Anomalous Exponents at the Onset of an Instability

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Critical exponents are calculated *exactly* at the onset of an instability, by using asymptotic expansion techniques. When the unstable mode is subject to multiplicative noise whose spectrum at zero frequency vanishes, we show that the critical behavior can be anomalous; i.e., the mode amplitude X scales with departure from onset μ as $\langle X\rangle \propto \mu^{\beta}$ with an exponent β different from its deterministic value. This behavior is observed in a direct numerical simulation of the dynamo instability, and our results provide a possible explanation for recent experimental observations.

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In the vicinity of a continuous phase transition, the amplitude of the order parameter, say, M, increases with the departure from the critical point, say, ϵ, as a power law, i.e., $M \propto \epsilon^{\beta}$. Mean-field theories predict simple rational numbers for the exponent β (for instance, $1/2$ for systems with cubic nonlinearities). It has been realized for a long time that, because of thermal fluctuations, the power law may differ from this mean-field prediction [1]. The exponents are then said to be anomalous. By using renormalization-group techniques, their value can be calculated as a perturbative expansion in the critical dimension minus the spatial dimension of the system [2].

Similarly, in the vicinity of a continuous instability in an out-of-equilibrium system, the amplitude of the unstable mode, say, X, grows with the departure from onset, say, μ, as a power law $\langle X\rangle \propto \mu^{\beta}$ (where the angular brackets denote time average). Dynamical systems obtained by using normal form theory [3] provide simple rational values for β (usually, $1/2$ when the problem has the $X \rightarrow -X$ symmetry, $1/4$ at the tricritical point where the cubic nonlinearity vanishes, and so on). Guided by the phase transition observations, one may expect that fluctuations shift the exponent β away from its mean-field value. Somehow surprisingly, the overwhelming majority of experiments on instabilities report simple rational values in agreement with the mean-field prediction for β: Anomalous exponents seem not to be measured in this context [4,5]. In a recent experiment in a turbulent flow of liquid sodium, the dynamo instability has been observed, and some measurements indicate that the first moment of the magnetic field displays an exponent 0.77 [6]. It is possible that experimental biases are responsible for this observation: The instability is slightly imperfect, and the numerical value of the exponent is then highly sensitive to the accuracy of determination of the onset. Another appealing possible explanation is that the turbulent fluctuations of the flow lead to the anomalous exponent [7]. With the latter in mind, we now describe a canonical model that leads to anomalous behavior similar to the one measured in the dynamo instability.

In the dynamo context, the turbulent fluctuations act as a multiplicative term in the equation for the magnetic field. In contrast to the case of equilibrium phase transition where additive thermal fluctuations prohibit phase transition in small dimensions, bifurcations are not destroyed by multiplicative fluctuations even for small (possibly zero) dimensions. We thus start with a zero-dimensional system subject to multiplicative noise. For a multiplicative white noise, on-off intermittency is a generic behavior close to the threshold of instability [8,9]. Then the averaged amplitude scales as $\langle X\rangle \propto \mu$. Although the exponent differs from the mean-field prediction, its value $\beta = 1$ is in disagreement with the one measured in the experimental dynamo. It has been shown that on-off behavior is observed when the departure from onset is smaller than half of the value of the noise spectrum at zero frequency [10]. In the dynamo experiment, on-off intermittency is not observed. We suggest that it is due to the absence of a noise component at zero frequency, and, to strengthen this hypothesis, we investigate the effect of a noise whose spectrum at zero frequency vanishes. We thus consider the dynamics of the unstable mode X given by

$$\dot{X} = \mu X - X^{n+1} + \dot{Y}X, \qquad \dot{Y} = -F_Y + \zeta. \tag{1}$$

Here ζ is a Gaussian white noise with $\langle \zeta(t)\zeta(t')\rangle = 2\delta(t-t')$. F is a (potential) function of Y, and the subindex denotes differentiation with respect to this variable. $\dot{Y}$ acts as a multiplicative noise (for X) whose frequency spectrum is controlled by the function $F(Y)$. When the potential F is such that the second moment of Y is finite, the spectrum of $\dot{Y}$ vanishes at low frequency (it behaves as the square of the frequency f, for small f). Standard estimates of the effect of noise on the onset of instability (for instance, by calculating the evolution of the ensemble average of $\log X$ from the linear part of the first equation [11]) show that the onset of instability of the solution $X = 0$ is not affected by the noise and remains at $\mu = 0$. In contrast, the nonlinear regime above onset is strongly affected. We display in Fig. 1 a time series of X for different functions F in the vicinity of the onset of instability (unless otherwise stated, numerical simulations are performed in the case of cubic nonlinearities: $n = 2$). For Fig. 1(a), we used white noise, $F = 0$, and on-off intermittency is observed: Short bursts of finite amplitude (on phases) alternate with long durations with negligible amplitude (off phases). In Fig. 1(b), the case $F = F_{OU} \equiv \gamma Y^2/2$ is presented. For this choice, Y is the Ornstein-Uhlenbeck process. There is no off phase, and we expect a behavior for the moments that differs from the one of on-off intermittency. Figure 1(c) displays a time series for $F = F_{AN} \equiv \nu|Y|$ that results in an intermediate behavior.

In Fig. 2, the first moment is displayed as a function of μ for the two functions F_{OU} with $\gamma = 0.2$ and F_{AN} with various values of ν. For the F_{OU} case, we observe for $\mu \in [3.10^{-4}, 10^{-1}]$ an evolution that seems compatible with a power law. A best fit determination of the associated exponent results in the value 0.69, thus different from 1 and 1/2. However, when μ is very small, the slope changes and the deterministic exponent 1/2 is recovered: The apparent anomalous behavior disappears at criticality [12]. This is confirmed by a perturbative expansion performed on the Fokker-Planck equation (not presented here). This expansion predicts that X is concentrated around the value X^* at which a weighted average of the nonlinear effect balances the linear growth rate $\mu = X^{*n} \int_{-\infty}^{\infty} \Pi(Y) \exp(nY) dY$, where $\Pi(Y) \propto e^{-F}$ is the stationary probability density of Y. Thus, in this case and for $n = 2$, the first moment scales as $\sqrt{\mu}$ as observed numerically.

A simple potential F for which this expansion can break down is $F_{AN} = \nu|Y|$. Indeed, if $\nu > n$, the expansion holds and results in normal scaling $\beta = 1/n$ but breaks down (because X^* vanishes) when $\nu < n$. In Fig. 2, where the first moment for this potential is displayed, we observe that $\langle X \rangle \propto \mu$ for small ν and $\langle X \rangle \propto \sqrt{\mu}$ for large ν. Anomalous behavior with an exponent between 1/2 and 1 is observed for ν of the order of 1. In this regime and in contrast to the F_{OU} case, the exponent remains anomalous for the smallest achievable values of μ. This numerical result is confirmed by a new perturbative expansion that we now sum up.

By using $\Omega = \log X - Y - \log \mu/n$, the Fokker-Planck equation for P, the stationary probability density function of Ω and Y is

$$0 = -\mu \partial_\Omega (1 - e^{n\Omega + nY}) P + \partial_Y (F_Y P) + \partial_Y^2 P. \quad (2)$$

Since the derivative in Ω is multiplied by a small parameter (we are interested in the limit $\mu \to 0$), we introduce a WKB-like expansion and search for $P(\Omega, y) = \exp[\sum_{m=-1} \mu^m S_m]$, where the first term S_{-1} depends only on Ω. At lowest order we obtain

$$\partial_Y^2 r_0 + \partial_Y (F_Y r_0) + S_{-1,\Omega}(e^{n\Omega + nY} - 1) r_0 = 0, \quad (3)$$

where $r_0 = \exp[S_0]$. This equation can be solved exactly for positive and negative Y. The two solutions are then matched at $Y = 0$, which selects the value of S_{-1}:

$$n = 2\nu I_\kappa[\lambda e^{n\Omega/2}] K_\kappa[\lambda e^{n\Omega/2}], \quad (4)$$

where $\lambda^2 = -4S_{-1,\Omega}/n^2$, $\kappa = \sqrt{\nu^2/n^2 - \lambda^2}$, and I_κ and K_κ are modified Bessel functions of order κ. The solution for r_0 is then

$$r_0 = \begin{cases} A(\Omega) e^{-F/2} I_\kappa[\lambda e^{n(\Omega+Y)/2}] K_\kappa[\lambda e^{n\Omega/2}] & (Y < 0) \\ A(\Omega) e^{-F/2} K_\kappa[\lambda e^{n(\Omega+Y)/2}] I_\kappa[\lambda e^{n\Omega/2}] & (Y > 0). \end{cases} \quad (5)$$

The amplitude $A(\Omega)$ is determined from the solvability condition at the next order. Up to this order, we have then obtained the expression $P = \exp[\mu^{-1} S_{-1}(\Omega)] r_0(\Omega, Y)$, where all the dependence in μ is in the exponential. As displayed in Fig. 3, this asymptotic result is in good

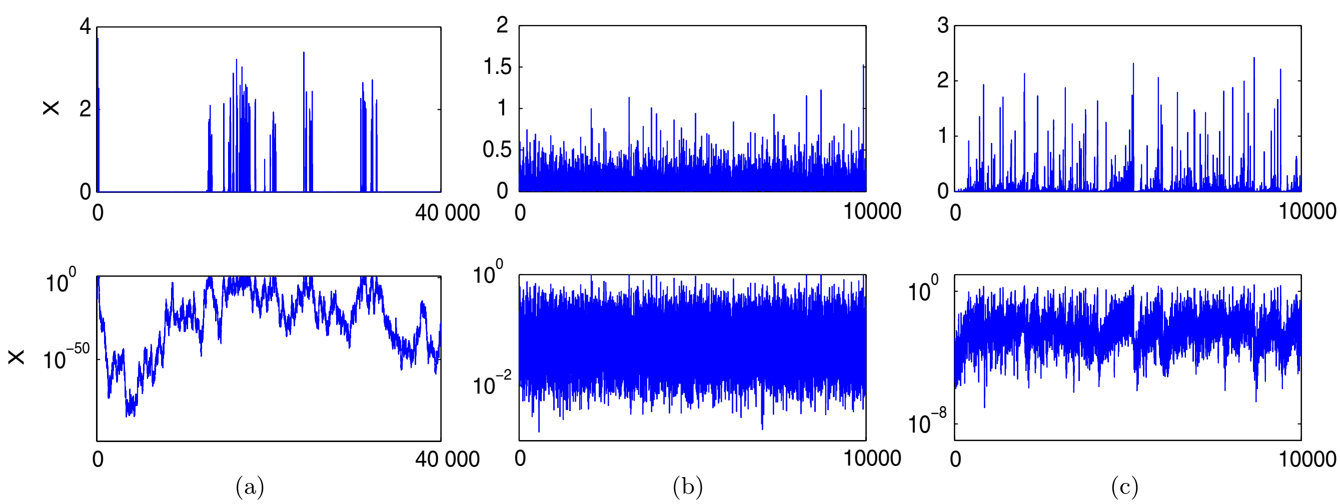

FIG. 1 (color online). Time series of the solution of Eq. (1) for $\mu = 0.01$. Top, linear scale; bottom, log scale. (a) $F = 0$ corresponding to a white noise; (b) $F_{OU} = \gamma Y^2/2$, Ornstein-Uhlenbeck noise with $\gamma = 1.5$; (c) $F_{AN} = \nu|Y|$, with $\nu = 0.75$. Note the differences in the y-coordinate values.

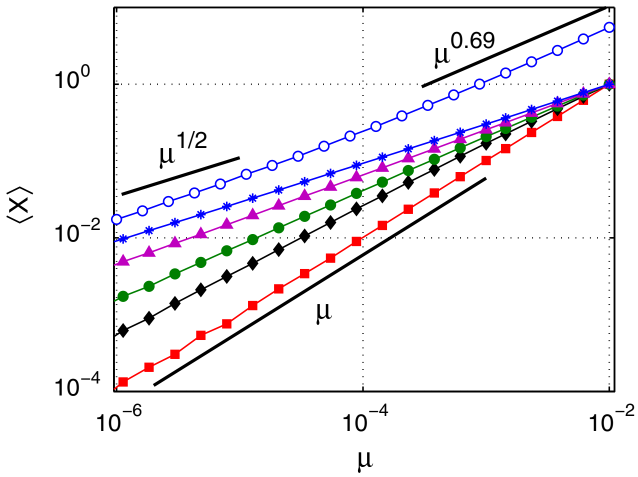

FIG. 2 (color online). First moment $\langle X\rangle$ as a function of μ for the solution of Eq. (1) for $F = F_{AN}$ with (■) $\nu = 0.125$, (◆) $\nu = 1.125$, (●) $\nu = 1.375$, (▲) $\nu = 1.75$, and (★) $\nu = 2.5$. The data are presented in loglog scale and have been normalized by their value at $\mu = 0.01$. The results for $F = F_{OU}$ with $\gamma = 0.2$ (○) are presented and shifted for comparison. The thick continuous lines indicate the exponents 1/2, 0.69, and 1.

agreement with the numerical simulations of the Langevin equations (1).

From this formulation, we can calculate the moments. The exponential term acts as a cutoff for large Ω and is of the form $\exp\{-\mu^{-1}\exp[n\nu\Omega/(n-\nu)]\}$. Therefore for $\mu \to 0$ and $\nu < n$, only very negative Ω have to be considered. In this limit the amplitude $A(\Omega)K_\kappa[\lambda e^{n\Omega/2}]$ tends to a constant, and, after several standard estimates of the asymptotic behavior of the Bessel functions, we obtain for $\nu < n$

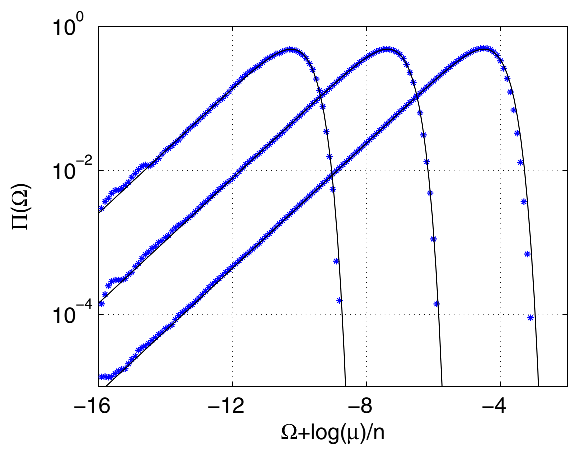

FIG. 3 (color online). Probability density function Π of $\Omega + \log(\mu)/n = \log(X) - Y$. The continuous line is the theoretical prediction and the symbols are the probability density function calculated from the numerical solutions of the Langevin equation. Here $\nu = 1$, and the three curves are associated (from left to right) to $\mu = 1.78 \times 10^{-5}$, $\mu = 3.16 \times 10^{-4}$, and $\mu = 5.6 \times 10^{-3}$.

$$\beta = \min\left[\frac{1}{\nu}, 1\right]. \tag{6}$$

We tested our prediction by numerically calculating the first moment for different values of ν and for $n = 2$ and $n = 3$. The results are shown in Fig. 4. For all cases, the predictions are within the error bars of the numerically calculated values of β, and thus the predictions are verified. To discuss one particular value, the numerically computed exponent for $\nu = 1.5$ and $n = 2$ is $\beta = 0.66 \pm 0.02$, which is in perfect agreement with the theoretical prediction 2/3. We have also performed several numerical simulations using potentials of the form $F = -\nu\sqrt{Y_0^2 + Y^2}$. We have observed that only the behavior of F for large values of $|Y|$ is important. In other words, the universality classes of the problem (i.e., the models having the same critical exponents) are determined by the behavior of the tails of $\Pi(Y)$. Incidentally, this shows that the anomalous scaling is not caused by the nonanalyticity of F at $Y = 0$.

At this stage, we emphasize that our perturbative expansion (in μ) allows us to calculate an exact (nonperturbative) expression for the value of the anomalous exponent. This exponent transitions from its on-off value 1 for $\nu \le 1$ to its deterministic value $1/n$ for $\nu \ge n$. In the simple case of cubic nonlinearities, we predict an exponent between 1/2 and 1. Interestingly enough, the scaling reported in the dynamo experiment belongs to this range.

We have focused here on the first moment of the unstable mode. The behavior of higher moments is also of interest. It can be characterized by the set of exponents β_p defined by $\langle X^p\rangle \propto \mu^{\beta_p}$. In the absence of fluctuations or at usual equilibrium phase transitions, monoscaling is observed, which means that $\beta_p = p\beta_1$. The situation is richer here: There is no linear relation between the exponents (for instance, it can be easily proved that $\beta_n = 1$).

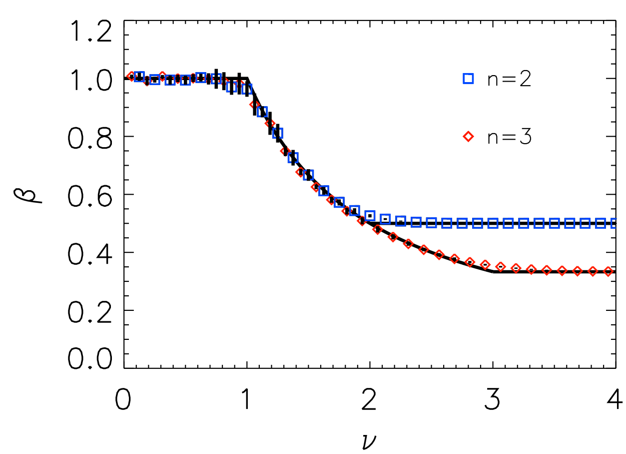

FIG. 4 (color online). Exponents of the first moment as a function of ν for (□) $n = 2$ and (◇) $n = 3$. The continuous lines are the theoretical predictions, and the symbols are obtained from the numerical solutions of the Langevin equation.

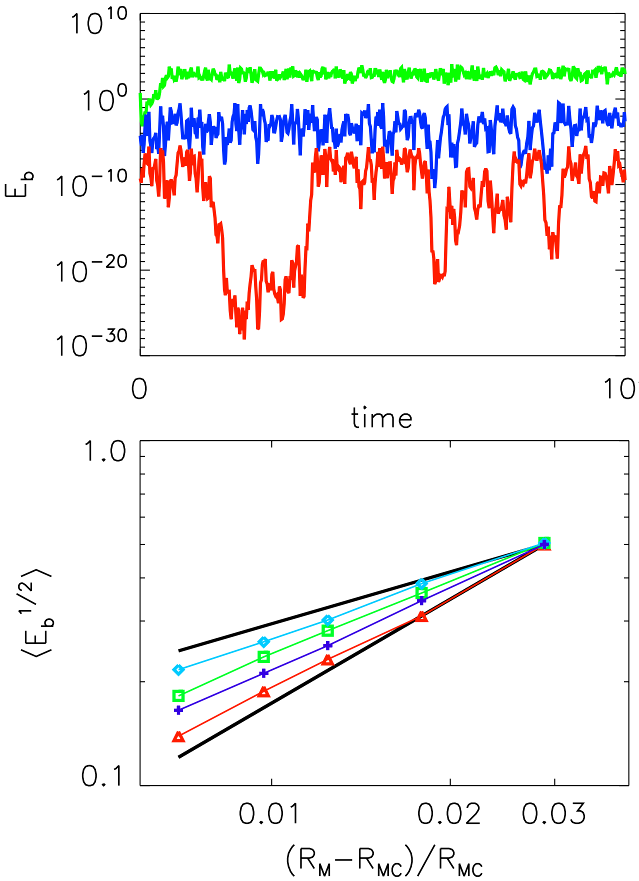

FIG. 5 (color online). (Top) Time series of the space-averaged magnetic energy $E_b = \bar{B}^2$ above the dynamo onset for (from top to bottom) $F = F_{OU}$ with $\gamma = 1$, $F = F_{AN}$ with $\nu = 0.4$, and white noise (see the discussion in the text). The curves have been shifted for clarity. (Bottom) First moment as a function of the departure from onset $(Rm - Rm_c)/Rm_c$ for $F = F_{AN}$ with (△) $\nu = 0$, (+) $\nu = 0.1$, (□) $\nu = 0.4$, and (◇): $\nu = 0.8$. The two thick lines indicate the exponents 1/2 and 1.

Thus, the solutions of model (1) display multiscaling. This is related to the complex structure of the probability density function of X. In particular, it cannot be expressed as a simple one-parameter distribution characterized by its first moment in contrast to the scaling hypothesis close to the critical point of an equilibrium phase transition [13].

Another important issue is the effect of spatial dimension. The model (1) is zero-dimensional (X depends only on time and not on space) while the magnetic field in magnetohydrodynamics (MHD) depends on three spatial dimensions. Analytical predictions for the critical behavior at larger (nonzero) dimensions would be of great interest but are still out of reach at present. To investigate further the pertinence of our model to the dynamo instability, we have performed direct numerical simulations of the MHD equations. To increase our control on the velocity temporal behavior, we used the infinite Prandtl number limit [14]. In this limit, the velocity is slaved to an external mechanical forcing and the Lorentz force

$$\nabla^2 \mathbf{u} = \mathbf{F} + \mathbf{b} \cdot \nabla \mathbf{b} - \nabla \mathbf{P},$$

where $\mathbf{b}$ is the magnetic field and $\mathbf{F}$ is the body force. It is proportional to the ABC flow $\mathbf{F} = A_n[5\sin(z) + 2\cos(y), 2\sin(x) + 5\cos(z), 2\sin(y) + 2\cos(x)]$ (see, for instance, [15]). A_n is an amplitude that changes every time interval τ based on a discrete version of our model $A_{n+1} = A_0 + (Y_{n+1} - Y_n)$ and $Y_{n+1} = Y_n - \tau F(Y_n) + r_n$, where r_n is a random number. The magnetic field satisfies the induction equation

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + Rm^{-1} \nabla^2 \mathbf{b}.$$

The MHD equations were solved in a periodic box of size $2\pi L$ by using a standard pseudospectral code [16] on a grid 32^3. The magnetic Reynolds number defined by $R_m = \langle \|u\|^2 \rangle^{1/2} L/\nu$ was varied above the onset value $Rm_c \simeq 11.65$. In Fig. 5(a), we display time series of the magnetic energy and note that they are similar to those presented in Fig. 1. The first moments are displayed in Fig. 5(b) for several values of ν. We observe that the exponent of the first moment decreases from 1 to 1/2 when ν increases. Estimates of the exponent are computationally demanding so that a quantitative comparison with our model is out of reach. Nevertheless, the results reported here support the robustness of the behavior we have identified.

In summary, we have presented a simple model that results in anomalous exponents which lie between the deterministic value and the on-off intermittent one. The exact value of these exponents was calculated by using an asymptotic expansion. The model emphasizes the role of the noise spectrum at zero frequency. It remains to be understood whether and when turbulent fluctuations can be modeled as the noise considered here [17]. In addition, how such a noise affects other phase transitions and whether the present expansion can capture other critical exponents are interesting open questions.

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