Optimal lengthscale for a turbulent dynamo

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We demonstrate that there is an optimal forcing length scale for low Prandtl number dynamo flows, that can significantly reduce the required energy injection rate. The investigation is based on simulations of the induction equation in a periodic box of size $2\pi L$. The flows considered are the laminar and turbulent ABC flows forced at different forcing wavenumbers k_f, where the turbulent case is simulated using a subgrid turbulence model. At the smallest allowed forcing wave number $k_f = k_{min} = 1/L$ the laminar critical magnetic Reynolds number Rm_c^{lam} is more than an order of magnitude smaller than the turbulent critical magnetic Reynolds number Rm_c^{turb} due to the hindering effect of turbulent fluctuations. We show that this hindering effect is almost suppressed when the forcing wavenumber k_f is increased above an optimum wavenumber $k_f L \simeq 4$ for which Rm_c^{turb} is minimum. At this optimal wavenumber Rm_c^{turb} is smaller by more than a factor of ten than the case forced in $k_f = 1$. This leads to a reduction of the energy injection rate by three orders of magnitude when compared to the case where the system is forced at the largest scales and thus provides a new strategy for the design of a fully turbulent experimental dynamo.

Dynamo is the mechanism by which magnetic fields are amplified in stars and planets due to their stretching by the underlying turbulent flow [1]. In the last two decades several experimental groups have attempted to reproduce the dynamo instability in the laboratory [2–4]. The first successful dynamos were achieved in Riga [2] and Karlsruhe [3]. The flow in these dynamos were highly constrained and did not allow for turbulence to fully develop at large scales. Thus the results could be successfully modelled by laminar flow dynamos. The first fully turbulent dynamo was achieved in [4] where the flow was driven by two counter rotating propellers. However, in this experiment a dynamo was only obtained when at least one ferromagnetic iron propeller was used. So far other attempts to achieve dynamo are not successful and unconstrained dynamos driven just by the turbulent flows have not been achieved.

One of the major difficulties to achieve liquid metal dynamos are the low values of magnetic Prandtl numbers P_M (the ratio of viscosity ν to magnetic diffusivity η) that characterizes liquid metals which is less than 10^{-5}. This implies that very large values of the Reynolds number $Re = UL/\nu$ (where U is the rms velocity and L is the domain size) are needed to reach even order one values of the magnetic Reynolds numbers $Rm = UL/\eta = P_M Re$. The magnetic Reynolds number is the critical parameter that determines the dynamo onset. For small values of Rm no dynamo instability exists and Rm should be larger than a critical value Rm_c in order to generate the spontaneous growth of the magnetic field. The large Re needed to reach values of Rm above Rm_c implies that the flow is turbulent and large power consumption is needed which scales like $I \propto \rho U^3 L^3/\ell_f$ (where ℓ_f is the length scale of the forcing).

For laminar flows Rm_c and its dependence on the forcing lengthscale has been calculated in [5–7]. The laminar threshold Rm_c^{lam} however strongly underestimates Rm_c for large Reynolds numbers and cannot be used to predict the dependence of Rm_c on the parameters of the system in the presence of turbulence. Turbulent fluctuations considerably inhibit dynamo action and increase Rm_c. The dependence of the dynamo threshold Rm_c on the Reynolds number was investigated by different groups [8–11] for different flows with the use of numerical simulations. These studies showed that as Re is increased (P_M is decreased) the critical magnetic Reynolds number Rm_c is initially increased. The turbulent fluctuations generated at large values of Re were preventing dynamo, raising Rm_c to values much larger than the case of organized laminar flows. However when sufficiently large Reynolds numbers are reached this increase saturates and a finite value of Rm_c is reached in the limit of $Re \to \infty$. We will refer to this value as the turbulent critical magnetic Reynolds number and define it as $Rm_c^{\mathrm{turb}} \equiv \lim_{Re\to\infty} Rm_c$, The opposite limit defines the laminar critical value $Rm_c^{\mathrm{lam}} \equiv \lim_{Re\to 0} Rm_c$. The afore mentioned studies managed to reach this asymptote only by using subgrid scale models (either hyperviscocity, α model LES, or a dynamical turbulent viscosity) that model the high Reynold number flows. It is worth pointing out that the different flows considered led to different values of Rm_c^{turb} implying that it is possible to optimize the flow to reduce Rm_c^{turb}. In this work we try to determine the dependence of the turbulent Rm_c^{turb} on the length scale of the forcing ℓ_f with respect to the domain size L in order to find its optimal value. The study is based on the results of numerical simulations using a pseudospectral method in a triple periodic domain [12, 13] and analytic estimates based on scale separation arguments.

In their simplest form the governing equation for the evolution of the magnetic field is given by

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \eta \Delta \mathbf{b} \tag{1}$$

where $\mathbf{b}$ is the magnetic field, and η the magnetic diffusivity. $\mathbf{u}$ is the velocity field that is determined by solving the independent incompressible Navier-Stokes equation of a unit density $\rho = 1$ fluid,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla P + \nu \Delta \mathbf{u} + \mathbf{f}. \tag{2}$$

where $\mathbf{f}$ is an external forcing. In the present study the domain considered is a triple periodic box of size $2\pi L$ and $\mathbf{f}$ is taken to be the ABC forcing

$$\mathbf{f} = \begin{bmatrix} A\sin(k_f z) + C\cos(k_f y), \\ B\sin(k_f x) + A\cos(k_f z), \\ C\sin(k_f y) + B\cos(k_f x) \end{bmatrix}$$

with $A = B = C = 1$. k_f is the forcing wave number, and we define the forcing lengthscale as $\ell_f \equiv k_f^{-1}$. The ABC flow has been the subject of many dynamo studies both in the laminar [14, 15] and turbulent state [9].

In this set up some analytical progress can be made in the case that $k_f L \gg 1$. Then one can use standard mean-field approximations to estimate the critical onset Rm_c [14, 16, 17]. Splitting the magnetic field in a large scale component $\mathbf{B}$ and a fluctuating part $\tilde{\mathbf{b}}$ for scale separation we obtain to first order for the fluctuating field

$$\eta \Delta \tilde{\mathbf{b}} = -\mathbf{B} \nabla \mathbf{u} \tag{3}$$

and for the large scale field

$$\partial_t \mathbf{B} = \nabla \times (\alpha \mathbf{B}) - \eta \Delta \mathbf{B}. \tag{4}$$

Here equation (3) has been derived assuming that the magnetic Reynolds number based on the forcing length scale $Rm_f = Rm(k_f L)^{-1}$ is small. α is a tensor such that $\alpha_{ij} B_j = \langle \mathbf{u} \times \tilde{\mathbf{b}} \rangle_i$. The angular brackets stand for small scale average and summation over the index j is implied. For the particular forcing chosen the α-tensor is diagonal and isotropic. The non-zero diagonal elements α_{ii} can be calculated by solving (3) for a given velocity field and are

$$\alpha_{ii} = \frac{-1}{\eta} \langle \mathbf{u} \times \Delta^{-1} \nabla_i \mathbf{u} \rangle_i = a \frac{U^2}{\eta k_f}, \tag{5}$$

(no summation over i is implied). a is the non-dimensional α_{ii} element that is independent of U, η and k_f and needs to be calculated from the flow. For the laminar case a can be calculated directly and gives $a^{\text{lam}} = 1/3$, while for the turbulent case it needs to be calculated from eq. 5 using the turbulent solutions obtained numerically (see e.g. [18]). The growth rate γ for a helical large-scale mode of wavenumber $K = 1/L$ is then

$$\gamma = a \frac{U^2}{\eta k_f L} - \eta L^{-2} \tag{6}$$

that leads to a critical Reynolds number $Rm_c = \sqrt{k_f L/a}$. The critical magnetic Reynolds number based on the forcing scale is $Rm_{fc} = \sqrt{1/k_f L a} \ll 1$ verifying the original assumption of small Rm_f. This result was discussed in [19]. Note that this argument is true both for turbulent and laminar flows although the value of the coefficient a will depend on the level of turbulence. At large Re however a will reach an asymptotic value a^{turb} that will determine the value of Rm_c^{turb} to be

$$Rm_c^{\text{turb}} = \sqrt{k_f L/a^{\text{turb}}}. \tag{7}$$

This scaling implies that in the large scale separation limit $k_f L \gg 1$ for fixed L and η as k_f is increased it becomes more difficult to obtain a dynamo.

To calculate Rm_c^{turb} in the absence of scale separation we performed numerical simulations of equations 1,2 varying the forcing wavenumber k_f for fixed L. In order to mimic the large Reynolds number flow that requires large grid size N we do not use an ordinary viscosity ν for the dissipation but rather a dynamical wavenumber-dependent turbulent viscosity [20] defined in spectral space as

$$\nu^{\text{turb}}(k,t) = 0.27[1 + 3.58(k/k_c)^8]\sqrt{E_K(k_c,t)/k_c} \tag{8}$$

where $k = |\mathbf{k}|$ is the wavenumber, $k_c = N/3L$ is the maximum wavenumber after de-aliasing and $E_K(k,t)$ is the kinetic energy spectrum of the flow. The same modeling was also used in [8] to obtain Rm_c^{turb}. The dynamical viscosity ν^{turb} depends on the grid size N, but the dependence of the large scale components of the flow on N are expected to die off much faster than in the case of ordinary viscosity.

To calculate Rm_c^{turb} as a function of k_f the following procedure was used (see also [8, 11]): For a given k_f and grid size N a series of simulations was performed varying Rm and the exponential growth rate of the magnetic energy was measured. The onset Rm_c was determined by linearly interpolating the values of Rm between the slowest growing dynamo and the slowest decaying dynamo. The series of runs was then repeated for higher values of N until either the value of Rm_c remained unchanged in which case this determined Rm_c^{turb} or the maximum of our attainable resolution was reached. Typically convergence was reached at grid sizes $N = 256$ but a few runs at $N = 512$ were also performed for verification. In addition we solved eq. 3 (with $\mathbf{u}$ given by eq. 2) with an imposed uniform magnetic field to calculate the elements of the α tensor and determine a^{turb}.

The results for the critical Reynolds number as a function of the forcing wave number k_f are shown in figure 1. The predicted scaling behavior (7) is shown with a dashed line in the same figure with a^{turb} computed using eq. 5. We get $a^{\text{turb}} \simeq 0.22$ that is of the order of the laminar value $a^{\text{lam}} = 1/3$. The filled circle indicates the

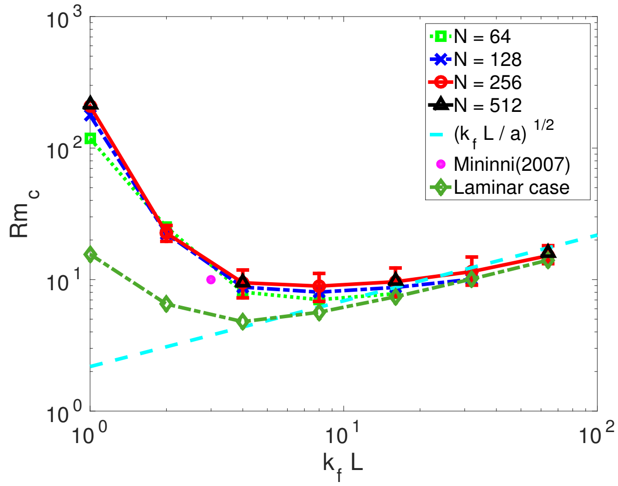

FIG. 1. Rm_c as a function of $k_f L$ obtained from numerical simulations with different resolutions $N = 64 - 512$. The data indicate that for $N \geq 128$, Rm_c does not vary as N is increased further and thus it approximates well $Rm_c^{\rm turb}$. The error-bars correspond to the maximum/minimum value of Rm for which we obtained a clear positive/negative growth rate for the simulations with $N = 256$. The dashed-dotted line corresponds to the dynamo threshold of the laminar flow. The filled circle corresponds to the value of $Rm_c^{\rm turb}$ obtained from direct numerical simulations in [9]. The dashed line shows the mean field (alpha) prediction valid in the limit $kL \to \infty$.

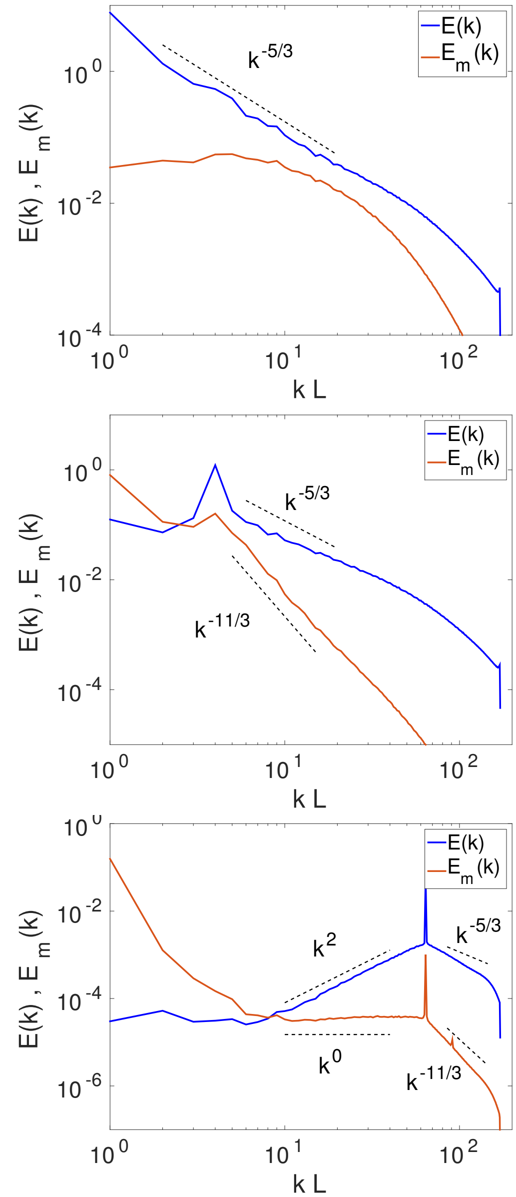

FIG. 2. Kinetic $E(k)$ and magnetic $E_m(k)$ energy spectra for the marginally unstable modes for $k_f L = 1$ (top), $k_f L = 4$ (middle), $k_f L = 64$ (bottom).

the value of $Rm_c^{\rm turb}$ calculated in [9] using simulations of higher resolutions and α-model LES.

The results are very motivating for future laboratory experiments. Although for large $k_f L$ the asymptotic scaling of (7) seems to be verified indicating that making the forcing length scale very small will not benefit dynamo experiments, at intermediate length scales $Rm_c^{\rm turb}$ appears to reach a minimum around $k_f L = 4$ to 8. In fact the value of $Rm_c^{\rm turb}$ at this optimal wavenumber is one order of magnitude smaller than the value of $Rm_c^{\rm turb}$ at $k_f L = 1$.

The kinetic and magnetic energy spectra for the slowest growing mode for three different forcing wavenumbers at highest resolution $N = 512$ are shown in the three panels of figure 2. When the flow is forced at the largest scale $k_f L = 1$ the kinetic energy shows a clear $k^{-5/3}$ spectrum while no clear power law scaling can be observed for the magnetic field. Most of the magnetic energy is concentrated at the small scales with very weak energy at the largest scale.

At the other extreme where the flow is forced at the small scales $k_f L = 64$ most of the kinetic energy is at small scales with a $k^{-5/3}$ scaling at the sub-forcing scales and a k^2 power-law scaling at scales larger than the forcing scale, suggesting equipartition of energy among all modes as predicted by equilibrium-statistical-mechanics [21]. In addition a small peak at large scales $kL \simeq 1 - 2$ is observed. The k^2 energy spectrum has been observed before in numerical simulations of the truncated Euler equation [22], and more recently for the large scales in simulations of the Navier-Stokes equation forced at small scales [23, 24]. The role and cause of this peak and its effect on dynamo is the subject of current investigations. The magnetic field on the other hand has a dominant peak at $kL = 1$, caused by the α dynamo that is followed by a flat spectrum k^0, a peak at the forcing scale and then by a $k^{-11/3}$ power law until the dissipation scales. The two peaks at $kL = 1$ and $k = k_f$ are in agreement with the mean field dynamo prediction. The two power-laws can also be explained by a balance between the stretch-

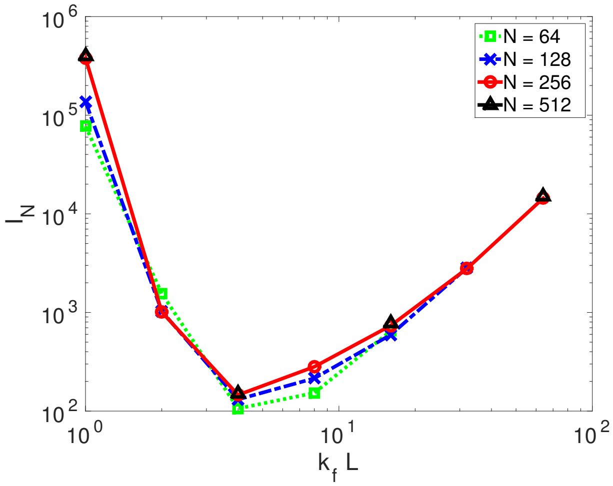

FIG. 3. The non-dimensional critical injection rate I_N as a function of $k_f L$.

ing rate S of the large scale field B_L by the fluctuations u_ℓ that is proportional to $S \propto B_L u_\ell/\ell$ and the Ohmic dissipation that is proportional to $\eta b_\ell/\ell^2$. Substituting $u_\ell \propto \ell^{1/3}$ for the turbulent scales and $u_\ell \propto \ell^{-3/2}$ for the scales in equipartition one recovers the two exponents $k^{-11/3}$ and k^0 respectively. The $k^{-11/3}$ spectrum was predicted in [25, 26] and has been observed in experiments [27] and numerical simulations [28]. The flat spectrum k^0 up to our knowledge is reported for the first time here.

The case $k_f L = 4$ that is close to the optimal wavenumber seems to be somewhere in between the two extreme cases. The magnetic field at the largest scale appears neither dominant as in the mean field case nor negligible as in the $k_f L = 1$ case. At the large scales there is not enough scale separation to observe any power-law, but at the small scales a power law close to $k^{-5/3}$ for the kinetic spectrum and $k^{-11/3}$ for the magnetic spectrum can be seen.

Our results have shown that future dynamo experiments can benefit from forcing at scales smaller than the domain size by a factor of 4 to 8. To further demonstrate this fact in figure 3 we plot the minimum energy injection rate $I = (2\pi L)^3\langle \mathbf{f}\cdot\mathbf{u}\rangle$ to achieve dynamo, normalized by the domain size L, the mass density ρ and magnetic diffusivity η $I_N = IL/(\rho\eta^3)$. The reason we have chosen this non-dimensionalization is because the mass density and magnetic diffusivity are properties of the liquid metals that vary only with temperature, while the domain size is typically fixed. In other words by normalizing it this way we ask the question: *in a laboratory experiment of a given domain size what is the optimal forcing scale to achieve dynamo with a minimal energy injection rate?*

The result is very encouraging! The optimal injection rate is almost three orders of magnitude smaller than the case for which the forcing was at the largest scale. This large drop in the injection rate can be partly explained by considering the turbulent scaling for the energy injection rate $I \propto \rho U^3 L^3/\ell$. Substituting U from the definition of Rm we obtain $I \propto \rho Rm^3\eta^3/\ell$ and thus $I_N \propto Rm^3 k_f L$. Thus the energy injection rate is very sensitive to changes in Rm and the beneficial factor of 20 that was observed in figure 1 translates to a factor 2000 for the energy injection rate. It is also worth pointing out that while for optimizing $Rm_c^{\rm turb}$ the optimal forcing wavenumber was between $k_f L = 4$ and $k_f L = 8$, when I_N is optimized the optimal wave number is more clearly given by $k_f L = 4$.

In the light of this result we can envision the design of new dynamo experiments where the flow is forced by an array of propellers so as to result in the small scale forcing required while the large scales are left unconstrained for turbulence to develop. Such experiments will challenge long-standing theoretical assumptions about mean field dynamo theories.

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