Shell-to-shell energy transfer in magnetohydrodynamics. I. Steady state turbulence

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(Received 19 May 2005; published 4 October 2005)

We investigate the transfer of energy from large scales to small scales in fully developed forced three-dimensional magnetohydrodynamics (MHD) turbulence by analyzing the results of direct numerical simulations in the absence of an externally imposed uniform magnetic field. Our results show that the transfer of kinetic energy from large scales to kinetic energy at smaller scales and the transfer of magnetic energy from large scales to magnetic energy at smaller scales are local, as is also found in the case of neutral fluids and in a way that is compatible with the Kolmogorov theory of turbulence. However, the transfer of energy from the velocity field to the magnetic field is a highly nonlocal process in Fourier space. Energy from the velocity field at large scales can be transferred directly into small-scale magnetic fields without the participation of intermediate scales. Some implications of our results to MHD turbulence modeling are also discussed.

DOI: 10.1103/PhysRevE.72.046301 PACS number(s): 47.65.+a, 47.27.Gs, 95.30.Qd

I. INTRODUCTION

Most astrophysical and planetary systems—e.g., solar and stellar winds, accretion disks, and interstellar medium—are in a turbulent state and coupled to magnetic fields. Understanding and quantifying the statistical properties of magnetohydrodynamical (MHD) turbulence is crucial to explain many physical processes in the cosmos and in industrial flows as well [1]. Although the phenomenology of hydrodynamical (HD) turbulence is understood to some extent and the theory has been able to make predictions like Kolmogorov's 4/5 law and the functional form of the energy spectrum in the inertial range, which have been well verified in experiments and numerical simulations, a similar statement cannot be made for MHD turbulence at the same level. In MHD flows, the two fields (velocity and magnetic) and two associated energies involved in the dynamical processes allow for many possibilities for the energy to transfer between smaller or larger scales, making the dynamics more complex to address in both theory and modeling.

We briefly describe some phenomenological aspects of HD turbulence to point out some of the difficulties usually encountered when the formulation of HD turbulence is applied in the MHD case. To follow the Kolmogorov theory [2] (hereafter, K41), we need to assume a statistically isotropic and homogeneous flow in steady state in which the energy is cascading from eddies of scale l to smaller eddies and so on until the energy reaches dissipation scales. Since we are considering a statistically steady state, the flux of energy to smaller scales has to be constant. We can further assume that the flux at some scale can depend only on the scale l and the amplitude of the velocity field $|\mathbf{u}_l|$ at this scale. This assumption is justified by the argument that larger eddies will only advect smaller eddies without significantly altering their scale and only when eddies of similar size interact do they produce a cascade. Therefore, only "local" interactions among the different scales control the cascade. Here we use the term "local" in terms of the different scales involved (i.e., scales of similar size) and not as locality in physical space. With these assumptions we obtain that the energy $|\mathbf{u}_l|^2$ at the scale l will cascade to smaller scales in a time $l/|\mathbf{u}_l|$, and since the energy cascade rate ϵ is constant, we obtain $\epsilon \sim |\mathbf{u}_l|^3/l$, which implies $|\mathbf{u}_l| \sim l^{1/3}$, which finally leads to the well-verified K41 spectrum $dE/dk \sim k^{-5/3}$ to within small intermittency corrections.

The assumptions of the HD theory of turbulence have been tested in the literature. Reference [3] first tested the assumption of locality using direct numerical simulations (DNS's) of 64^3 grid points. Their work has been followed by a number of authors with higher-resolution simulations [4–10]. References [8,11,12] have also investigated the effect of long-range interactions and anisotropy induced by an anisotropic large-scale flow. Although some issues still remain regarding the effect of long-range interactions, the locality of the energy transfer has been confirmed.

However, there are two important assumptions used in the HD case that are not necessarily true for the MHD case. First, the assumption of isotropy breaks down if an imposed uniform magnetic field is considered. We will not investigate such effects in the present work and will only consider flows with $\int \mathbf{b} d\mathbf{x}^3 = 0$. The second assumption, that of the locality of interactions among the different scales, is what motivates our work. Unlike the HD case where the effect of larger eddies on smaller ones is the advection of the latter ones (an effect that can be taken away by a Galilean transformation), in MHD the effect of a large-scale fluctuation of the magnetic field cannot be so eliminated. Therefore, in MHD it is possible for small scales to interact directly with large scales. If this is the case, we cannot consider a "contiguous" transfer of energy in wave number space and cannot *a priori* follow the same arguments Kolmogorov used for HD turbulence. Therefore knowledge of the energy transfer among different scales is important for the construction of any phenomenological models of MHD turbulence.

Present phenomenological models follow Kolmogorov-like arguments that take into account the effect of the mag-

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netic field. Iroshnikov [13] and Kraichnan [14] (IK) proposed the first models to describe isotropic-MHD turbulence, predicting a spectrum of $k^{-3/2}$. Goldreich and Shridar [15] proposed a new model for anisotropic MHD turbulence that takes into account the anisotropy introduced by a uniform magnetic field $\mathbf{B}_0$, predicting a spectrum of $k_\perp^{-5/3}$, where $k_\perp$ refers to the direction perpendicular to $\mathbf{B}_0$. Several models have been proposed that combine the two spectra (see, e.g., [16–18]), suggesting that the index of the energy spectrum is sensitive to the presence and intensity of $\mathbf{B}_0$. Some aspects of nonlocality of interactions are taken into account in the aforementioned models by considering that large-scale fluctuations of the magnetic field act as a uniform magnetic field to the smaller scales, and as a result they speed up or slow down the rate at which the energy is cascading. However, in these models, although nonlocal interactions are taken into account, the energy is transferred locally from one scale to a slightly smaller scale, like in Kolmogorov's HD turbulence model.

The locality of the interactions and the energy transfer in MHD turbulence has been investigated through various closure models. The energy transfer has been studied within the EDQNM (Eddy Damping Quasi Normal Markovian) closure model by [19] and more recently by [20] where nonlocal interactions have been noted. Using field-theoretical calculations the transfer of energy has been estimated by [21–23]. As far as we know, the locality of the energy transfer in MHD has been investigated through three-dimensional DNS's only very recently [24] (see also [25] for the two-dimensional case). These authors measured the transfer of energy between different scales and fields using free-decaying MHD turbulence simulations with 512^3 grid points. Their results showed that there is local transfer of energy between the same fields, while the transfers involving the two different fields showed a less local behavior, in the sense that a wider range of scales was involved in the interactions.

In our work we use the results of the DNS's of mechanically forced MHD turbulence (unlike the free-decaying case studied in [24]) to study the locality of the energy transfer between different scales and fields. In all the cases studied we consider a mechanical external forcing that generates a well-defined large-scale flow and small-scale turbulent fluctuations. This is a regime of interest for several astrophysical and geophysical flows where magnetic fields are believed to be sustained against Ohmic dissipation by a dynamo process [26] and the only external source of energy driving the system is mechanical (e.g., convection and rotation). There is an important difference between the case studied in [24] and the case considered in our work. In our case energy is forced through the velocity field and the system reaches a steady state with equipartition between the two fields. For this to happen there must be a nonzero flux for all times from the velocity field to the magnetic field. This is not necessarily true for the case of decaying turbulence, and as our results show this significantly modifies the energy transfers from the velocity field to the magnetic field.

In Sec. II we introduce the definitions of the transfer terms for MHD, and in Sec. III we present the code we use for the numerical simulations as well as the results of the analysis. Finally, in Sec. IV we summarize the main results of our work.

II. THEORY AND DEFINITIONS

The equations that describe the dynamics of an incompressible conducting fluid coupled to a magnetic field in the MHD approximation are given by

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f}, \tag{1}$$

$$\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{b}, \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0, \tag{3}$$

where $\mathbf{u}$ is the velocity field and $\mathbf{b}$ is the magnetic field. p is the (total) pressure, and ν and η are the viscosity and magnetic diffusivity, respectively. Here, $\mathbf{f}$ is the external force that drives the turbulence and the dynamo. The largest wave number of the Fourier transform of f is going to be denoted as $\mathbf{k}_F$, and we are going to refer to $|\mathbf{k}_F|^{-1}$ as the forced scale. We are also going to define the viscous dissipation scale as $k_\nu^{-1}=(\epsilon/\nu^3)^{-1/4}$ and resistive dissipation scale as $k_\eta^{-1}=(\epsilon/\eta^3)^{-1/4}$ where ϵ is the energy dissipation rate. A large separation between the two scales ($|\mathbf{k}_F|^{-1} \gg \max\{\{k_\nu^{-1},k_\eta^{-1}\}$) is required for the flow to reach a turbulent state.

To investigate the transfer of energy among different scales of turbulence we use the Fourier transforms of the fields:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \tilde{\mathbf{u}}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int \mathbf{u}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}^3$$

and

$$\mathbf{b}(\mathbf{x}) = \sum_{\mathbf{k}} \tilde{\mathbf{b}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \tilde{\mathbf{b}}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int \mathbf{b}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}^3,$$

where the domain is taken to be a triply periodic cube of size $L=2\pi$. We can now introduce the shell filter decomposition

$$\mathbf{u}(\mathbf{x}) = \sum_K \mathbf{u}_K(x), \quad \mathbf{b}(\mathbf{x}) = \sum_K \mathbf{b}_K(x),$$

where

$$\mathbf{u}_K(x) = \sum_{K<|\mathbf{k}|\le K+1} \tilde{\mathbf{u}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},$$

and similarly for the field $\mathbf{b}$,

$$\mathbf{b}_K(x) = \sum_{K<|\mathbf{k}|\le K+1} \tilde{\mathbf{b}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The fields $\mathbf{u}_K$ and $\mathbf{b}_K$ are therefore defined as the part of the velocity and magnetic field, respectively, whose Fourier transform contains only wave numbers in the shell $(K,K+1]$ (hereafter called shell K) and represent "eddies" of scale K^{-1}. The evolution of the kinetic energy in a shell K, $E_u(K)=\int \mathbf{u}_K^2/2dx^3$ is given by

$$\partial_t E_u(K) = \int \sum_Q [-\mathbf{u}_K\cdot(\mathbf{u}\cdot\nabla)\cdot\mathbf{u}_Q + \mathbf{u}_K\cdot(\mathbf{b}\cdot\nabla)\cdot\mathbf{b}_Q] - \nu|\nabla\mathbf{u}_K|^2 + \mathbf{f}\cdot\mathbf{u}_K d\mathbf{x}^3, \tag{4}$$

and for the magnetic energy $E_b(K)=\int \mathbf{b}_K^2/2dx^3$ we obtain

$$\partial_t E_b(K) = \int \sum_Q [-\mathbf{b}_K \cdot (\mathbf{u} \cdot \boldsymbol{\nabla}) \cdot \mathbf{b}_Q + \mathbf{b}_K \cdot (\mathbf{b} \cdot \boldsymbol{\nabla}) \cdot \mathbf{u}_Q] - \eta |\nabla \mathbf{b}_{\mathbf{K}}|^2 d\mathbf{x}^3. \tag{5}$$

The above equations can be written in the more compact form

$$\partial_t E_u(K) = \sum_Q [\mathcal{T}_{uu}(Q,K) + \mathcal{T}_{bu}(Q,K)] - \nu \mathcal{D}_u(K) + \mathcal{F}(K), \tag{6}$$

$$\partial_t E_b(K) = \sum_Q [\mathcal{T}_{ub}(Q,K) + \mathcal{T}_{bb}(Q,K)] - \eta \mathcal{D}_b(K). \tag{7}$$

Here we have introduced the functions $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{ub}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, and $\mathcal{T}_{bu}(Q,K)$, which express the energy transfer between different fields and shells.

$\mathcal{T}_{uu}(Q,K)$ expresses the transfer rate of kinetic energy lying in shell Q to kinetic energy lying in shell K through the velocity advection term and is defined as

$$\mathcal{T}_{uu}(Q,K) \equiv -\int \mathbf{u}_{\mathbf{K}}(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}_{\mathbf{Q}} d\mathbf{x}^3. \tag{8}$$

We similarly define

$$\mathcal{T}_{bb}(Q,K) \equiv -\int \mathbf{b}_{\mathbf{K}}(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{b}_{\mathbf{Q}} d\mathbf{x}^3, \tag{9}$$

which expresses the rate of energy transfer of magnetic energy lying in shell Q to magnetic energy lying in shell K through the magnetic advection term. The Lorentz force is responsible for the transfer of energy from the magnetic field to the velocity field. The resulting transfer rate is defined as

$$\mathcal{T}_{bu}(Q,K) \equiv \int \mathbf{u}_{\mathbf{K}}(\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{b}_{\mathbf{Q}} d\mathbf{x}^3. \tag{10}$$

Finally the term responsible for the stretching of the magnetic field lines results in the transfer from kinetic energy to magnetic energy, given by

$$\mathcal{T}_{ub}(Q,K) \equiv \int \mathbf{b}_{\mathbf{K}}(\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{u}_{\mathbf{Q}} d\mathbf{x}^3. \tag{11}$$

In summary, the functions $\mathcal{T}_{vw}(Q,K)$ (for arbitrary fields $\mathbf{v}$ and $\mathbf{w}$) represent the rate of transfer of energy from the field $\mathbf{v}$ (first index) in shell Q (first argument) into energy of the field $\mathbf{w}$ (second index) in shell K (second argument). If $\mathcal{T}_{vw}(Q,K) > 0$, then a positive amount of v energy is transferred from shell Q to w energy in shell K. If $\mathcal{T}_{vw}(Q,K) < 0$, then a negative amount of v energy is transferred from shell Q to w energy in shell K, or in other words, energy is transferred backwards from shell K to shell Q.

In Eqs. (6) and (7) we have also introduced two dissipation functions: the kinetic energy dissipation rate

$$\nu \mathcal{D}_u(K) \equiv \nu \int |\nabla \mathbf{u}_K|^2 d\mathbf{x}^3 \tag{12}$$

and the magnetic energy dissipation rate

$$\eta \mathcal{D}_b(K) \equiv \eta \int |\nabla \mathbf{b}_K|^2 d\mathbf{x}^3. \tag{13}$$

Finally,

$$\mathcal{F}(K) \equiv \int \mathbf{f} \cdot \mathbf{u}_K d\mathbf{x}^3 \tag{14}$$

is the energy injection rate to the velocity field through the forcing term.

Before presenting the results from numerical simulations, let us discuss some of the properties of the transfer functions. If $\mathcal{T}_{vw}(Q,K)$ (where v,w can be either u or b) is expressing the rate of energy transfer from field $\mathbf{v}$ in shell Q to field $\mathbf{w}$ in shell K, then the following identity should hold:

$$\mathcal{T}_{vw}(Q,K) = -\mathcal{T}_{wv}(K,Q). \tag{15}$$

The interpretation of Eq. (15) is that the rate at which shell Q is giving energy to shell K must be equal to rate shell K is receiving energy from shell Q. Equation (15) can be easily shown to hold for all transfer functions we defined [Eqs. (8)–(11)]. It is this property that allows us to interpret the functions $\mathcal{T}_{uu}$, $\mathcal{T}_{bu}$, $\mathcal{T}_{ub}$, and $\mathcal{T}_{bb}$ as the energy transfer between different scales and fields.

For a turbulent flow in a statistically steady state, Eqs. (6) and (7) imply that

$$\sum_Q \langle \mathcal{T}_{uu}(Q,K) + \mathcal{T}_{bu}(Q,K) \rangle = \langle \mathcal{D}_u(K) \rangle - \langle \mathcal{F}(K) \rangle \tag{16}$$

and

$$\sum_Q \langle \mathcal{T}_{ub}(Q,K) + \mathcal{T}_{bb}(Q,K) \rangle = \langle \mathcal{D}_b(K) \rangle, \tag{17}$$

where $\langle \cdot \rangle$ stands for a time average or an ensemble average. For fixed K outside the forcing band and in the limit of $\nu, \eta \to 0$, we have that

$$\sum_Q \langle \mathcal{T}_{uu}(Q,K) + \mathcal{T}_{bu}(Q,K) \rangle = 0 \tag{18}$$

and

$$\sum_Q \langle \mathcal{T}_{ub}(Q,K) + \mathcal{T}_{bb}(Q,K) \rangle = 0. \tag{19}$$

However, limited resolution will allow us to be in the regime where these last two equations hold only for a small range of wave numbers.

Finally we need to comment on the definitions of the various transfer functions we are using in this paper and the connection to the triad of wave numbers $(\mathbf{k},\mathbf{p},\mathbf{q})$ that satisfy the relation $\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}$ (because of the convolution term resulting from the quadratic nonlinearities of the primitive equations); such a triad is the basis for mode-to-mode interactions (see, e.g., [27]). Our approach is equivalent to considering all triad interactions with the one wave number $\mathbf{k} \in K$ and $\mathbf{q} \in Q$ and summing over all $\mathbf{p}$ satisfying $\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}$ in all shells, where $\mathbf{p}$ is the wave number of the advecting field and $\mathbf{k}$ and $\mathbf{q}$ are the wave numbers of the modes energy is transferred to and from. Although the approach we are using gives us information on whether the energy is trans-

TABLE I. Simulations. L is the integral length scale of the flow, defined as $L=\int E_u(k)dk/\int E_u(k)k^{-1}dk$, ν the kinematic viscosity, and η the magnetic diffusivity. The kinetic and magnetic Reynolds numbers Re and R_M are based on L and the rms velocity, while the ratio of magnetic to kinetic energy, E_b/E_u, is the average in the turbulent steady state.

Forcing	L	ν	η	Re	R_M	E_b/E_u
ABC	1.64	2×10^{-3}	2×10^{-3}	820	820	0.84
TG	1.35	2×10^{-3}	5×10^{-3}	675	270	0.72

ferred locally or not, it cannot give definite conclusions on whether the interactions themselves are local. For example, even if energy is transferred locally from a wave number $\mathbf{k}$ to a wave number $\mathbf{q}\sim\mathbf{k}$, the wave number $\mathbf{p}$ that is responsible for the transfer is not necessarily of the same order of magnitude as $|\mathbf{k}|$ and $|\mathbf{q}|$. Ideally, one would investigate transfer terms of the form $\mathbf{T}_{uu}(K|P|Q)\equiv\int\mathbf{u}_K(\mathbf{u}_P\cdot\boldsymbol{\nabla})\mathbf{u}_Q d\mathbf{x}^3$, which contain information about the third wave number involved in the interactions taking place. However, the difficulty of manipulating data from high-resolution runs and the difficulty of interpreting the results of transfer functions that depend on three arguments restricts us, for the present time, to examine just the locality of the energy transfer.

III. RESULTS

To study the transfer of energy in MHD turbulence we use the turbulent steady state of several mechanically forced three-dimensional MHD direct numerical simulations. The simulations and details of the code can be found in [28,29]. The runs were performed in a triply periodic domain with a resolution of 256^3 grid points using a pseudospectral scheme with the 2/3 rule for dealiasing. The equations were evolved in time using a second-order Runge-Kutta method.

Turbulence was generated by two different types of forcing. In the first case a nonhelical Taylor-Green (TG) force was used, $\mathbf{f}_{\mathrm{TG}}(k_0)=(\sin(k_0x)\cos(k_0y)\cos(k_0z),-\cos(k_0x)\times\sin(k_0y)\cos(k_0z),0)$ with $k_0=2$ [28]. In the second case a helical ABC force was used, $\mathbf{f}_{ABC}(k_0)=(B\cos(k_0y)+C\sin(k_0z),C\cos(k_0z)+A\sin(k_0x),A\cos(k_0x)+B\sin(k_0y))$ with $k_0=2$ [29]. All simulations were done with constant-in-time external force. First a hydrodynamic simulation was carried using each force to reach a turbulent steady state. Both external forces generate a well-defined large-scale flow at $|K_F|\sim3$ and small-scale turbulent fluctuations following to a good approximation a 5/3 Kolmogorov law. Then MHD simulations were carried, and a small magnetic field was amplified and sustained to equipartition by a dynamo process. The results in this paper are based on the saturated stage of the dynamo, which we will refer in the following as the MHD turbulent steady state.

The transfers were calculated based on the definitions (8)–(11). The transfer of energy during the early stages of the MHD simulations, when the magnetic energy is small and the velocity field is not modified by the Lorentz force (often referred to as the kinematic dynamo regime), is examined in a companion paper [30] (hereafter referred as paper II). Table I gives several relevant parameters for each run, and Fig. 1 shows the resulting energy spectra.

Both simulations display a large-scale magnetic field, although the spectrum of magnetic energy in the ABC simulation shows a stronger peak at $k=1$. This peak is related to the dynamo α effect and the inverse cascade of magnetic helicity. Details of this process will be discussed in paper II. However, it is important to note that in the ABC simulation the large-scale magnetic field is strongly helical, while in the TG simulation the magnetic helicity is negligible. This large-scale magnetic field is self-sustained by the turbulence. In both simulations, the net cross helicity (correlation between the velocity and magnetic field) is small and can be neglected.

A. Hydrodynamic turbulence

The locality of interactions in hydrodynamic turbulence has been investigated before in the literature [4–10]. Although some open issues still remain [8,11,12], it has been shown that energy is transferred mostly locally. Here, for reasons of comparison we show the transfer $\mathcal{T}_{uu}(Q,K)$ from hydrodynamical simulations using the same external forces and parameters used in the MHD simulations. The results are in good agreement with previous works.

In Fig. 2 we show the energy transfer for a few modes for the TG flow and in Fig. 3 the energy transfer for the ABC flow. In both cases the transfer of energy is direct and local: all the curves (with the exception of the forced mode $Q=3$) are negative for K smaller than Q and positive for K larger than Q. As a result, all inertial-range modes receive energy

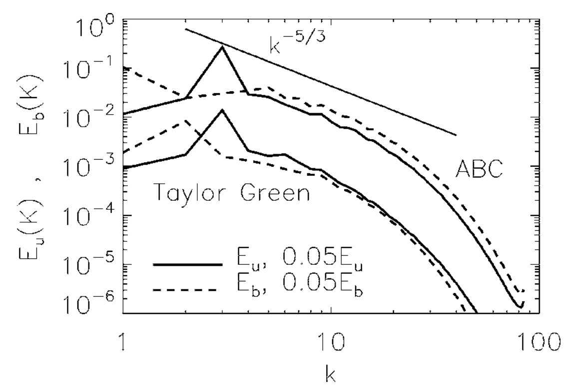

FIG. 1. Spectra of kinetic energy (solid line) and magnetic energy (dashed line) of the ABC and Taylor-Green runs, where the Taylor-Green spectra have been shifted down by a factor of 20 for clarity. The Kolmogorov slope is showed as a reference. Note that the magnetic Prandtl number $P_M\equiv\nu/\eta$ differs for the two runs.

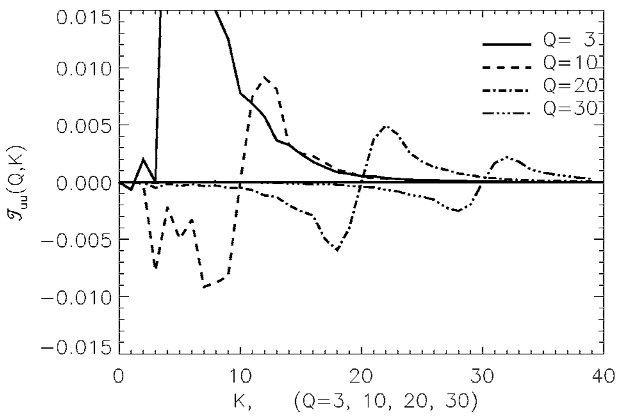

FIG. 2. The transfer of energy $\mathcal{T}_{uu}(Q,K)$ for the Taylor-Green run. The figure shows the rate that energy is transferred from modes $Q=3,10,20,30$ to all other modes K.

from modes with slightly smaller wave numbers (negative $\mathcal{T}_{uu}$) and give energy to modes with slightly larger wave numbers (positive $\mathcal{T}_{uu}$). The locality of the transfer is expressed from the fact that the transfer of energy from the modes in shell Q to modes in shells K with $K \ll Q$ or $K \gg Q$ is very small and decreases fast with the separation of the two wave numbers. Finally, as the shell wave number K and Q is increased, there is a drop in the amplitude of the transfer. If the transfer functions were self-similar, then an increase of the wave numbers K and Q to λK and λQ would imply $\mathcal{T}_{uu}(\lambda Q,\lambda K)=\lambda^{-2}\mathcal{T}_{uu}(Q,K)$ [27]. This scaling could explain this drop of amplitude. However, the inertial range in our DNS's is too small to test self-similarity and a large part of the drop is due to the presence of viscosity.

The forced mode has a slightly different behavior. The transfer rate from the forced wave number to its nearby shells has a considerably larger amplitude. Also, for both flows there is some backscattering from the forced wave number to shells with smaller wave number. This is clearer in the helical (ABC) flow.

B. Magnetohydrodynamic turbulence

We are now ready to examine results from the energy transfer for MHD turbulence. First we examine the transfer

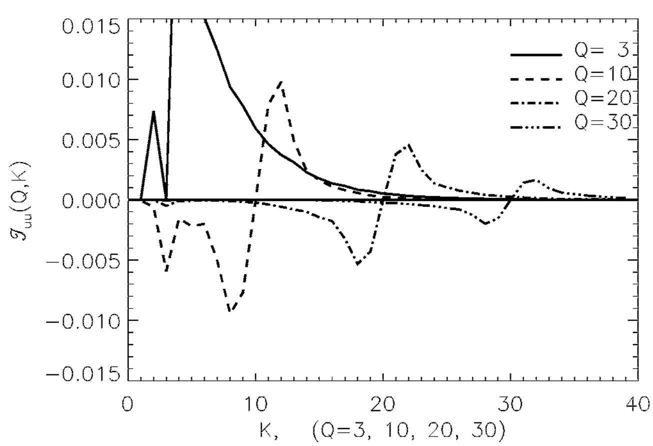

FIG. 3. The transfer of energy $\mathcal{T}_{uu}(Q,K)$ for the ABC run. The figure shows the rate that energy is transferred from modes $Q=3,10,20,30$ to all other modes K.

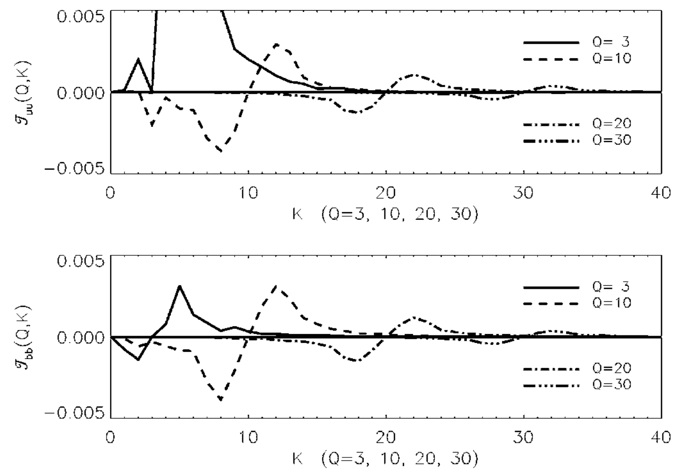

FIG. 4. Top panel: the transfer of energy $\mathcal{T}_{uu}(Q,K)$ for the Taylor-Green run. The figure shows the rate that kinetic energy is transferred from modes $Q=3,10,20,30$ to kinetic energy to all other modes K. Bottom panel: the transfer of energy $\mathcal{T}_{bb}(Q,K)$ for the same flow. The figure shows the rate that magnetic energy is transferred from modes $Q=3,10,20,30$ to magnetic energy to all other modes K.

of kinetic energy from large scales to kinetic energy in small scales through the term $\mathcal{T}_{uu}(Q,K)$ and magnetic energy from large scales to magnetic energy in small scales through the term $\mathcal{T}_{bb}(Q,K)$. These two transfer functions bare some significant similarities with the hydrodynamic case.

In Figs. 4 and 5 we show $\mathcal{T}_{uu}$ (top panel) and $\mathcal{T}_{bb}$ (bottom panel) for the nonhelical TG flow and the helical ABC flow. The velocity-to-velocity transfer has not changed drastically (other than a decrease in amplitude) from the pure hydrodynamic case. As in Sec. III A, the transfer implies a local direct cascade. All curves are negative for K smaller than Q and positive for K larger than Q. Each mode is therefore receiving energy from the larger scales (negative transfer) and giving energy to the smaller scales (positive transfer). The decrease in amplitude (when compared with the hydrodynamic case) is partly because the magnitude of the velocity field is decreased when the magnetic field comes to equipartition and partly because now there is a net transfer of

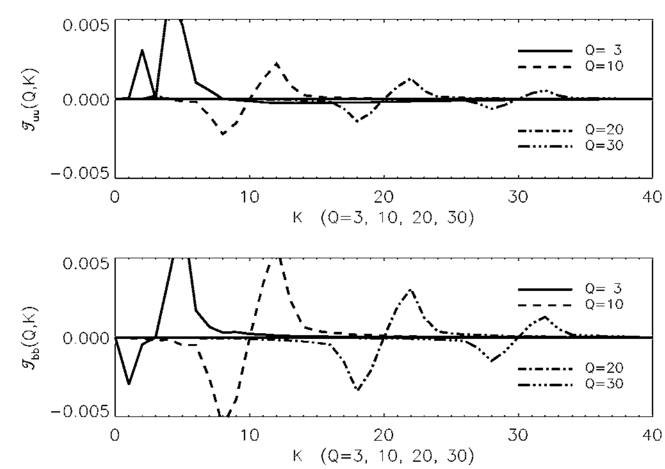

FIG. 5. Same as Fig. 4 for the ABC run.

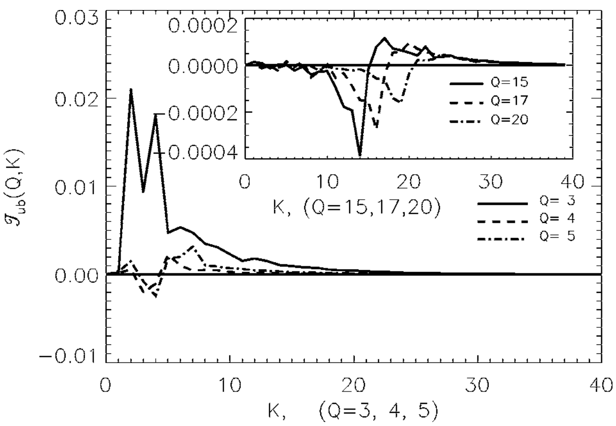

FIG. 6. The transfer of kinetic energy to magnetic energy $\mathcal{T}_{ub}(Q,K)$ for the Taylor-Green run. The figure shows the rate that kinetic energy is transferred from modes $Q=3,4,5$ (inset modes $Q=15,17,20$) to magnetic energy in modes K.

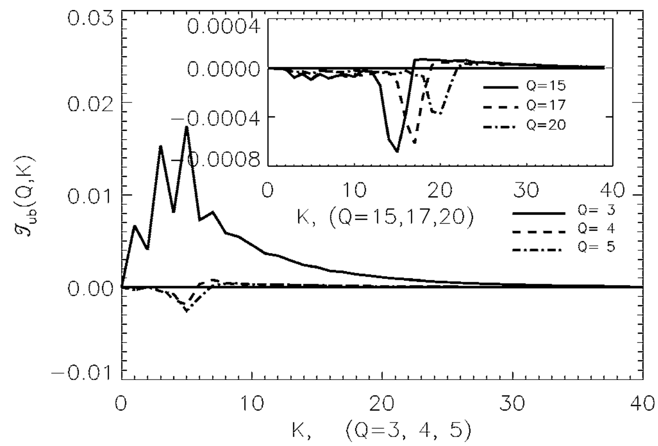

FIG. 7. The transfer of kinetic energy to magnetic energy $\mathcal{T}_{ub}(Q,K)$ for the ABC run. The figure shows the rate that kinetic energy is transferred from modes $Q=3,4,5$ (inset modes $Q=15,17,20$) to magnetic energy in modes K.

energy from the velocity field to the magnetic field, making the available energy to cascade to small velocity scales smaller.

The transfer of magnetic energy to magnetic energy $\mathcal{T}_{bb}(Q,K)$ seems to follow the same behavior as the velocity field transfer. The results show a direct cascade with local transfer of energy from large scales to small scales. We note that for the helical case the transfer of magnetic energy is larger than the transfer of kinetic energy. The likely reason for this behavior is that in the ABC flow the magnetic energy at large scales and intermediate scales saturates at higher values than in the TG flow, due to the presence of helicity or the dynamo α effect. This process will be discussed in more detail in paper II.

Next we investigate the transfer of energy from one field to the other by examining the terms $\mathcal{T}_{ub}$ and $\mathcal{T}_{bu}$. Because of the antisymmetric property $\mathcal{T}_{ub}(Q,K)=-\mathcal{T}_{bu}(K,Q)$, it is sufficient to just study the transfer of energy from the velocity field to the magnetic field. However, we need to remark that unlike the $\mathcal{T}_{uu}(Q,K),\mathcal{T}_{bb}(Q,K)$ terms that their dependence on K and Q is the same up to a minus sign, the behavior of $\mathcal{T}_{ub}(Q,K)$ as we vary K is not the same as if we vary Q. Therefore the two behaviors need to be studied separately (i.e., the transfer of energy from a velocity mode to two different magnetic modes is different from the transfer of energy from two different velocity modes to a magnetic mode). In Figs. 6 (TG) and 7 (ABC), we show the transfer of kinetic energy from velocity modes Q=3, 4, 5, 15, 17, and 20 to all examined magnetic modes K.

A few things should be noted. First, in both runs (ABC and TG) the modes associated with the large-scale flow ($Q=3$) seem to play a dominant role in the transfer of energy from the velocity field to the magnetic field. Note also that there is a wider range of magnetic field modes into which the forced velocity field modes input energy.

This is more apparent for the helical flow, which seems better at stretching and folding the magnetic field. The cascade in the modes inside the inertial range is direct in both cases but with a small difference. In both cases the large-scale velocity field is transferring energy to a smaller-scale magnetic field and receiving energy from a larger-scale magnetic field. However, for the Taylor-Green case there is a very small transfer from one field to the other in the same shell. On the other hand, in the ABC flow the peak of the transfer from the magnetic field to the velocity field (the negative peaks in Fig. 7) is for the same shell. Note also that for K shells larger than Q, the transfer for all Q follows the same curve. This implies that all small-scale velocity modes give energy to the magnetic field modes at the same rate. This is clearer when we examine the dependence with Q.

In Figs. 8 and 9 we show the same transfer function $\mathcal{T}_{ub}(Q,K)$ for three values of $K=10,20,30$. The energy cascade is also direct (energy going from large scales to small scales); however, it is clear from these figures that the transfer from the velocity field to the magnetic field is a highly nonlocal process. Each magnetic field mode Q is receiving energy (positive $\mathcal{T}_{ub}$) from all the velocity modes with wave number K smaller than Q, with the same rate. The only ex-

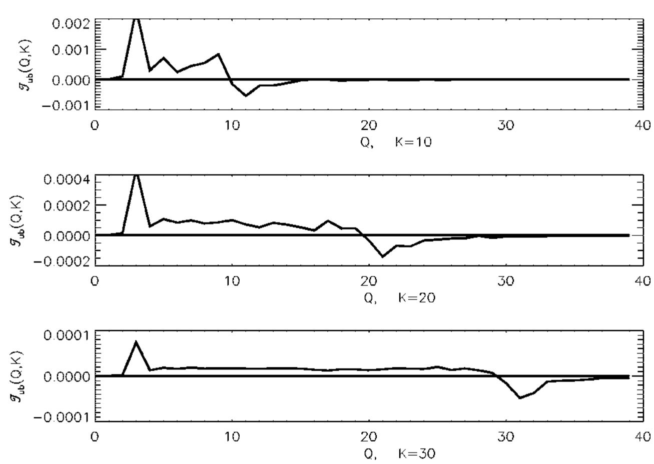

FIG. 8. The transfer of kinetic energy to magnetic energy $\mathcal{T}_{ub}(Q,K,)$ for the Taylor-Green run. The figure shows the rate that kinetic energy is transferred from modes Q (x axis) to magnetic energy in modes $Q=10$ (top panel), $Q=20$ (middle panel), and $Q=30$ (bottom panel).

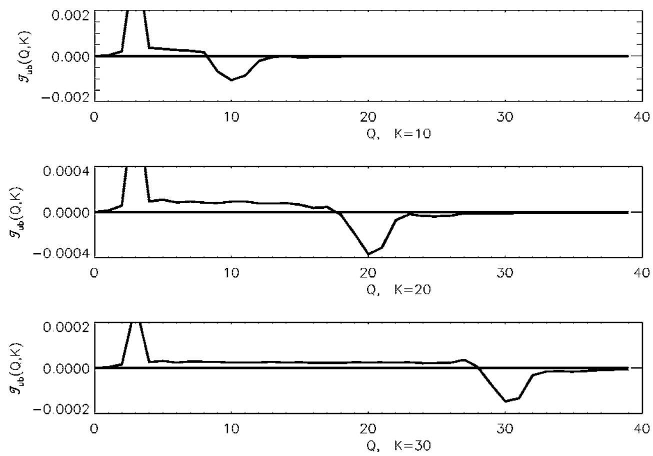

FIG. 9. The transfer of kinetic energy to magnetic energy $\mathcal{T}_{ub}(Q,K,)$ for the ABC run. The figure shows the rate that kinetic energy is transferred from modes Q (x axis) to magnetic energy in modes $Q=10$ (top panel), $Q=20$ (middle panel), and $Q=30$ (bottom panel).

ception is the mechanically sustained large-scale velocity field, which gives even more energy (observe the peak at $k=3$). In fact, most of the energy that is transferred from the velocity field to the magnetic field originates from the velocity field modes at $Q=3$ (around 60% for the TG run and 75% for the ABC run). Similar behavior has been observed in two-dimensional MHD flows [25]. This energy turns into magnetic energy at several wave numbers K which locally cascades to smaller scales through the $\mathcal{T}_{bb}$ term. This bigger contribution of the large-scale flow to $\mathcal{T}_{ub}$ (compared with the contribution of the turbulent components) is in good agreement with the suppression of small-scale velocity fluctuations by the large-scale magnetic field, as observed in [28]. However, we need to note that the fraction of energy input from the large-scale flow might be a function of the Reynolds number. As the scale of the magnetic field becomes smaller, there is more energy input from the turbulent components of the velocity field than from the large-scale (forced) flow. This just follows from the fact that for K large enough, the area below the curve with constant $\mathcal{T}_{ub}$ is larger than the peak at $Q=3$. It is possible therefore that in the limit of large inertial range the effect of the forced velocity scales in small magnetic scales will not be as strong. Finally we note that this mechanism described above is different in a kinematic dynamo regime, as is shown in paper II.

In summary, the existence of the long plateau with constant $\mathcal{T}_{ub}(Q,K)$ at each fixed value of K and the fact that all magnetic wave numbers K receive energy from the large-scale flow at $Q=3$ points out that interactions between the velocity field and magnetic field are nonlocal in Fourier space.

This nonlocal behavior of energy transfer from the velocity field to the magnetic field seems to be absent from the decaying MHD turbulence case studied by [24]. In that case, although the $\mathcal{T}_{ub}$ and $\mathcal{T}_{bu}$ were more nonlocal than the $\mathcal{T}_{bb}$ and $\mathcal{T}_{uu}$ terms (since energy was transferred from the former ones in a wider range of shells than the latter ones), eventually at large separation of wave numbers the transfer goes to

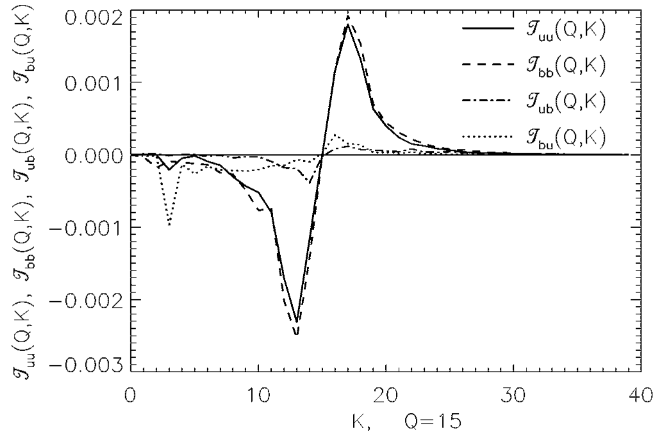

FIG. 10. A comparison of the transfers $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, $\mathcal{T}_{ub}(Q,K)$, and $\mathcal{T}_{bu}(Q,K)$ for $Q=15$ for the Taylor-Green flow.

zero. This is very different from the plateau behavior we observe in the forced turbulence runs. We suspect that this difference is due to the fact that in the mechanically forced turbulence there is a net flux of energy from the velocity field to the magnetic field that is responsible for the formation of the plateau which does not exist in the decaying turbulence case.

C. Comparison between the transfers

In the previous section we showed that the transfer of energy from velocity field to velocity field and from magnetic field to magnetic field exhibits a local behavior similar to the transfer in hydrodynamic turbulence, and the transfer from one field to the other is exhibiting a nonlocal behavior. In order to draw conclusions we need to compare the magnitude of these transfers. Figures 10 and 11 show a comparison of the transfers $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, $\mathcal{T}_{ub}(Q,K)$, and $\mathcal{T}_{bu}(Q,K)$ with $Q=15$ for the TG and ABC runs, respectively. The local transfers u to u and b to b appear to be of larger magnitude than the nonlocal transfers u to b and b to u. In the case of the ABC flow the magnitude of the b-to-b transfer seems to be twice the magnitude of the u-to-u transfer. This

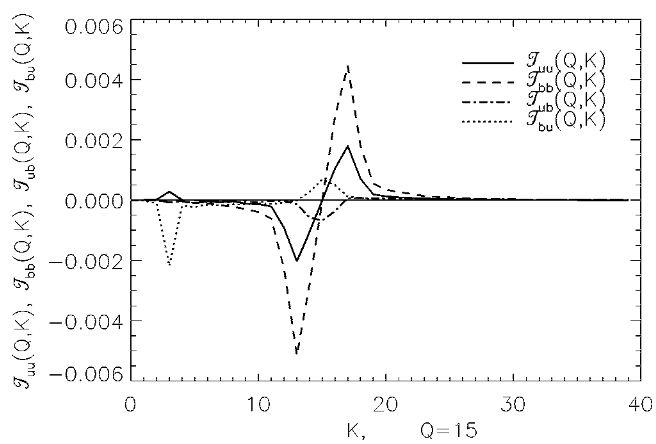

FIG. 11. A comparison of the transfers $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, $\mathcal{T}_{ub}(Q,K)$, and $\mathcal{T}_{bu}(Q,K)$ for $Q=15$ for the ABC flow.

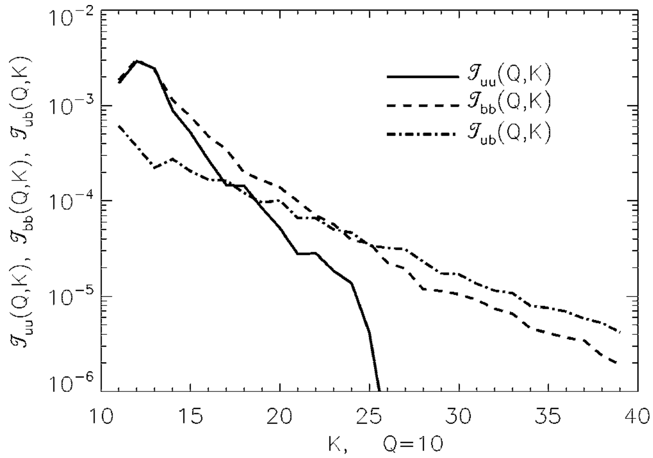

FIG. 12. A comparison of the large K tails of transfers $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, and $\mathcal{T}_{ub}(Q,K)$ in a log-linear plot for $Q=10$ for the Taylor-Green flow flow.

is due to the fact that the magnetic energy in this run is larger than in the TG run at large and intermediate scales.

Figure 12 illustrates the transfer functions $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, and $\mathcal{T}_{ub}(Q,K)$ as in Fig. 10 (TG flow), but we focus here on the large-K tail of the transfer and we consider $Q=10$. The fastest drop is for the transfer $\mathcal{T}_{uu}(Q,K)$, making it the most "local" one, next comes the $\mathcal{T}_{bb}(Q,K)$ transfer, and finally $\mathcal{T}_{ub}(Q,K)$ has the slowest drop. The same result was obtained for the ABC flow (not shown here).

Figures 10–12 (as well as a comparison of the nonlocal transfers shown in Figs. 8 and 9 with the local transfers in Figs. 4 and 5, respectively) show that local interactions between the same fields are much stronger than nonlocal interactions between different fields. However, nonlocal interactions spread over several shells, and the magnetic field at a given scale K can receive (give) energy from (to) several velocity field Q wave numbers (instead of mostly the nearest neighbors as is the case for local interactions). Figure 13 shows the ratio

$$\frac{NL}{L}(K) = \sum_{Q=1}^{K} \mathcal{T}_{ub}(Q,K) \Bigg/ \sum_{Q=1}^{K} \mathcal{T}_{bb}(Q,K). \tag{20}$$

This is the ratio of the total energy that the magnetic field at shell K receives from the velocity field through nonlocal

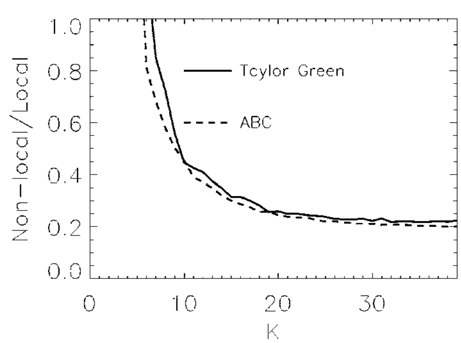

FIG. 13. The ratio NL/L of energy received through the nonlocal transfer $\mathcal{T}_{ub}$ to local $\mathcal{T}_{bb}$ for the ABC and TG simulations. The small scales receive 20% of their energy through the nonlocal transfer $\mathcal{T}_{ub}$.

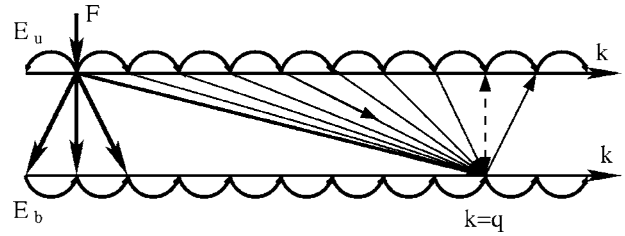

FIG. 14. A sketch of the energy transfer between different scales and different fields. The thickness of the lines is an indication of the magnitude of the transfers. The figure illustrates how energy is transferred to magnetic modes with wave number $k=q$ in the inertial range. The transfer between same fields is always local and direct. Each magnetic mode receives energy from all larger-scale velocity modes and gives to slightly smaller-scale velocity modes.

transfer to the total magnetic energy received at the same scale through the local direct cascade of (magnetic) energy. Although in individual shells the local interactions are one order of magnitude larger than the nonlocal transfer, the net amount of energy received at a given scale K by the two processes is comparable (this ratio is different in a kinematic dynamo regime, as will be shown in paper II). At small scales, the ratio seems to settle to a value close to 0.2, indicating that 20% of the energy received by these scales is through the nonlocal transfer $\mathcal{T}_{ub}$. We note that this ratio can possibly depend on the Reynolds number since local terms are more sensitive to viscosity.

IV. CONCLUSIONS

In this paper we examined the transfer of energy in forced MHD turbulence between the different scales and fields involved using the results from numerical simulations in a turbulent steady state sustained by a mechanical external force. No qualitative differences in the transfer of kinetic-to-kinetic (or magnetic-to-magnetic) energy has been observed when compared against the transfer of energy in a hydrodynamic simulation. These transfers were found to be always local and direct. However, all kinetic energy modes have been observed to give energy to magnetic modes nonlocally, in the sense that a small-scale magnetic field receives the same amount of energy from all larger scales of the velocity field in the inertial range. Also each magnetic mode was found to receive a significant amount of energy from the large-scale flow at $|k_F|\sim 3$ (the scale of the forcing), an effect that seems to become smaller as we move to smaller scales in the inertial range. We note that it is the nonlocal interactions that actually sustain the magnetic field against Ohmic dissipation. No qualitative difference in the transfers was observed between the different forcing runs, suggesting that the nonlocality of the energy transfer is a general feature of mechanically forced MHD turbulence. A summary of our results is sketched in Fig. 14.

We have already noted that a different behavior for the nonlocal transfers $\mathcal{T}_{ub}$ and $\mathcal{T}_{bu}$ was obtained for the mechanically forced turbulence investigated in this work when compared with the decaying turbulence case studied by [24]. Compared with incompressible hydrodynamic turbulence, involving only one field and one transfer function, MHD turbulence is richer and more complex. It involves two interacting fields and several transfer functions, and as a result the energy injected at large scales can travel to small scales through several channels. Also the number of quadratic ideal invariants is larger, and inverse cascades (not present in three-dimensional hydrodynamics) can take place. This suggests that in MHD flows the particular way the system is set up (e.g., mechanically or magnetically forced, free-decaying cases without external forces), or even the scale at which the energy is injected (compared with the length of the box), might have a direct effect in the evolution of the flow and lead to different transfers.

We would also like to comment on the implications of our results to the different models of magnetohydrodynamic turbulence. In the present phenomenological models of MHD turbulence [13–17], locality of the energy transfer is assumed. That is to say, these models are derived assuming that scales of different magnitude do not strongly interact. While this assumption seems to be valid for HD turbulence, this is not necessarily true for MHD. As we have shown nonlocal interactions are present in MHD turbulence and control the *u*-to-*b* transfers of energy. However, these nonlocal interactions are smaller in amplitude and most of the input of energy to the magnetic field comes from the large-scale flow and then cascades to smaller scales, making the assumption of locality justified to some extent. However, nonlocal *u-to-b* transfers need to be considered to have a proper description of the energy cascade. To illustrate this we show in Fig. 15 the energy transfer in terms of the Elsässer variables $\mathbf{z}^{\pm}=\mathbf{u}\pm\mathbf{b}$, often used in turbulence models, and we compare it with the contributions from u to u, b to b, b to u, and u to b. In the figure we plot $\mathcal{T}_{z^+z^+}\equiv-\int\mathbf{z}^+_K\mathbf{z}^-\nabla\mathbf{z}^+_Q d\mathbf{x}$ and compare it with the energy transfer due to the local transfer terms $\mathcal{T}_{uu}+\mathcal{T}_{bb}$ and the nonlocal transfer terms $\mathcal{T}_{ub}+\mathcal{T}_{bu}$. The local transfer terms appear to be dominant, except in the tails where the transfer of Elsässer variables is dominated by the nonlocal transfers between the magnetic and kinetic energies. This tails, although with small amplitude, cannot be completely neglected, as shown by the NL/L ratio of Fig. 13. The nonlocal tail in the transfer gives a net contribution of energy at magnetic small scales of roughly 1/5 when compared with the local transfer.

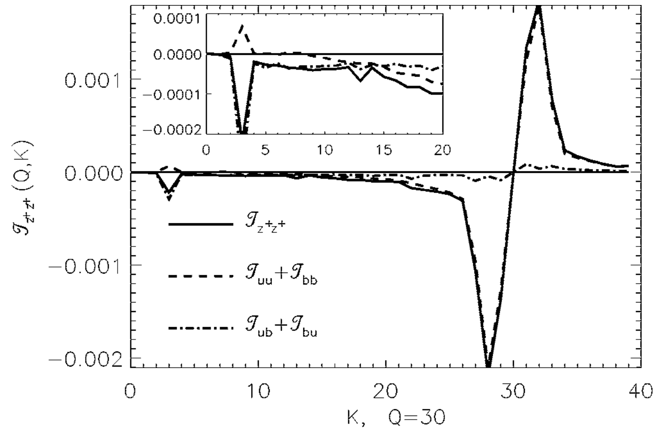

FIG. 15. A comparison of the Elsässer energy transfer $\mathcal{T}_{z^+z^+}(Q,K)$ with the transfers from the local $\mathcal{T}_{uu}+\mathcal{T}_{bb}$ and the nonlocal $\mathcal{T}_{ub}+\mathcal{T}_{bu}$ contributions. The inset shows a blowup of the small-K tail where the nonlocal interactions are more dominant.

Finally, we would like to say that our results were based on numerical simulations of moderate Reynolds number much smaller than what is observed in most physical phenomena. We already noted that due to the small inertial range, we cannot test self-similarity, which would require one to compare the transfers [i.e., $\mathcal{T}_{uu}(Q,K_1)$, $\mathcal{T}_{uu}(Q,K_2)$] to wave numbers that are both significantly away from each other ($K_1 \ll K_2$) and away from the forced and dissipative scales ($K_F \ll K_1$ and $K_2 \ll k_\eta$). We plan to address this issue with higher-resolution runs in our future work.

ACKNOWLEDGMENTS

The authors are grateful to J. Herring for valuable discussions and his careful reading of the manuscript. Computer time was provided by NCAR and Pittsburgh Supercomputing Center. NSF Grant No. CMG-0327888 at NCAR supported this work in part and is gratefully acknowledged.

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