathbf{k}| = k_f$. Then the enstrophy production-dissipation balance (15) implies

$$\chi = \langle \omega \phi \rangle = \langle \mathbf{u} \cdot (-\nabla^2 \mathbf{f}) \rangle = k_f^2 \langle \mathbf{u} \cdot \mathbf{f} \rangle = k_f^2 \epsilon. \tag{37}$$

Combining this with (34), we observe that

$$\epsilon^2 \leqslant \nu U^2 \chi = \nu k_f^2 U^2 \epsilon \tag{38}$$

so that

$$\epsilon \leqslant \nu k_f^2 U^2 \quad \text{and} \quad \chi \leqslant \nu k_f^4 U^2 \tag{39}$$

implying that both β and γ are bounded by Re^{-1}. Note that this kind of monochromatic forcing is a special case that has been shown in the literature for some cases to lead to a laminar flow that never looses stability [39].

Another type of forcing that results in this scaling without the ultra-narrow band restriction is

$$\mathbf{f}(x, y, t) = \epsilon \frac{\mathcal{P}\mathbf{u}}{L^{-2} \|\mathcal{P}\mathbf{u}\|^2}, \tag{40}$$

where $\mathcal{P}$ is the projector onto spatial modes of wavenumber $\mathbf{k}$ with $|\mathbf{k}| \in [k_{\min}, k_{\max}]$, and the coefficient ϵ is now the control parameter. This type of forcing is often applied in numerical simulations of homogeneous isotropic turbulence. With this forcing in the Navier–Stokes equations constitutes an autonomous dynamical system with kinetic energy injected at a constant rate ϵ at wavenumbers with $|\mathbf{k}| \in [k_{\min}, k_{\max}]$. The rms speed U and the enstrophy dissipation rate χ are then emergent quantities determined soley by the imposed energy flux ϵ. The mean power balance for solutions is still

$$\nu\langle|\nabla\mathbf{u}|^2\rangle = \nu\langle\omega^2\rangle = \epsilon, \tag{41}$$

and the enstrophy production-dissipation balance reads

$$\chi = \nu\langle|\nabla\omega|^2\rangle = \epsilon\left\langle\frac{\|\nabla\mathcal{P}\mathbf{u}\|^2}{\|\mathcal{P}\mathbf{u}\|^2}\right\rangle. \tag{42}$$

Because forcing only involves wavenumbers in $[k_{\min}, k_{\max}]$ with positive energy injection at each wavenumber, at each instant of time

$$k_{\min}^2\|\mathcal{P}\mathbf{u}\|^2 \leqslant \|\nabla\mathcal{P}\mathbf{u}\|^2 \leqslant k_{\max}^2\|\mathcal{P}\mathbf{u}\|^2. \tag{43}$$

Then (42) implies that

$$k_{\min}^2\epsilon \leqslant \chi \leqslant k_{\max}^2\epsilon. \tag{44}$$

Using this with (34) we see that

$$\epsilon^2 \leqslant \nu U^2\chi \leqslant \nu U^2 k_{\max}^2\epsilon, \tag{45}$$

and we conclude

$$\epsilon \leqslant \nu k_{\max}^2 U^2 \quad \text{and} \quad \chi \leqslant \nu k_{\max}^4 U^2. \tag{46}$$

Hence also in this case both β and γ are bounded $\sim Re^{-1}$.

Note that in both these derivations a condition like (44) or the stronger condition (37) was used. It is an open question whether such a condition holds for more general and more "realistic" forcing functions.

5. Discussion

These quantitative bounds show that for 2d turbulence sustained by forces as described in the previous sections, there is no residual energy dissipation in the vanishing viscosity limit defined by $Re \to \infty$ at fixed U, L, k_f and Ω_f. To be precise, ϵ vanishes at least as fast as $Re^{-1/2}$ in this limit. This confirms that there is no forward energy cascade in the steady state in the inviscid limit. On the other hand the residual enstrophy dissipation allowed by (32) in this limit does not rule out a forward enstrophy cascade. This combination, $\epsilon \to 0$ with $\chi = \mathcal{O}(1)$ in the inviscid limit, is consistent with the dual-cascade picture of 2d turbulence developed by Kraichnan [9], Leith [10] and Batchelor [11]. Of course a forward energy cascade is not prohibited for finite values of Re. The $\beta \sim Re^{-1/2}$ scaling allowed by the bound is less severe than what a laminar flow (or a flow with only inverse cascade of energy) would predict, and as a result (23) does not exclude the presence of a direct "subdominant" cascade of energy when the Reynolds number is finite [17].

On the other hand the direct cascade of enstrophy is necessarily absent for some forcing functions [16,27,28]. When the forcing acts at a single scale or constant power is injected in a finite band of wavenumbers, both ϵ and χ vanish $\sim Re^{-1}$. This suggests an essentially laminar behavior for these flows: if the energy spectrum follows a power law $E(k) \sim k^{-\alpha}$ for large wavenumbers then the exponent must be $\alpha \leqslant -5$ for χ to vanish in the vanishing viscosity limit. These results have been interpreted as absence of enstrophy cascade in finite domains. However, both of these results rely on the condition (44) which is not guaranteed for a general forcing functions. Whether (44) might hold for more general forcing functions is an open question; the results (32) and (35) give the restrictions on the energy and enstrophy dissipation rate for a general forcing. Note that the $\beta \sim Re^{-1/2}$ does not impose significant restriction on the energy spectrum given the bound on χ. These considerations suggest that it is possible therefore that in 2d turbulence the steady state energy spectrum depends on the type of forcing used, even within the class of relatively narrow-band driving. In order to investigate this issue it would be useful to perform high resolution direct numerical simulations driven by forces that produces flows that do *not* satisfy (44). We do not, however, propose any specific forcing mechanism for achieving such flows here; the analysis in this Letter just rules out some strategies.

We conclude by noting that an interesting question that follows from these results is that of the *Re*-scaling of the energy dissipation in 3d systems that almost have 2d behavior like strongly rotating, strongly stratified or conducting fluids in the presence of a strong magnetic field. For example, is there a critical value of the rotation such that the scaling of the energy dissipation rate with the Reynolds number makes a transition from $\epsilon \sim Re^0$ to $\epsilon \sim Re^{-1/2}$? If so, how does the critical rotation rate depend on *Re* and/or details of the forcing driving the flow? These and related questions remain for future studies.

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