ÉCOLE DE PHYSIQUE DES HOUCHES

New Challenges in Turbulence Research V

École de Physique des Houches, April 7-12, 2019

Critical Transitions in Turbulence

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11 April 2019

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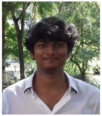
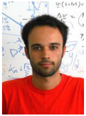
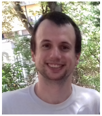
with many thanks to the students and postdocs that worked with me on this subject...

K. Sesashayanan, S. Benavides, A. Van Kan, G. Sahoo,

V. Dallas, T. Nemoto, A. Cameron

Most results presented in this talk are reviewed in:

A. Alexakis, L. Biferale

"**Cascades and transitions in turbulent flows**"

Physics Reports **767-769**, 1-101 (2018)

Physics Reports 767–769 (2018) 1–101

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Cascades and transitions in turbulent flows

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Check for updates

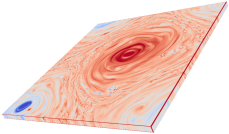
I. Introduction

Coexistence of "coherent structures at large scales" & "small scale turbulence"

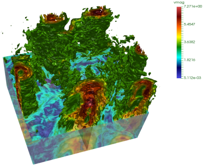
Picture from: `https://earthobservatory.nasa.gov`

Thin layers/Rotating/Stratified/Magnetic fields ...

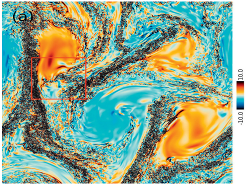
There is a variety of systems that have the ability to generate both large and small scale structures...



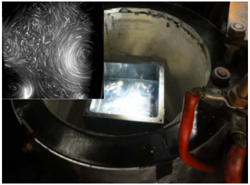
Thin Layers



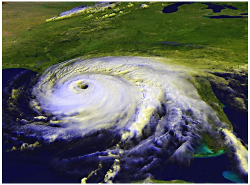
Rotating Flows



Rotating & Stratified



in the Lab



in the atmosphere

in planets

Pictures from: A^2 & LB Phys. Rep. **767-769**, 1-101 (2018), L. Biferale et al Phys. Rev. X **6** 041036 (2016), D. Rosenberg, et al Phys. Fluids **27** 055105 (2015) , J. Herault, et al, Europhys. Lett. **111** 44002 (2015), https://earthobservatory.nasa.gov

Thin layers/Rotating/Stratified/Magnetic fields ...

Thin Layers

- L. Smith, J. Chasnov, & F. Waleffe, Phys. Rev. Lett. **77**, 2467 (1996)
- A. Celani, S. Musacchio, and D. Vincenzi, Phys. Rev. Lett. **104**, 184506 (2010)

Rotating flows

- A. Sen, et al Phys. Rev. E **86**, 036319 (2012)
- E. Deusebio, et al Phys. Rev. E **90**, 023005 (2014)

Rotating and Stratified flows

- A. Pouquet and R. Marino, Phys. Rev. Lett. **111**, 234501 (2013)
- R. Marino, et al European Phys. Lett. **102** 44006 (2013)
- A. Sozza, et al Phys. of Fluids **27**, 035112 (2014)

Magnetic fields

- A. Alexakis, Phys. Rev. E **84**, 056330 (2011)
- K. Seshasayanan, S.J. Benavides, A. Alexakis Phys. Rev. E **90** 051003(R) (2014)

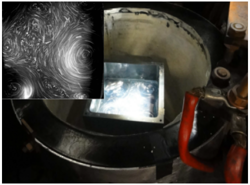
Experiments

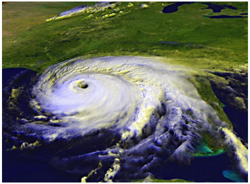
- M. Shats, D. Byrne, H. Xia Phys. Rev. Lett. **105**, 264501 (2010)
- H. Xia, D. Byrne, G. Falkovich, M. Shats Nature Physics 7, 321-324 (2011)
- A. Potherat, R. Klein, J. Fluid Mech. **761** 168 (2014)

And in more exotic systems...

- **atmosphere** D. Byrne et al - Geophys.R.L (2013) "*Height dependent transition from 3D to 2D turbulence in the hurricane boundary layer*"
- **ocean** G. P. King, et al - J. Geophys. Res. (2015) "*Upscale and downscale energy transfer over the tropical Pacific revealed by scatterometer winds*"
- **Venus** M. N. Izakov, Solar System Research (2013) "*Large-scale quasi-2D turbulence and a inverse spectral flux of energy in the atmosphere of Venus*"
- **Jupiter** R. Young et al Nature Physics (2017) "*Forward and inverse kinetic energy cascades in Jupiters turbulent weather layer*"
- **plasma flows** G. Miloshevich at al, Plasma Physics (2018) "*Direction of cascades in a magnetofluid model*"
- **optical turbulence** V. Malkin et al, Phys. Rev. E (2018) "*Transition between inverse and direct energy cascades in multiscale optical turbulence*"
- **acoustic turbulence** A. Ganshin, et al Phys.Rev.Lett. (2008) "*Observation of an inverse energy cascade in acoustic turbulence in superfluid helium*"
- **capillary turbulence** Abdurakhimov et al Phys.Rev.E (2015) "*Bidirectional energy cascade in surface capillary waves*"

A need for a unified treatment of these problems





- There is a large number of diverse systems that display a simultaneous cascade of energy in the large and in the small scales (split cascades).
- Such split cascades are present due to different mechanisms (confinement, rotation, magnetic fields, ...).
- **is there a unified treatment of these problems?**

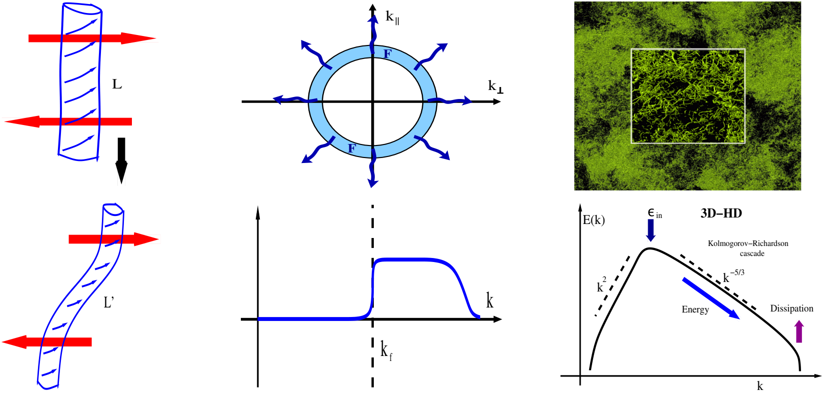
What this talk is about

- Define as precisely as I can the problem.
- Present all possible scenarios of "cascade transitions".
- Demonstrate with examples each possibility.
- Describe our current state of understanding of these systems.
- Present open problems!

II. Setting up the problem

Forward Cascade: 3D turbulence

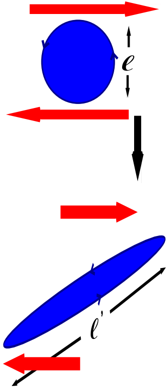
Vortex tube stretching ⇒ Forward cascade

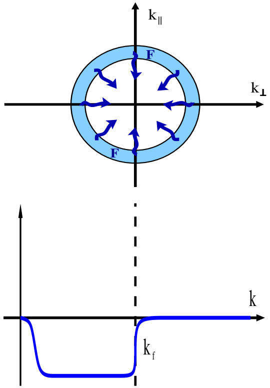


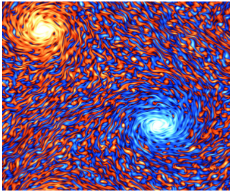
Picture from: M. Yokokawa et al. Proceedings of the 2002 ACM/IEEE Conference on Supercomputing

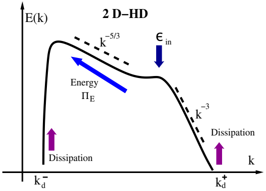
Inverse Cascade: 2D turbulence

Vortex patch shearing $\Rightarrow$ Inverse cascade





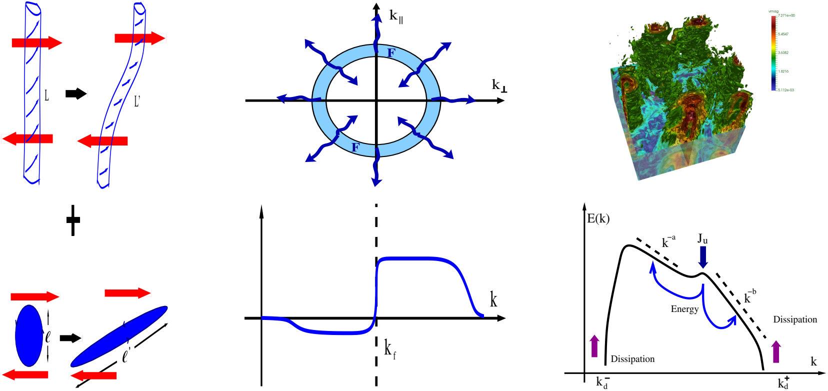




Picture from: C-K Chan Phys. Rev. E 85, 036315

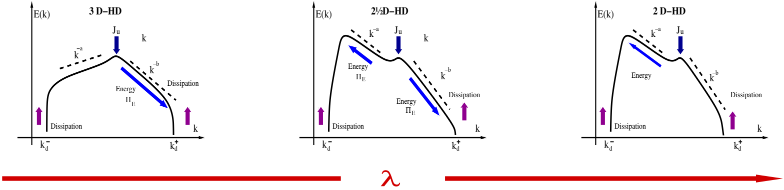
Split Cascade: Thin layers, Rotating turbulence, ...

A balance between the two $\Rightarrow$ Split cascade?



Picture from: L. Biferale et al Phys. Rev. X **6** 041036 (2016)

A turbulence to turbulence transition ...

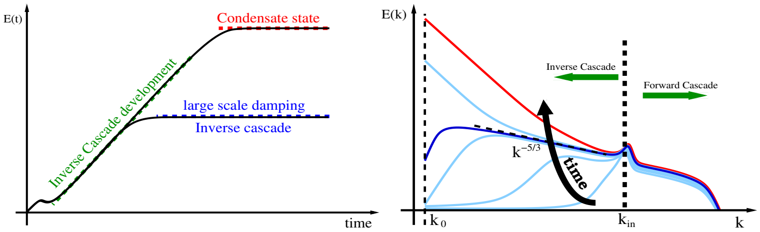


- the system transitions from one turbulent state (forward cascading) to an other (inverse cascading) varying a parameter λ.
- the transition occurs in the presence of turbulence ($\lambda \neq Re$).
- through a state that cascades energy both forward and inversely:

Split Cascade!

Two stages of an inverse cascade

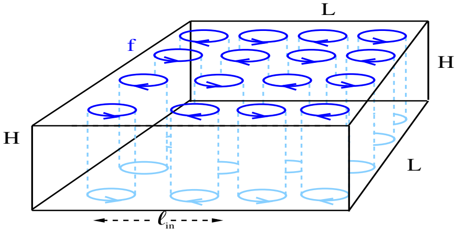
Inverse cascade and condensates



- At early stages energy cascades inversely to the large scales.
- In the presence of a **sufficiently strong large scale dissipation** this process will saturate at a scale ℓ_α smaller than the box size L and a **cascade** from the forcing scales ℓ_f to ℓ_α is build.
- For **weak large scale dissipation or in its absence** energy will pile up in the largest scales forming a **condensate**.

In this talk we will focus on the former case!

A General System



$$\partial_t \mathbf{V} = \omega \mathcal{L}[\mathbf{V}] + \mathcal{NL}[\mathbf{V},\mathbf{V}] - \nu(-\Delta)^n \mathbf{V} - \alpha(-\Delta)^{-m} \mathbf{V} + \mathbf{F}$$

ℓ_{in} = energy injection scale, ϵ_{in} = energy injection rate,
ν = hyper-viscosity ($n = 1$), α = hypo-viscosity ($m = 0$),
ω = wave frequency (eg rotation rate, Brunt-Väisälä frequency, etc)

$$\epsilon_\alpha = \alpha \left\langle |\nabla^{-m} \mathbf{V}|^2 \right\rangle, \qquad \epsilon_\nu = \nu \left\langle |\nabla^n \mathbf{V}|^2 \right\rangle$$

$$\epsilon_{in} = \epsilon_\alpha + \epsilon_\nu$$

Control and order Parameters

Forcing scale Control parameters:

$$\lambda_1 = \frac{\ell_{in}}{H}, \qquad \lambda_2 = \frac{\epsilon_{in}^{1/3}\ell_{in}^{-2/3}}{\omega}, \qquad \dots$$

Viscous & Domain size parameters

$$Re = \frac{\epsilon_{in}^{1/3}\ell_{in}^{4/3}}{\nu} \to \infty, \quad R_\alpha = \frac{\epsilon_{in}^{1/3}\ell_{in}^{-2/3}}{\alpha} \to \infty, \quad \Lambda = \frac{L}{\ell_{in}} \to \infty$$

Order parameters

$$Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}, \qquad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \qquad \text{with} \qquad Q_\alpha + Q_\nu = 1$$

Control and order Parameters

Forcing scale Control parameters:

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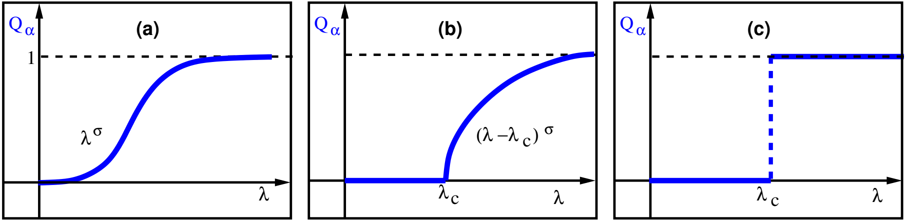
Order parameters

$$Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}, \qquad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \qquad \text{with} \qquad Q_\alpha + Q_\nu = 1$$

We would like to know how Q_α, Q_ν change as $\lambda_1, \lambda_2, \dots$ vary, in the limit $Re, R_\alpha, \Lambda \to \infty$

Classification: Smooth, 2nd order and 1st order transitions

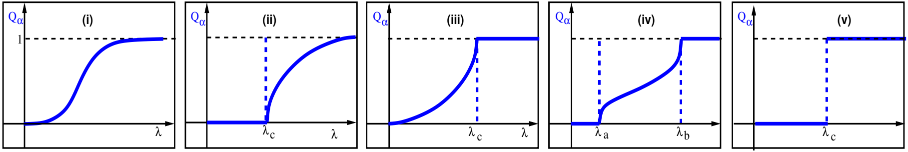


Smooth, 2nd order phase transition, 1st order phase transition

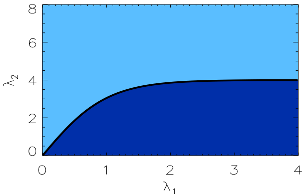
Cases (b) & (c) will be referred as "**critical**"

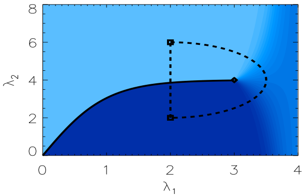
Phase Space Diagrams

One parameter systems: λ



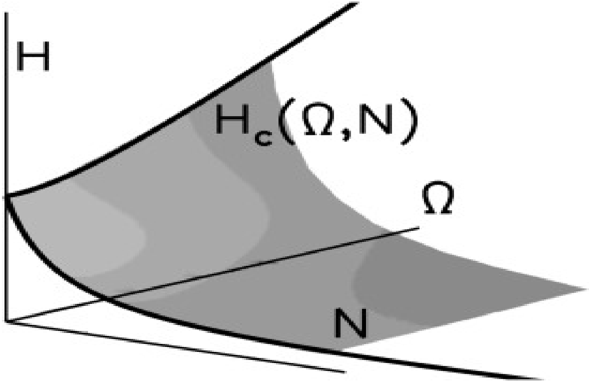
Two parameter systems: λ_1, λ_2





Phase space Diagrams

Three parameter systems: $\lambda_1 = H, \lambda_2 = \Omega, \lambda_3 = N$
eg Rotating and stratified turbulence



Questions

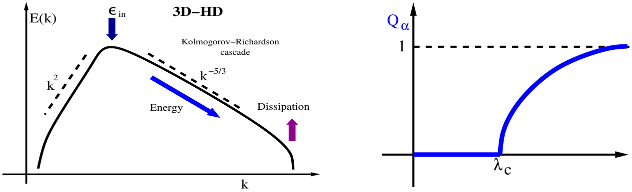
1. In what sense can these transitions be critical?
   (*ie*, what does having exactly zero flux means?)

2. How can split cascades even exist?
   (*ie*, how can a system cascade energy both to large and small scales at the same time?)

3. When the transition is:
   - (i) smooth?
   - (ii) 1st order (discontinuous)?
   - (iii)2nd order (continuous with discontinuous derivatives)?

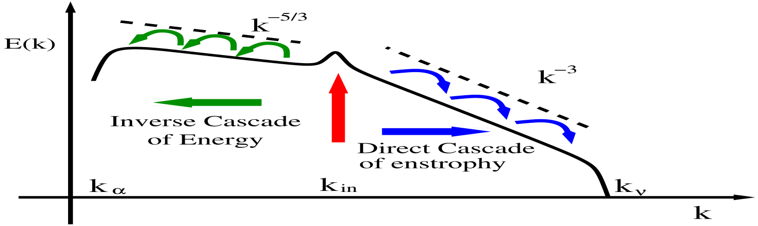
Criticality

1. In what sense can these transitions be critical?
   (*ie,* what does having exactly zero flux means?)



$$\partial_t \mathbf{V} = \omega \mathcal{L}[\mathbf{V}] + \mathcal{NL}[\mathbf{V},\mathbf{V}] + \nu \Delta \mathbf{V} - \alpha \mathbf{V} + \mathbf{F}$$

2D Turbulence



$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\mathbf{P} + \mathbf{f} - \nu_\mathbf{n}(-\mathbf{\Delta})^\mathbf{n}\mathbf{u} - \alpha_\mathbf{m}(-\mathbf{\Delta})^{-\mathbf{m}}\mathbf{u}$$

Ideal invariants:

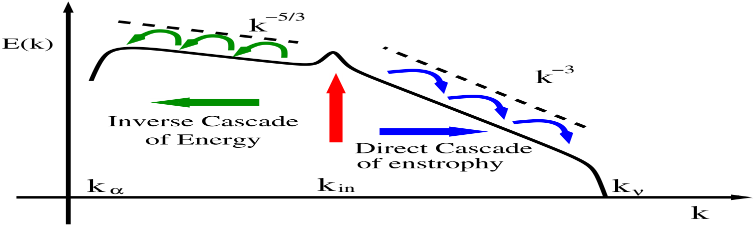
$$\mathcal{E} = \frac{1}{2}\langle|\mathbf{u}|^2\rangle, \qquad \mathcal{Z} = \frac{\mathbf{1}}{\mathbf{2}}\langle|\omega|^2\rangle \quad \text{with} \quad \omega = \nabla\times\mathbf{u}$$

Injection and Dissipation rates:

$$\epsilon_{in} = \langle\mathbf{u}\cdot\mathbf{f}\rangle, \qquad \epsilon_\nu = \nu\langle|\nabla^n\mathbf{u}|^2\rangle, \qquad \epsilon_\alpha = \alpha\langle|\nabla^{-\mathbf{m}}\mathbf{u}|^2\rangle,$$

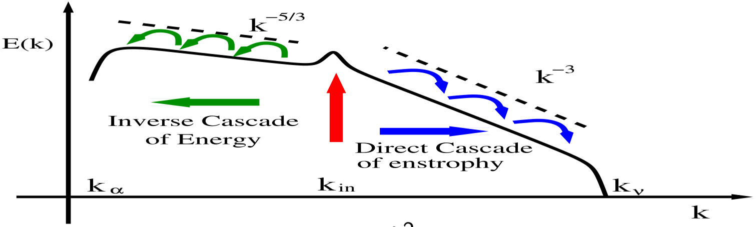
$$\eta_{in} = \langle\omega\cdot\nabla\times\mathbf{f}\rangle, \qquad \eta_\nu = \nu\langle|\nabla^n\omega|^2\rangle, \qquad \eta_\alpha = \alpha\langle|\nabla^{-m}\omega|^2\rangle,$$

2D Turbulence



$$E(k) = \begin{cases} \epsilon_{in}^{2/3} k^{-5/3} & k_\alpha \ll k \ll k_{in} \\ \eta_{in}^{2/3} k^{-3} & k_{in} \ll k \ll k_\nu \end{cases}$$

2D Turbulence



$$\eta_{in} = \epsilon_{in} k_{in}^2$$

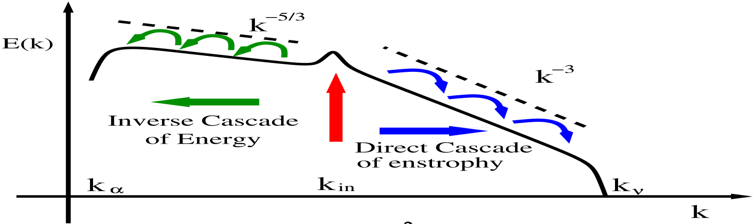
$$\epsilon_\alpha \simeq \epsilon_{in}$$
$$\eta_\alpha \simeq \epsilon_\alpha k_\alpha^2$$
$$\eta_\alpha \simeq \eta_{in} \left(\frac{k_\alpha}{k_{in}}\right)^2$$

$$\boxed{\eta_\alpha \simeq \eta_{in} R_\alpha^{-3}}$$

$$\eta_\nu \simeq \eta_{in}$$
$$\epsilon_\nu \simeq \eta_\nu k_\nu^{-2}$$
$$\epsilon_\nu \simeq \epsilon_{in} \left(\frac{k_{in}}{k_\nu}\right)^2$$

$$\boxed{\epsilon_\nu \simeq \epsilon_{in} Re^{-1}}$$

2D Turbulence



$$\eta_{in} = \epsilon_{in} k_{in}^2$$

$$\epsilon_\alpha \simeq \epsilon_{in}$$
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$$\eta_\alpha \simeq \eta_{in} \left(\frac{k_\alpha}{k_{in}}\right)^2$$

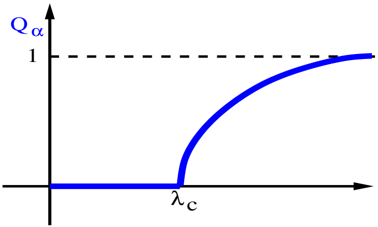
$$\eta_\nu \simeq \eta_{in}$$
$$\epsilon_\nu \simeq \eta_\nu k_\nu^{-2}$$
$$\epsilon_\nu \simeq \epsilon_{in} \left(\frac{k_{in}}{k_\nu}\right)^2$$

$$\boxed{\eta_\alpha \simeq \eta_{in} R_\alpha^{-3}}$$

$$\boxed{\epsilon_\nu \simeq \epsilon_{in} Re^{-1}}$$

- Zero inverse/forward flux is realized only in the large box, zero ν and zero α limit.

Criticality



$$Q_\alpha = \lim_{Re\to\infty} \lim_{R_\alpha\to\infty} \lim_{L/\ell_{in}\to\infty} \left(\frac{\epsilon_\alpha}{\epsilon_{in}}\right)$$

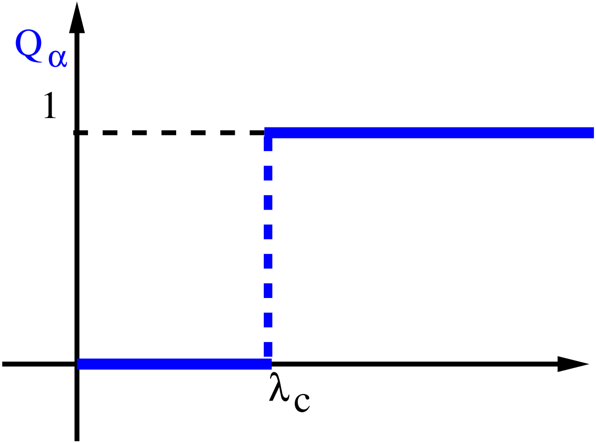
$$Q_\nu = \lim_{Re\to\infty} \lim_{R_\alpha\to\infty} \lim_{L/\ell_{in}\to\infty} \left(\frac{\epsilon_\nu}{\epsilon_{in}}\right)$$

To conclude criticality the $L/\ell_{in}, Re, R_\alpha \to \infty$ limits have to be taken first.

III. Examples

Example I

1st order transition



Example I: Helical Decomposition

$$\tilde{\mathbf{u}}(\mathbf{k}) = \frac{1}{(2\pi L)^3}\int e^{i\mathbf{k}\mathbf{x}}\mathbf{u}d\mathbf{x}^3, \quad \mathbf{u}(x) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}\tilde{\mathbf{u}}(\mathbf{k})$$

$$\boxed{\tilde{\mathbf{u}}(\mathbf{k}) = u^+(\mathbf{k})\,\mathbf{h}_{\mathbf{k}}^+ + u^-(\mathbf{k})\,\mathbf{h}_{\mathbf{k}}^-}$$

$$\mathbf{h}_{\mathbf{k}}^{\pm} = \frac{\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})}{\sqrt{2}|\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})|} \pm i\frac{\mathbf{k}\times\hat{\mathbf{e}}}{\sqrt{2}|\mathbf{k}\times\hat{\mathbf{e}}|}$$

$$i\mathbf{k}\times\mathbf{h}_{\mathbf{k}}^{\pm} = \pm k\mathbf{h}_{\mathbf{k}}^{\pm}, \quad \mathbf{h}_{\mathbf{k}}^{s}\cdot\mathbf{h}_{\mathbf{q}}^{-s'} = \delta_{\mathbf{k},\mathbf{q}}\delta_{s,s'}$$

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- M. Lesieur, Revue ‘Turbulence’, Observatoire de Nice (1972)
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- P. Constantin, & A. Majda, Commun. Math. Phys. 115, 435-456 (1988)
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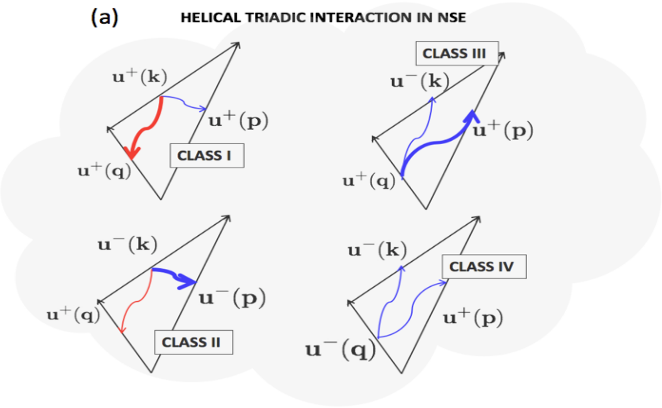
- F. Waleffe, Phys. Fluids 4, 350 (1992)

Example I: Helical Decomposition

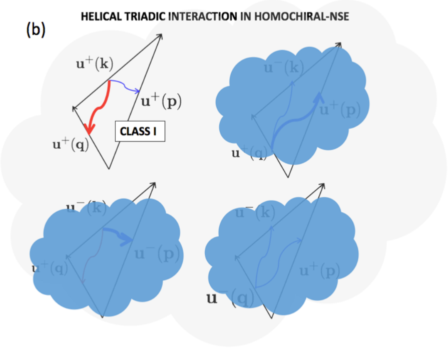
The nature of triad interactions in homogeneous turbulence

Fabian Waleffe
Center for Turbulence Research, Stanford University–NASA Ames, Building 500, Stanford, California 94305-3030

(Received 24 July 1991; accepted 22 October 1991)



Example I: Helical Decomposition



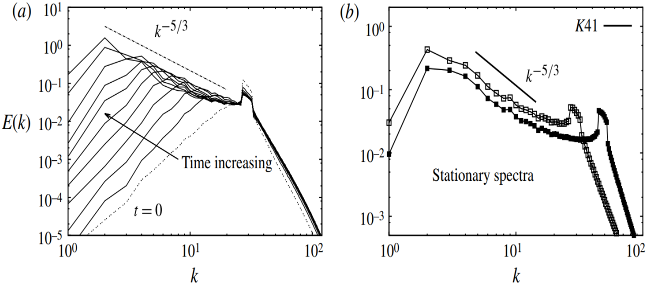
Ideal Invariants:

$$E^{\pm} = \frac{1}{2}\langle \mathbf{u}^{\pm} \cdot \mathbf{u}^{\pm} \rangle, \qquad H^{\pm} = \pm\frac{1}{2}\langle \mathbf{w}^{\pm} \cdot \mathbf{u}^{\pm} \rangle > 0$$

Biferale, Musacchio, Toschi, Phys. Rev. Lett. **108** 164501 (2012), J. Fluid Mech. **730**, 309 (2013)

Example I: Helical Decomposition

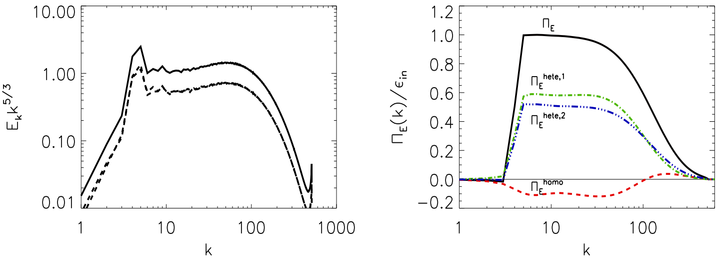
Homochiral turbulence



When interactions are restricted to homochiral the energy cascades inversely to large scales

Biferale, Musacchio, Toschi, Phys. Rev. Lett. **108** 164501 (2012), J. Fluid Mech. **730**, 309 (2013)

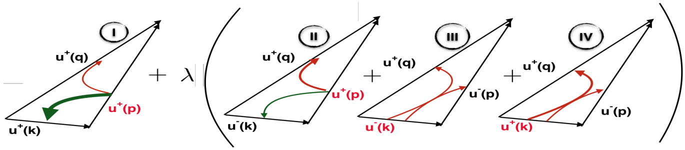
Example I: Helical Decomposition



Even in the full Navier-Stokes equations homochiral interactions cascade energy inversely! (but are sub-dominant)

Alexakis J. Fluid Mech. **812**, 752 (2017)

Example I: Helical Decomposition



$$\partial_t \mathbf{u} = [\mathcal{N}\mathcal{L}] - \nu \Delta^4 \mathbf{u} - \mu \Delta^{-2} \mathbf{u} + \mathbf{F}$$

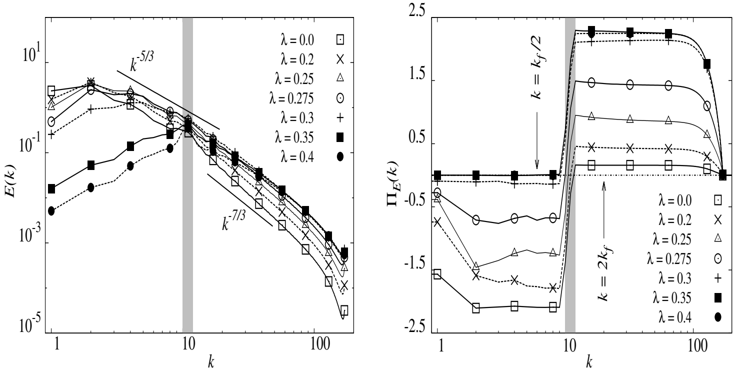
$$\mathcal{N}\mathcal{L} = \lambda \mathbb{P}(\mathbf{u} \times \mathbf{w}) + (1-\lambda)[\mathbb{P}^+(\mathbf{u}^+ \times \mathbf{w}^+) + \mathbb{P}^-(\mathbf{u}^- \times \mathbf{w}^-)]$$

$\lambda = 1$: Navier-Stokes, $\lambda = 0$ homochiral NS :

- the equations conserve E, H for any value of λ.
- the nonlinearity is self similar under scale transformations

Sahoo et al Phys. Rev. Lett. **118**, 164501 (2017)

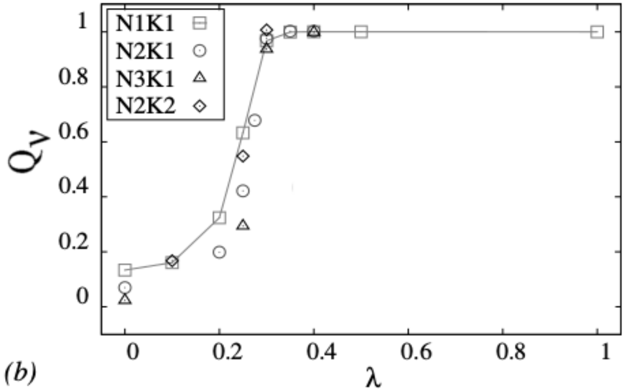
Example I: Helical Decomposition



Sahoo et al Phys. Rev. Lett. **118**, 164501 (2017)

Example I: Helical Decomposition

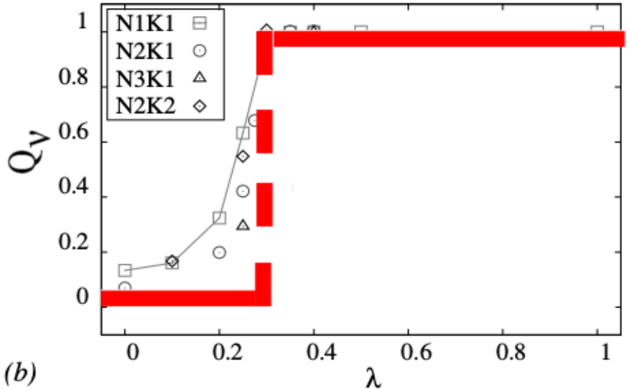
$$Q_\nu = \epsilon_\nu / \epsilon_{in}$$



Sahoo et al Phys. Rev. Lett. **118**, 164501 (2017)

Example I: Helical Decomposition

$$Q_\nu = \epsilon_\nu / \epsilon_{in}$$



The transition becomes critical at $Re, R_\alpha, L/\ell_{in} \to \infty$.

Sahoo et al Phys. Rev. Lett. **118**, 164501 (2017)

Why is the transition 1st order?

- The model is scale invariant
- Same physics control both the large and the small scales
- If there is enough scale separation so that forcing and damping effects can be neglected the system can cascade energy either forward or inverse not both!

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Why is the transition 1st order?

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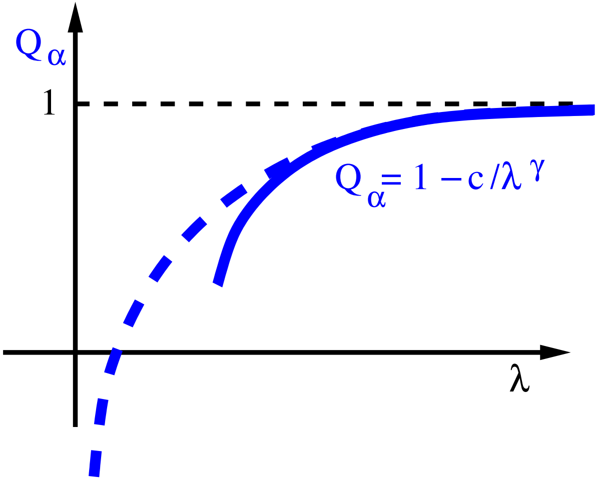
- Same physics control both the large and the small scales
- If there is enough scale separation so that forcing and damping effects can be neglected the system can cascade energy either forward or inverse not both!
- **Thus the transition (if present) has to be 1st order!**

Implication:

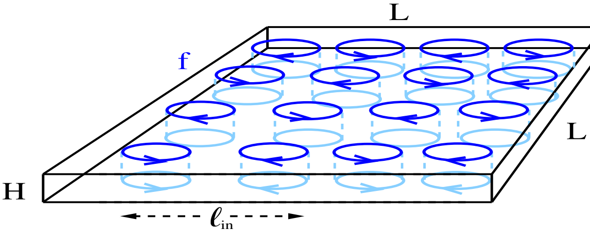
For a **split cascade** to be present the large scales have to follow different physics than the small scales!

Example II

smooth transition



Example II: Very thin layer $H \ll \ell_{in}$



$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P - \alpha \Delta^{-m} \mathbf{u} - \nu \Delta^{n} \mathbf{u} + \mathbf{f}$$

$$\epsilon_\alpha = \alpha \langle |\nabla^{-m} \mathbf{u}|^2 \rangle, \quad \epsilon_{\mathbf{in}} = \langle \mathbf{f} \cdot \mathbf{u} \rangle, \quad \epsilon_\nu = \nu \langle |\nabla^{\mathbf{n}} \mathbf{u}|^2 \rangle,$$

$$\epsilon_{in} = \epsilon_\alpha + \epsilon_\nu$$

$$\lambda = \frac{\ell_{in}}{H}, \qquad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \quad \& \quad Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}$$

Example II: Very thin layer $H \ll \ell_{in}$

PRL **104**, 184506 (2010) PHYSICAL REVIEW LETTERS week ending 7 MAY 2010

Turbulence in More than Two and Less than Three Dimensions

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[2]*Dipartimento di Fisica Generale and INFN, Università di Torino, via P. Giuria 1, 10125 Torino, Italy*

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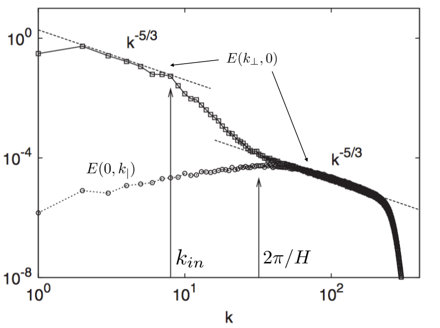
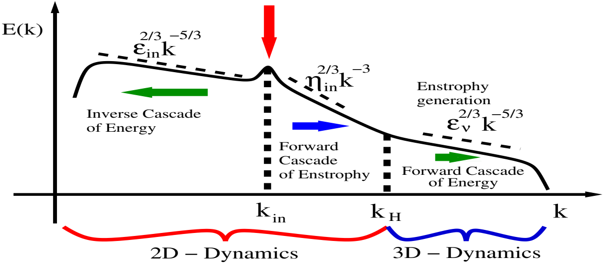
(Received 12 January 2010; published 7 May 2010)

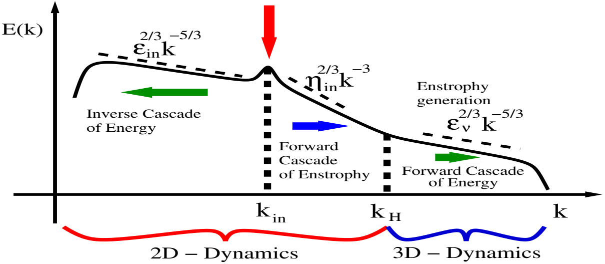


Example II: Very thin layer $H \ll \ell_{in}$



- 2D dynamics for $k \ll k_H$ $(\ell \gg H) \Rightarrow$ inverse energy cascade.
- 3D dynamics for $k \gg k_H$ $(\ell \ll H) \Rightarrow$ forward energy cascade

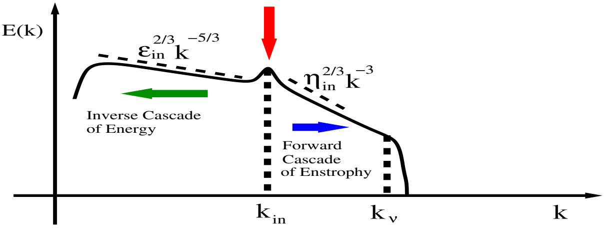
Example II: Very thin layer $H \ll \ell_{in}$



- 2D dynamics for $k \ll k_H$ $(\ell \gg H) \Rightarrow$ inverse energy cascade.
- 3D dynamics for $k \gg k_H$ $(\ell \ll H) \Rightarrow$ forward energy cascade

How can a forward energy cascade build up at $k > k_H$?

Example II: Very thin layer ($H = 0$, ie 2D)



Enstrophy arrives at k_ν at a rate

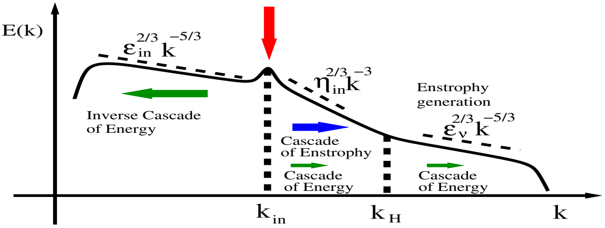
$$\eta_{in} = \eta_\nu = \epsilon_{in} k_{in}^2$$

Energy arrives at k_ν at a rate

$$\epsilon_\nu \propto \epsilon_{in} \left(\frac{k_{in}}{k_\nu} \right)^2$$

.

Example II: Very thin layer $H \ll \ell_{in}$, $(\lambda = \ell_{in}/H)$



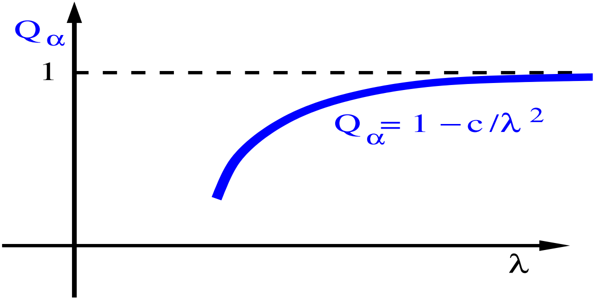
- Replace k_ν by k_H

Energy that arrives at small scales is given by:

$$\epsilon_\nu = \epsilon_{in} \left(\frac{k_{in}}{k_H}\right)^2 = \epsilon_{in} \left(\frac{H}{\ell_{in}}\right)^2 = \epsilon_{in} \frac{1}{\lambda^2}$$

.

Example II: Very thin layer $H \ll \ell_{in}$, ($\lambda = \ell_{in}/H$)



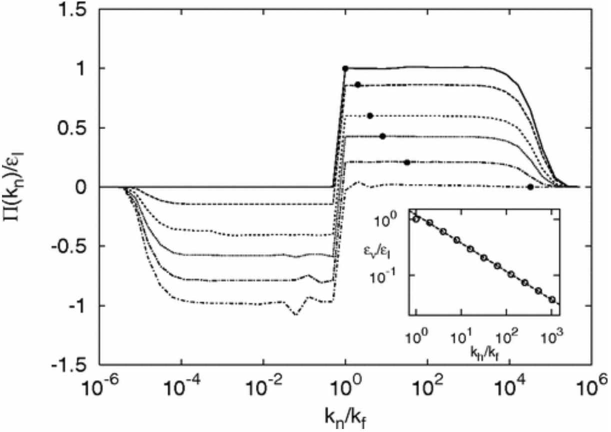
$$Q_\nu = \epsilon_\nu/\epsilon_{in} \propto 1/\lambda^2 \quad \Rightarrow \quad Q_\alpha = (1 - Q_\nu) \simeq 1 - c/\lambda^2$$

This provides an example for a '*smooth*' transition from a split cascade to a strictly forward cascade.

Boffetta, et al Phys. Rev. E 83, 066302 (2011)

Example II: Very thin layer $H \ll \ell_{in}$, $(\lambda = \ell_{in}/H)$

Scaling tested (up to my knowledge) only using shell models!

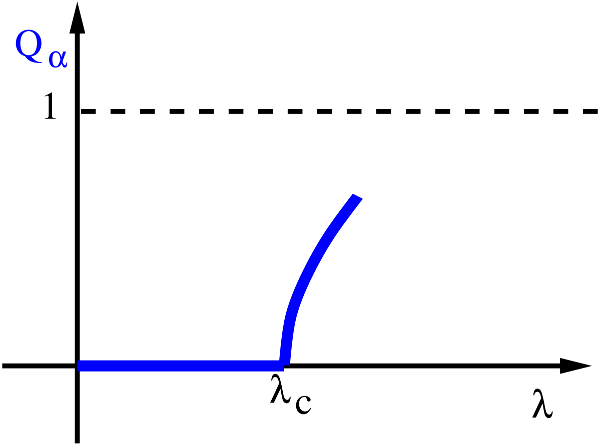


$$Q_\nu = \epsilon_\nu/\epsilon_{in} \propto 1/\lambda^2$$

Picture from: Boffetta, et al Phys. Rev. E 83, 066302 (2011)

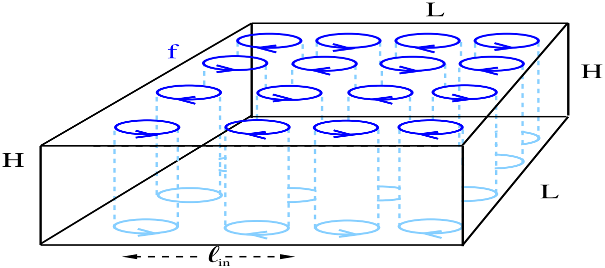
Example III

Second order phase transitions



Example III: Thick layer Turbulence $H \sim \ell_{in}$

From a strictly forward to an inverse cascade



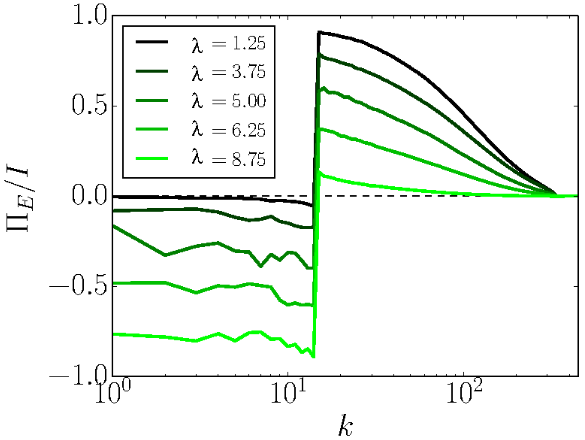
$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P - \alpha\Delta^{-m}\mathbf{u} - \nu\Delta^{n}\mathbf{u} + \mathbf{f}$$

$$\epsilon_\alpha = \alpha\langle|\nabla^{-m}\mathbf{u}|^2\rangle, \quad \epsilon_{\mathbf{in}} = \langle \mathbf{f}\cdot\mathbf{u}\rangle, \quad \epsilon_\nu = \nu\langle|\nabla^{\mathbf{n}}\mathbf{u}|^2\rangle,$$

$$\lambda = \frac{\ell}{H}, \qquad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \quad \& \quad Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}$$

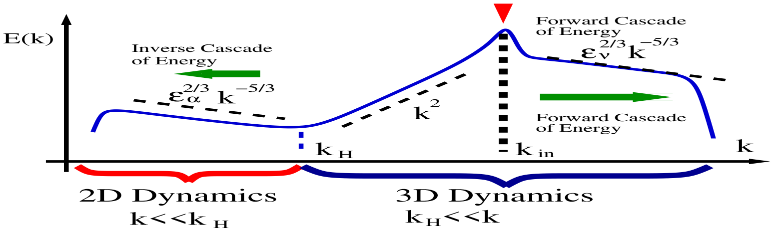
Example III: Thick layer Turbulence $H \sim \ell_{in}$

Fluxes for different values of $\lambda = \ell_{in}/H$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Thick layer Turbulence $(\lambda = \ell_{in}/H)$



$$E(k) \propto \epsilon_\alpha^{2/3} k^{-5/3} \qquad k \ll k_H \qquad (1)$$

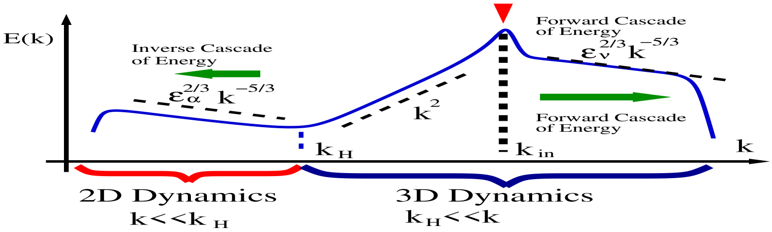
$$E(k) \propto (\epsilon_{in}^{2/3} k_{in}^{-11/3}) k^2 \qquad k_H \ll k \ll k_{in} \qquad (2)$$

$$E(k) \propto \epsilon_{in}^{2/3} k^{-5/3} \qquad k_{in} \ll k \qquad (3)$$

Equating $E(k)$ at $k = k_H$ we obtain

$$\boxed{\epsilon_\alpha = \epsilon_{in} \left(\frac{k_H}{k_{in}}\right)^{11/2} = \epsilon_{in} \lambda^{11/2}} \quad \text{smooth?}$$

Example III: Thick layer Turbulence $(\lambda = \ell_{in}/H)$



$$E(k) \propto \epsilon_\alpha^{2/3} k^{-5/3} \qquad k \ll k_H \tag{1}$$

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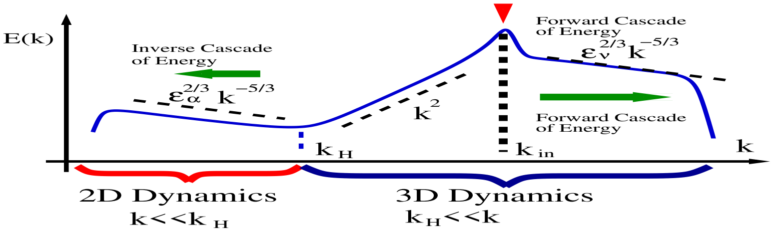
Equating $E(k)$ at $k = k_H$ we obtain

$$\boxed{\epsilon_\alpha = \epsilon_{in} \left(\frac{k_H}{k_{in}}\right)^{11/2} = \epsilon_{in} \lambda^{11/2}} \quad \text{smooth?}$$

However...

Example III: Thick layer Turbulence $(\lambda = \ell_{in}/H)$



at $\ell \simeq H$:

Inverse Flux due to local 2D interactions

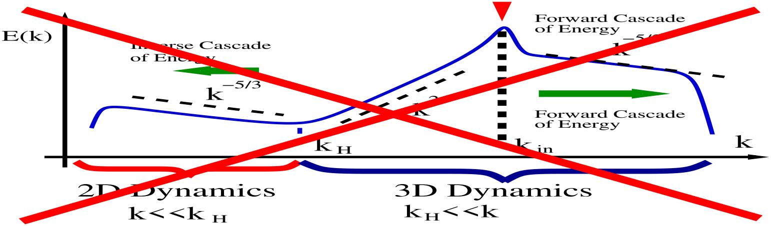
Forward flux due to a turbulent eddy viscosity

$$\nu_{eddy}\frac{U_H^2}{H^2}, \qquad (\nu_{eddy} \sim u_{in}\ell_{in})$$

$$\epsilon_{in}\left(\frac{k_H}{k_{in}}\right)^{11/2} \qquad\qquad \epsilon_{in}\left(\frac{k_H}{k_{in}}\right)^{5}$$

The effect of the eddy viscosity is larger by a factor of $(k_{in}/k_H)^{1/2}$!

Example III: Thick layer Turbulence $(\lambda = \ell_{in}/H)$

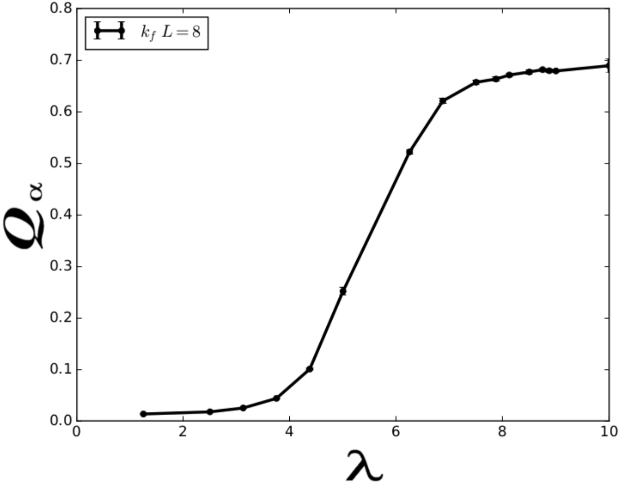


- Nonlocal interactions become important!
- Eddy-viscosity depends on the properties of the forcing scales
- if the forcing scales are sufficiently 2D (3D)
  there is (is not) an inverse cascade
- the transition occurs at

$$\boxed{H_c \propto \ell_{in}}$$

Example III: Thick layer Turbulence $(\lambda = \ell_{in}/H)$

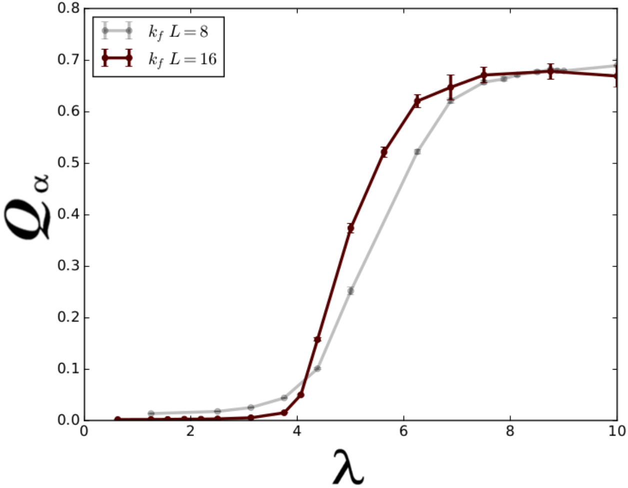
Inverse flux $(Q_\alpha = \epsilon_\alpha/\epsilon_{in})$ as a function of $(\lambda = \ell_{in}/H)$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Thick layer Turbulence $(\lambda = \ell_{in}/H)$

Inverse flux ($Q_\alpha = \epsilon_\alpha/\epsilon_{in}$) as a function of $(\lambda = \ell_{in}/H)$



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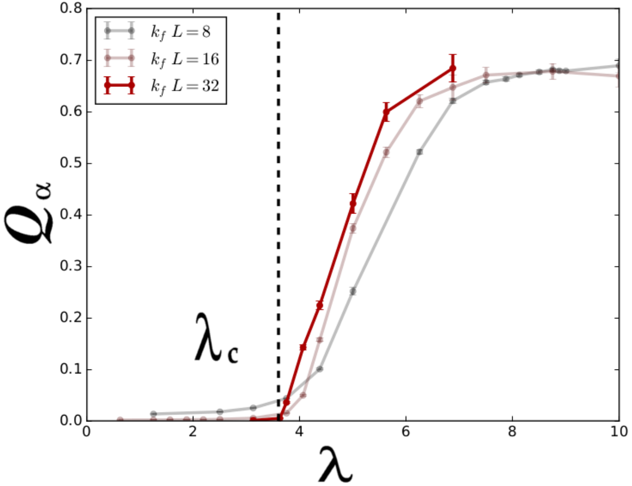
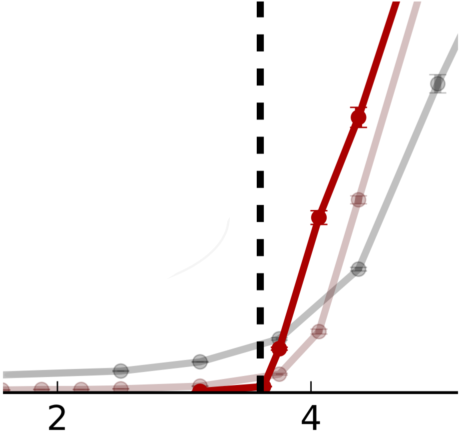
Inverse flux ($Q_\alpha = \epsilon_\alpha/\epsilon_{in}$) as a function of $(\lambda = \ell_{in}/H)$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

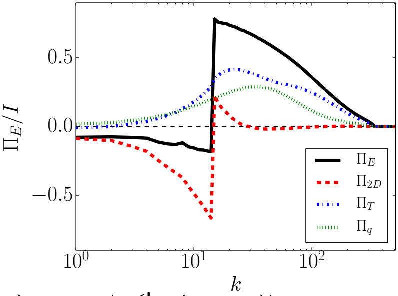
Example III: Thick layer Turbulence $(\lambda = \ell_{in}/H)$

Inverse flux $(Q_\alpha = \epsilon_\alpha/\epsilon_{in})$ as a function of $(\lambda = \ell_{in}/H)$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Decomposed fluxes



—— Total Flux $\Pi(k) = -\langle \mathbf{u}^{<\mathbf{k}} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \rangle,$

- - - 2D Flux $\Pi_{2D}(k) = -\langle \mathbf{u}_{2D}^{<k} \cdot (\mathbf{u}_{2D} \cdot \nabla \mathbf{u}_{2D}) \rangle,$

-·-·- 2D-3D Flux $\Pi_q(k) = -\langle \mathbf{u}_{2D}^{<k} \cdot (\mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D}) \rangle - \langle \mathbf{v}_{3D}^{<k} \cdot (\mathbf{v}_{3D} \cdot \nabla \mathbf{u}_{2D}) \rangle$

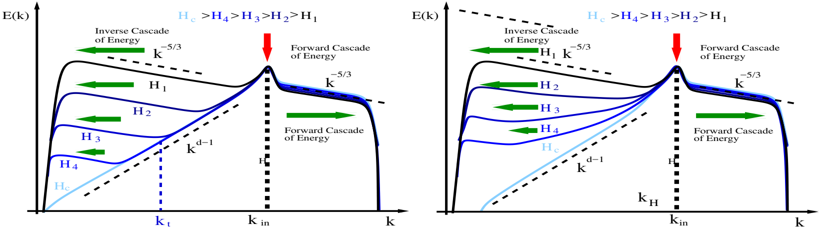
· · · 3D Flux $\Pi_q(k) = -\langle \mathbf{v}_{3D}^{<k} \cdot (\mathbf{u} \cdot \nabla \mathbf{v}_{3D}) \rangle$

Some processes move energy to large scales
and some processes move energy to small scales!

Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Spectrum transition

Transition from a ‘thermal’ to a ‘Kolmogorov’ spectrum



- Either the spectrum changes from a thermal spectrum

$$E(k) \propto (\epsilon_{in}^{2/3} k_{in}^{-8/3}) k$$

to a Kolmogorov spectrum

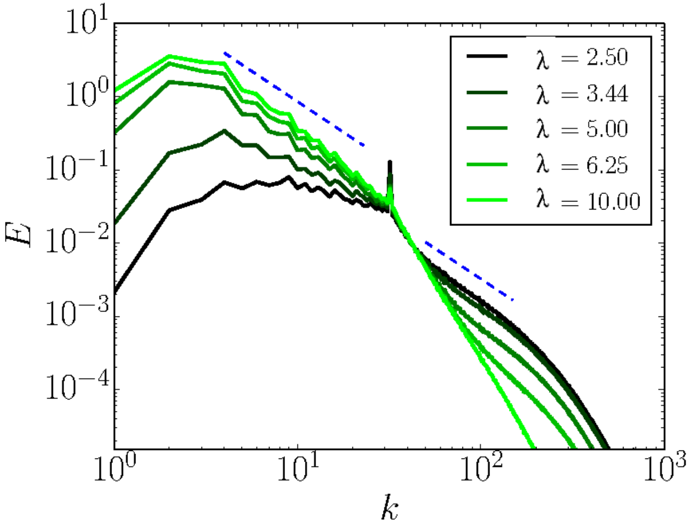
$$E(k) \propto \epsilon_\alpha^{2/3} k^{-5/3}$$

at $k_t = k_{in}(\epsilon_\alpha/\epsilon_{in})^{1/4}$

or

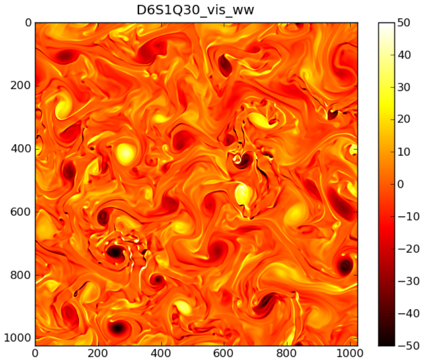
- New power-laws (and new physics) appear!

Example III: Spectrum transition



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

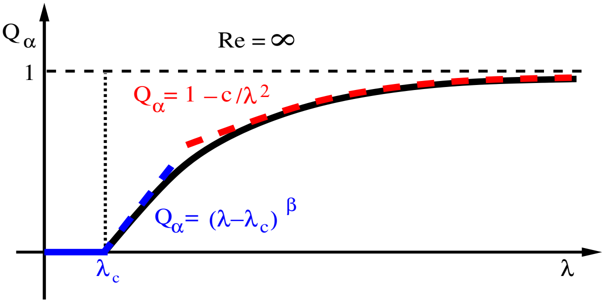
Example III: Physical space, 2D and 3D Dynamics



Predator-prey dynamics?

Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Cascade transitions in thin layers (at infinite *Re*)

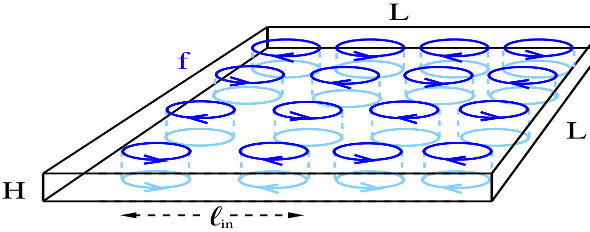


For infinite Reynolds displayed

- a "2nd order" transition (from a forward cascade to a split cascade)
- a "smooth" transition (from a split cascade to an inverse cascade)

VI.
Finite Re and
Finite size effects

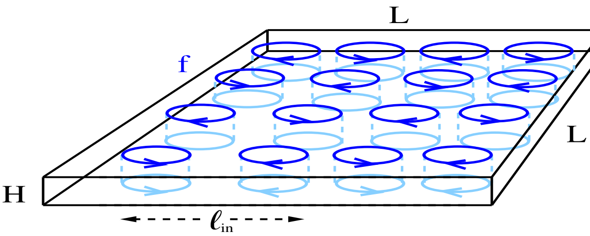
Finite Re: Transition to exactly 2D Dynamics



$$\mathbf{u}_{2D} = \overline{\mathbf{u}}, \qquad \mathbf{v}_{3D} = \mathbf{u} - \mathbf{u}_{\mathbf{2D}}$$

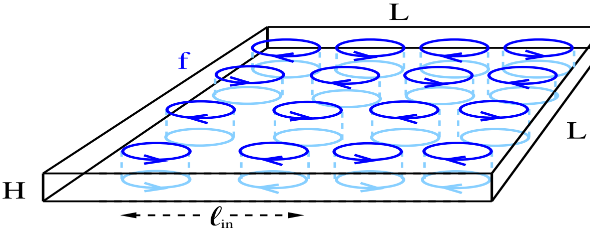
$$\partial_t \mathbf{u}_{2D} + \mathbf{u}_{2D} \cdot \nabla \mathbf{u}_{2D} = -\overline{\mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D}} - \overline{\nabla P} + \nu \Delta \mathbf{u}_{2D} + \eta \Delta^{-2} \mathbf{u}_{2D} + \mathbf{F}_{2D}$$
$$\partial_t \mathbf{v}_{3D} + \mathbf{u}_{2D} \cdot \nabla \mathbf{v}_{3D} + \mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D} = -\mathbf{v}_{3D} \cdot \nabla \mathbf{u}_{2D} + \nabla p' + \nu \Delta \mathbf{v}_{3D}$$

Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)



$$\frac{d}{dt}\langle {\mathbf{v}_{3D}}^2 \rangle = -\langle \mathbf{v}_{3D} \cdot (\nabla \mathbf{u}_{2D}) \cdot \mathbf{v}_{3D} \rangle - \nu \langle (\nabla_{2D} \mathbf{v}_{3D})^2 \rangle - \nu \langle (\partial_z \mathbf{v}_{3D})^2 \rangle$$

Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)



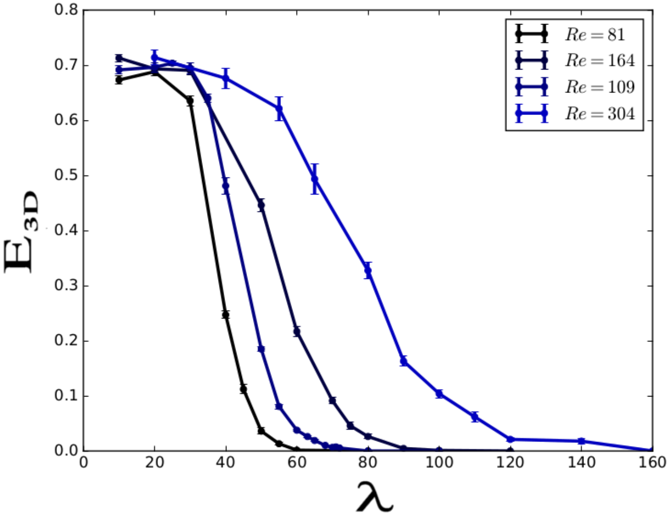
$$\frac{d}{dt}\langle {\mathbf{v}_{3D}}^2\rangle = -\langle \mathbf{v}_{3D}\cdot(\nabla \mathbf{u}_{2D})\cdot \mathbf{v}_{3D}\rangle - \nu\langle(\nabla_{2D}\mathbf{v}_{3D})^2\rangle - \nu\langle(\partial_z \mathbf{v}_{3D})^2\rangle$$

$$\text{if } \nu\left(\frac{2\pi}{H}\right)^2 > \|\nabla \mathbf{u}_{2D}\|_\infty \text{ then } \langle {\mathbf{v}_{3D}}^2\rangle \to 0$$

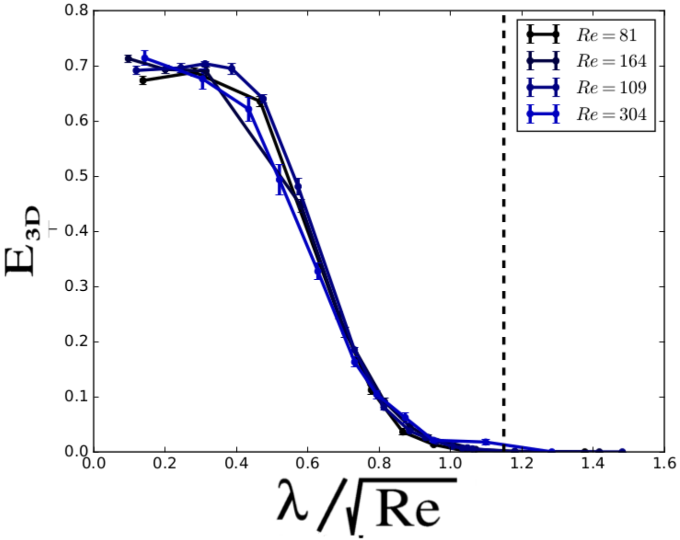
$$\boxed{H_c \sim \ell_\nu \sim \ell_f/\sqrt{Re}}$$

[see Gallet & Doering J. Fluid Mech. **773**, 154 (2015) for proof]

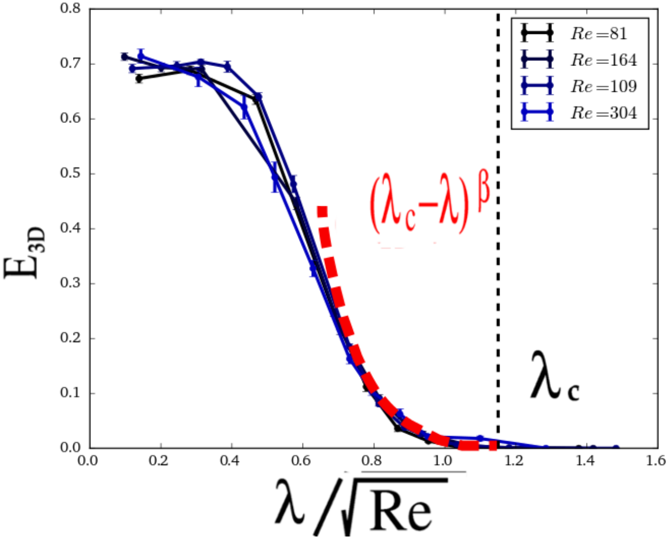
Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)



Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)

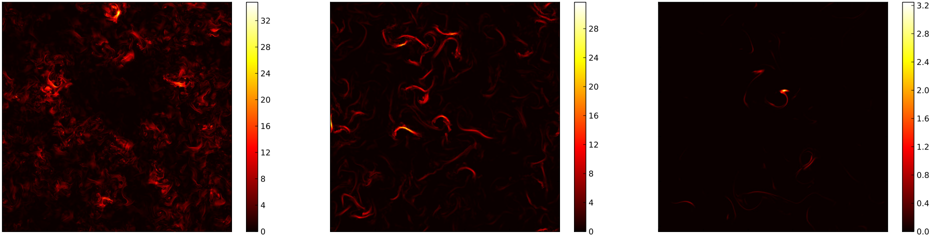


Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)



Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)

Spatial intermittency
3D energy density

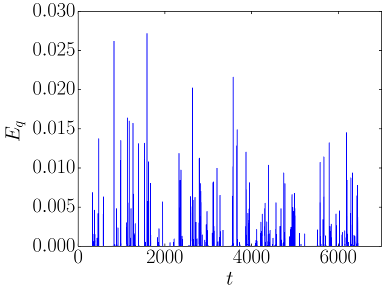


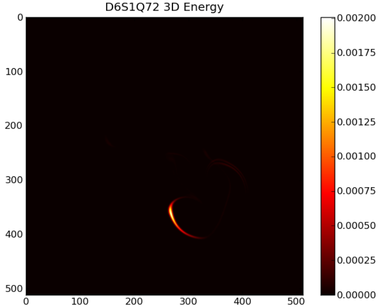
$$\frac{(\lambda_c - \lambda)}{\lambda_c} = 0.70, \qquad \frac{(\lambda_c - \lambda)}{\lambda_c} = 0.40, \qquad \frac{(\lambda_c - \lambda)}{\lambda_c} = 0.10$$

$$\langle E_{3D} \rangle = \frac{\varepsilon_{NL} V_{on}}{(V_{off} + V_{on})} \quad \text{where} \quad \varepsilon_{NL} \propto (\lambda_c - \lambda)^{\beta_1} \quad \text{and} \quad V_{on} \propto (\lambda_c - \lambda)^{\beta_2}$$

Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Spatio-Temporal intermittency

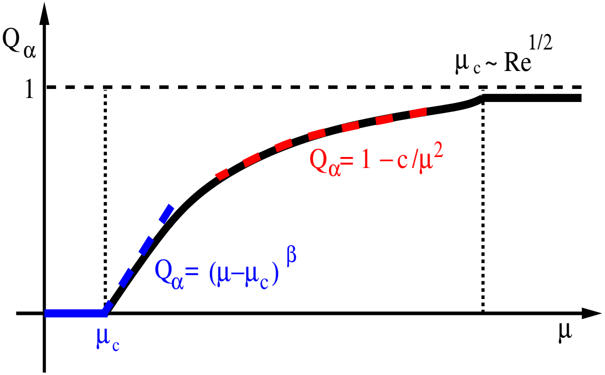




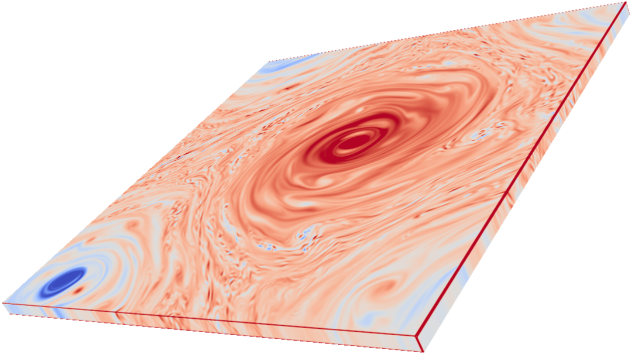
$$\langle E_{3D} \rangle = \frac{\varepsilon_{NL} V_{on} T_{on}}{(V_{off} + V_{on})(T_{on} + T_{off})}$$

Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Cascade transitions in thin layers (at finite *Re*)



Finite Size: Condensates



Finite Size: Condensates

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Finite Size: Condensates

Thin Layers

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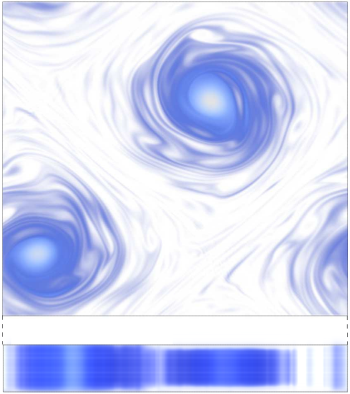
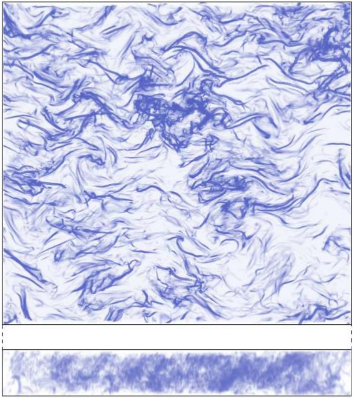
Two distinct states

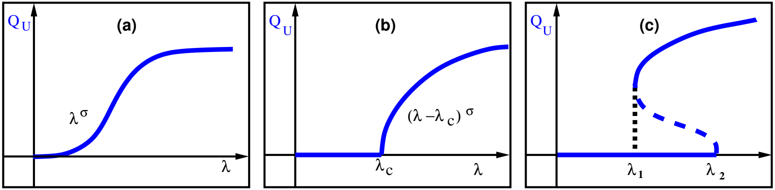


Hi-Energy
Organized state
$\lambda > \lambda_c$



Low-Energy
Disorganized state
$\lambda < \lambda_c$

Finite Size: Condensates

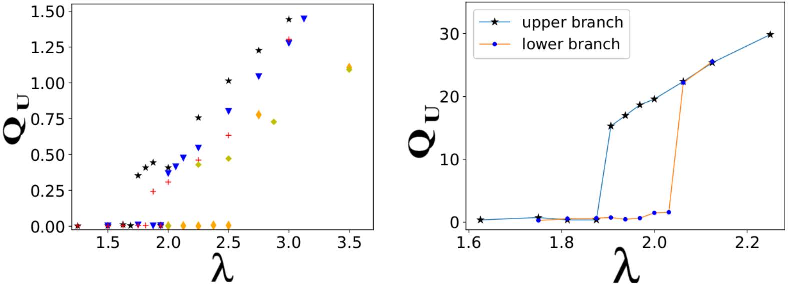


$$Q_U = \frac{E}{(\epsilon_{in}\ell_{in})^{2/3}}$$

- A more relevant order parameter is the energy of the condensate
- Same classification as with the cascade rate (Q_α, Q_ν).
- Continuity of Q_α, Q_ν does not imply continuity of Q_U

Finite Size: Condensates

Thin layer turbulence

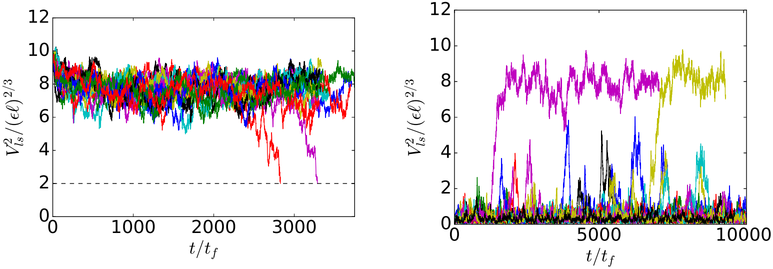


Same behavior observed in:
Rotating turbulence:N.Yokoyama et al Phys.Rev.Fluids 2, 092602 (2017)
Rotating convection:B.Favier et al J.Fluid Mech. 864, R1 (2019))

Figure from: A. van Kan, A. Alexakis Journal of Fluid Mechanics 864, 490-518 (2019)

Finite Size: Condensates

Trajectories close to the critical points



There are rare events that transition the flow from one state to an other

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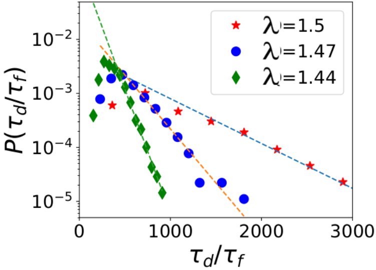
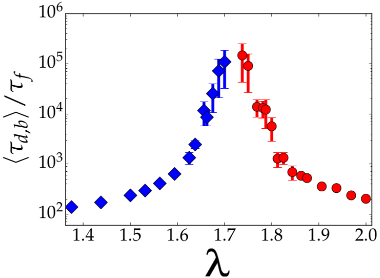
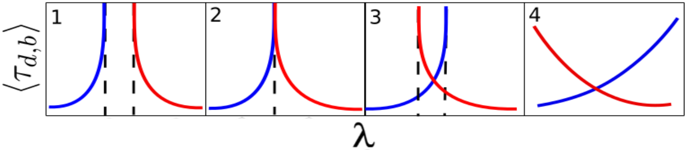
Figure from:A. van Kan, T. Nemoto, A. Alexakis, arXiv:1903.05578 (2019)

Finite Size: Condensates

Mean transition time







The transition to a condensate state is controlled by rare events!

V.
Conclusions

Conclusions

1. Criticality of the energy flux develops in the large L, Re, R_α limit.
2. A split cascade is possible if large scales follow different physics than the small scales.
3. 1st order (discontinuous) transitions are expected in self similar models.
4. Smooth transitions are met when the local interactions dominate and change from inverse cascading to forward cascading.
5. 2nd order transitions are the least understood transitions.
6. 2nd order transition are met when local and non-local interactions compete for the direction of cascade. It might be controlled by Predator - Prey dynamics.
7. For finite Re spatio-temporal intermittency appears
8. For finite size condensates controlled by rare events are present.

Open Problems

- **Critical transitions:**
  Criticality of the transitions has only been demonstrated in numerical simulations of moderate scale separations (moderate Λ, Re, R_α) and models and is still conjectural.
  There is very little theory and very few experiments on the subject.

- **Phase space diagram of rotating and stratified turbulence:**
  Understanding rotating and stratified turbulence in its different limits is crucial for understanding the atmosphere and the ocean.

- **Transition to exactly 2D flows and spatio-temporal intermittency:** Appearance of new exponents.

- **Condensates and thermal equilibrium states:** Rare event sampling methods could help.

Future work

- There is a lot more we need to understand!
- There is need for new experiments!
- There is need for new numerical simulations!
- There is need for a theory!
- **There is need for young researchers!**

Thank you very much

for your patience!

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