Eur. Phys. J. Special Topics **166**, 83–88 (2009)

DOI: 10.1140/epjst/e2009-00883-1

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Suction in Darcy and Stokes interfacial flows: Maximum growth rate versus minimum dissipation

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Abstract. Two-dimensional flows with suction or mass loss are investigated within Darcy's or Stokes' framework. Examples include a Hele-Shaw cell with a lifted plate or extraction of lipids from a lipid bilayer. An initially circular patch retracts due to the suction and might undergo an instability whereby it becomes undulating. The selection of the wavelength of undulations is investigated with the help of an extremum principle, the minimization of the generalized dissipation, from which derive the flow equations.

The selection mechanism of periodic patterns remains a challenging problem in hydrodynamics, especially for interfacial flows. The determination of the wavelength of Rayleigh-Bénard and the Bénard-Marangoni patterns have motivated an enormous literature [1]. In the present paper, we suggest that, for out-of-equilibrium problems which have a variational formulation, it is possible to predict the wavelength from the minimisation of the corresponding functional. We apply this formalism to viscous flows such as suction (or loss of mass) of a viscous liquid described within Stokes' or Darcy's equations. For simplicity, we restrict to a 2D treatment and illustrate our theory by two experimental configurations. The first one concerns the separation of two plates adhering through a viscous fluid, the second the extraction of lipids from an inhomogeneous lipid bilayer. In both cases, a continuum set of solutions is available, and the selection of the physical solution raises difficulties.

In the experiment with adhesion, the viscous liquid is confined between two parallel glass plates, separated by a tiny gap; air surrounds the viscous patch. The lifting of one plate induces an effective sucking; the size of the patch decreases with time. The interface between the liquid and air exhibits instantaneously a large number of undulations which can be explained qualitatively by a fingering instabiltiy analogous to the Saffman-Taylor instability [2–4]. A more quantitative explanation was given within the formalism of Darcy flows which result from the lubrication approximation to a 3D Stokes flow when one dimension (the gap between the plates) is tiny. In [5,6,8,9], the patterns were explained but the comparison of the theoretical wavelength with the experimental one was not satisfactory. The same situation occurs when lipids are extracted from an inhomogenous lipid bilayer containing two different phases [7,10–12]. In this case, since the thickness of the bilayer is fixed by the size of the lipids (around 5 nm), the flow is mostly described by a 2D Stokes flow [13]. As for Darcy flow, when the maximum growth rate criterion is applied to this Stokes flow, the wavelength is found to disagree with experiments.

To resolve this disagreement, our approach consists in transforming the hydrodynamic equations with their boundary conditions into the minimization of a generalized viscous dissipation. The search of a minimum among the continuous set of solutions selects a well-defined wavenumber. The perturbative treatement is restricted to short times. However, we believe that the use of a variational principle corresponding to the flow equations can be successfully applied

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to wavelength selection in other experimental situations such as coating flows. This method is commonly used in elasticity [14,15] where the total mechanical energy is conserved but is more unusual in hydrodynamics [4].

1 Formulation

We consider a suction problem in a two-dimensional flow, obeying either Stokes or Darcy's equations. Some pumping or sucking occurs in a patch of viscous fluid of viscosity μ_{i} (i like inner) in an infinite two-dimensional bath of viscosity μ_{o} (o like outer) such that the mass of the two-dimensional surface of the inner viscous blob decreases with time. Sucking can also occur in the outer phase, so this process is represented by $q(r)$, a density of mass loss per unit time at a point at r (we use polar coordinates r, θ). The suction remains isotropic so cannot explain simply any undulation of the boundary between the two phases. The velocity field V_α in each phase $\alpha = \mathrm{i}, \mathrm{o}$ satisfies the conservation of mass

$$\nabla \cdot V_\alpha = -q_\alpha(r). \tag{1}$$

For the sake of simplicity, we present the computations for $q_{\mathrm{i}} = q$ constant strictly inside, $q = 0$ at the interface and $q_{\mathrm{o}} = 0$ outside. This is not a limitation and the general result is given in the conclusion. We will also use the total suction in the inner phase $Q_{\mathrm{i}} = qR_0^2/2$ in the following.

The kinematic boundary conditions are related to the normal velocity U_N of the interface (outward unit normal N) and to the absence of flow at infinity,

$$U_N = V_{\mathrm{i}} \cdot N = V_{\mathrm{o}} \cdot N \quad \text{and} \quad V_{\mathrm{o}}(r = +\infty) = 0, \tag{2}$$

respectively.
Darcy flow. We consider a flow following Darcy's equation

$$V_\alpha = -\frac{b^2}{12\mu_\alpha} \nabla P_\alpha, \tag{3}$$

P_α being the pressure field in phase α. The jump in pressure at the interface is proportional to its curvature κ,

$$P_{\mathrm{i}} - P_{\mathrm{o}} = \gamma\kappa, \tag{4}$$

γ being the surface tension. For a comparison with experiments, the value of γ must be corrected by a numerical prefactor due to 3D geometrical effects [16].
Stokes flow. The two-dimensional Stokes equation is

$$\nabla P_\alpha = \mu_\alpha \nabla^2 V_\alpha. \tag{5}$$

In this case, the tangential component of the velocity is also continuous at the interface

$$V_{\mathrm{i}} \cdot T = V_{\mathrm{o}} \cdot T. \tag{6}$$

The last boundary condition involves the stress tensor, given in terms of the pressure and the strain rate tensor E_α,

$$\sigma_\alpha = -P_\alpha I + \mu \left(E_\alpha - \frac{1}{2} \mathrm{trace} E_\alpha I \right) \tag{7}$$

$$E_\alpha = \nabla V_\alpha + (\nabla V_\alpha)^t. \tag{8}$$

In Cartesian coordinates, $E_{\alpha,ij} = \partial_i V_{\alpha,j} + \partial_j V_{\alpha,i}$. The trace of the strain rate tensor E_α times the identity matrix I is subtracted [17] because the flow is not necessarily incompressible because of the mass loss. The stress balance at the interface gives

$$(\sigma_{\mathrm{i}} - \sigma_{\mathrm{o}}) \cdot N = \gamma\kappa N, \tag{9}$$

γ being proportional to the line tension of the raft/lipid interface.

Linear stability. The base state corresponds to a retracting circular inner patch of radius $R_0(t)$, such that

$$\frac{\mathrm{dR_0}}{\mathrm{d}t} = -\frac{Q_\mathrm{i}}{R_0}. \tag{10}$$

In the following, we consider a perturbation of the circular contour and investigate wether this perturbation will grow or not, and seek which growing perturbation will be selected. At the linear approximation, sinusoidal modes decouple so that the perturbation is restricted to a mode labeled by an integer n; the position of the interface $r = R(\theta, t)$ is given by

$$R(\theta, t) = R_0(t)(1 + \epsilon_n(t)\cos n\theta). \tag{11}$$

The contour becomes unstable under perturbation of wavenumber n as soon as the growth rate

$$\omega_n = \frac{1}{\epsilon_n(t)}\frac{\mathrm{d}\epsilon_n(t)}{\mathrm{d}t} \tag{12}$$

is positive. Eventually, note that $n = 0$ and $n = 1$ correspond to a change in radius of the interface and to a translation of the interface, respectively, so that $n = 2$ is the smallest physically acceptable wavenumber.

2 Darcy flow

Linear stability. The base state is axisymmetric: the pressure is given by $P_\mathrm{i}^{(0)} = 3\mu_\mathrm{i} q(r^2 - R_0^2)/b^2 + \gamma/R_0$ and $P_\mathrm{o}^{(0)} = 12\mu_\mathrm{o} Q_\mathrm{i}/b^2 \ln(r/R_0)$, hence the radial and orthoradial components of the velocity $V_{\mathrm{i},r}^{(0)} = -qr/2$, $V_{\mathrm{o},r}^{(0)} = -Q_\mathrm{i}/r$ and $V_{\alpha,\theta}^{(0)} = 0$.

The linear stability analysis of this base state was performed by [6,8] when $q_\mathrm{i} = q_\mathrm{o}$ are constant. The case $q_\mathrm{o} = 0$ is very similar and the perturbations to the pressure p_α and the velocity v_α are found to be

$$p_\alpha = s_\alpha \mu_\alpha \epsilon_n \frac{12 R_0 \omega_n}{n b^2} \left(r/R_0\right)^{s_\alpha n} \cos n\theta, \tag{13}$$

$$v_{\alpha,r} = \omega_n \epsilon_n \left(r/R_0\right)^{s_\alpha (n-1)} \cos n\theta, \tag{14}$$

$$v_{\alpha,\theta} = -s_\alpha \omega_n \epsilon_n \left(r R_0\right)^{s_\alpha (n-1)} \sin n\theta \tag{15}$$

$s_\mathrm{i} = +1$ for the inner phase and $s_\mathrm{o} = -1$ for the outer phase, the growth rate $\omega_n = 1/\epsilon_n\, \mathrm{d}\epsilon_n/\mathrm{d}t$ being

$$\omega_n = \frac{2Q_\mathrm{i}}{R_0^2} + \frac{Q_\mathrm{i}}{R_0^2}\frac{\mu_\mathrm{i} - \mu_\mathrm{o}}{\mu_\mathrm{i} + \mu_\mathrm{o}} n - \frac{\gamma b^2}{12 R_0^3(\mu_\mathrm{i} + \mu_\mathrm{o})} n(n^2 - 1). \tag{16}$$

Q_i is again the suction in the inner phase. As known for the Saffman-Taylor instability, there is a band of unstable wavenumbers if the inner fluid is more viscous, with the additional condition that $\omega_2 > 0$ as $n = 2$ is the smallest physically acceptable wavenumber.

Maximum growth rate. In order to find the selected perturbation, the usual procedure is to maximize ω_n with respect to n. The corresponding $n = n_\mathrm{G}$ has a simple expression in the limit $n \gg 1$,

$$n_\mathrm{G} = 2\sqrt{Q_\mathrm{i} R_0(\mu_\mathrm{i} - \mu_\mathrm{o})/\gamma h^2}. \tag{17}$$

Minimum dissipation. Here we introduce the generalized dissipation as the sum of the work of surface tension and viscous dissipation, with the constraint of imposed loss of mass

$$\varXi = \gamma \oint \kappa (V_\mathrm{i} \cdot N)\, \mathrm{d}\ell + \sum_{\alpha=\mathrm{i,o}} \left[\frac{6\mu_\alpha}{b^2}\int V_\alpha^2\, \mathrm{d}S + \int P_\alpha (\nabla \cdot V_\alpha + q_\alpha)\, \mathrm{d}S\right]. \tag{18}$$

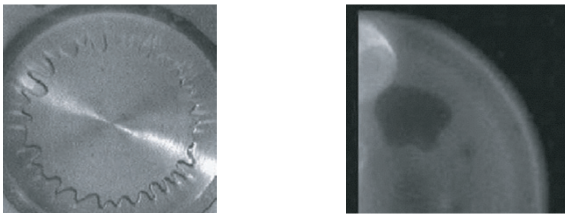

Fig. 1. Experimental situations of interest. Left – Darcy flow in a Hele-Shaw cell: when lifting the top plate of a cell containing a viscous blob, surrounding air invades the cell forming fingers (experiments by D. Bonn [6]). Right – Stokes flow in a vesicle made of a lipid bilayer: when lipids are sucked out of a raft (darker patch), the interface between the raft and the rest of the vesicle retracts and undulates (experiments by N. Puff and M. Angelova [7]).

The work of surface tension involves the normal velocity of the interface $V_\mathrm{i} \cdot N = V_\mathrm{o} \cdot N$ and its curvature κ, and is an integral along the boundary between the two fluids. The viscous dissipation was obtained from the standard three-dimensional formula [17] in terms of the strain tensor E (see Eq. 8),

$$\frac{1}{2}\mu \int \mathrm{trace}\left(E^2\right) \mathrm{d}\tau = \frac{1}{2}\mu \int \sum_{i,j} \left(\partial_i U_j + \partial_j U_i\right)^2 \mathrm{d}\tau, \tag{19}$$

in Cartesian coordinates (3D velocity components U_i), by using the small thickness approximation, and incompressible 3D flow, and a parabolic velocity profile in the cell transverse direction, i.e. the same conditions as that used to establish Darcy's equation for a Hele-Shaw cell. The pressure acts as a Lagrange multiplier. The stationarity of Ξ with respect to the velocities V_α indeed yields Darcy's equation (3) and Laplace's law (4).

We look for a minimum of $\Xi(n)$ as the wavenumber n of the contour varies. We use the velocity fields (Eqs. 14–15) at order 1 in ϵ_n and plug them into the generalized dissipation Ξ (it can be checked that higher order terms do not contribute to the value of Ξ),

$$\begin{aligned} \Xi - \Xi(\epsilon_n = 0) = \epsilon_n^2 \frac{\pi\gamma^2 b^2}{288(\mu_\mathrm{i} + \mu_\mathrm{o}) b^2 R_0^2 n} & [-12n^2(1-n^2)^2 - 6(2n^2 - 3n - 8)(N_\mathrm{i}^2 - N_\mathrm{o}^2) \\ & + n^2(2n^2 - n - 2)(N_\mathrm{i} - N_\mathrm{o}) + 24(n^2 + n + 4) N_\mathrm{i} N_\mathrm{o}], \\ & \text{with} \quad N_\mathrm{i} = \frac{\mu_\mathrm{i} Q_\mathrm{i} R_0}{\gamma b^2} \quad \text{and} \quad N_\mathrm{o} = \frac{\mu_\mathrm{o} Q_\mathrm{i} R_0}{\gamma b^2}. \end{aligned} \tag{20}$$

The functional Ξ has a minimum for a well-defined value of $n = n_\mathrm{D} < n_\mathrm{G}$. Its value in the limit $n \gg 1$ is

$$n_\mathrm{D} = \sqrt{3/5}\, n_\mathrm{G}. \tag{21}$$

In Hele-Shaw experiments with a lifted plate, the rate of suction $q(r)$ is constant everywhere. In the case of Fig.1, the outer fluid is air with zero viscosity so the value of q is irrelevant in this phase. Our analysis applies to this experimental situation when taking $q_\mathrm{o} = \mu_\mathrm{o} = 0$.

3 Two-dimensional Stokes flow

Linear stability. The base state is axisymmetric; the components of the velocity $V_{\mathrm{i,r}}^{(0)} = -qr/2$, $V_{\mathrm{o,r}}^{(0)} = -Q_\mathrm{i}/r$ and $V_{\alpha,\theta}^{(0)} = 0$ are imposed by mass conservation (1). The Stokes equation (5) and the the boundary conditions (2,9) fix the pressure, $P_\mathrm{o} = \mu_\mathrm{o} q$ and $P_\mathrm{i} = \gamma/R_0$.

The linear stability analysis is rather tedious but straightforward. We give here the main steps. Complex analysis can be used when the outer viscosity vanishes [18,19]. The perturbed pressure field satisfies the Laplace equation:

$$p_\alpha = \pi_\alpha \left(r/R_0\right)^{s_\alpha n} \cos n\theta. \tag{22}$$

Here $s_\mathrm{i} = +1$ and $s_\mathrm{o} = -1$, the π_α are two time-dependent amplitudes of order $\epsilon_n(t)$. The Stokes equation yields the component of the velocities

$$v_{\alpha,r} = \cos n\theta \frac{R_0}{\mu_\alpha} \left[\pi_\alpha \frac{s_\alpha n}{4(1+s_\alpha n)} (r/R_0)^{s_\alpha n+1} + s_\alpha a_\alpha (r/R_0)^{s_\alpha n-1} \right], \tag{23}$$

$$v_{\alpha,\theta} = \sin n\theta \frac{R_0}{\mu_\alpha} \left[-s_\alpha \pi_\alpha \frac{2+s_\alpha n}{4(1+s_\alpha n)} (r/R_0)^{s_\alpha n+1} - a_\alpha (r/R_0)^{s_\alpha n-1} \right]. \tag{24}$$

There are 4 degrees of freedom $(a_\mathrm{i}, a_\mathrm{o}, \pi_\mathrm{i}, \pi_\mathrm{o})$ but also 4 boundary conditions: the continuity of the velocities at the interface (2,6), the balance of normal and tangential stresses (9), so that we obtain

$$\begin{aligned} \frac{\pi_\alpha}{\epsilon_n} &= \frac{\mu_\alpha}{\mu_\mathrm{i}+\mu_\mathrm{o}} \left[4(n+s_\alpha)(\mu_\mathrm{o}-\mu_\mathrm{i}) \frac{Q_\mathrm{i}}{R_0^2} + s_\alpha \gamma \frac{n^2-1}{R_0} \right], \\ \frac{a_\alpha}{\epsilon_n} &= \frac{\mu_\alpha}{\mu_\mathrm{i}+\mu_\mathrm{o}} \left[s_\alpha n(\mu_\mathrm{o}-\mu_\mathrm{i}) \frac{Q_\mathrm{i}}{R_0^2} + \gamma \frac{n^2+s_\alpha n}{4R_0} \right]. \end{aligned} \tag{25}$$

At order 1 in ϵ_n, the kinematic boundary condition at the interface (2) yields $\mathrm{d}R/\mathrm{d}t = U_N = V_{\mathrm{i},r}^{(0)} + v_{\mathrm{i},r}$, hence the growth rate

$$\omega_n = 2\frac{Q_\mathrm{i}}{R_0^2} - \frac{n}{2} \frac{\gamma}{(\mu_\mathrm{o}+\mu_\mathrm{i})R_0}. \tag{26}$$

As long as $\omega_2 > 0$, there is a band of unstable wavenumbers, independently of the contrast of viscosities.

Maximum growth rate. Maximizing ω_n with respect to n yields the smallest physically acceptable wavenumber

$$n_\mathrm{G} = 2, \tag{27}$$

which is in contrast with the observations of $n > 2$ [7].

Minimum dissipation. Again we introduce the generalized dissipation as the sum of the work of surface tension and viscous dissipation, with the constraint of imposed loss of mass,

$$\Xi = \gamma \oint \kappa (V_\mathrm{i} \cdot N)\,\mathrm{d}\ell + \sum_{\alpha=\mathrm{i,o}} \left[\frac{\mu_\alpha}{2} \int \mathrm{trace}(\tilde{E}_\alpha^2)\,\mathrm{d}S + \int P_\alpha (\nabla \cdot V_\alpha + q_\alpha)\,\mathrm{d}S \right]. \tag{28}$$

The only difference with the functional (18) lies in the viscous dissipation which is the same as the three-dimensional one (19), except that the strain rate E_α was replaced by the traceless shear rate tensor

$$\tilde{E}_\alpha = E_\alpha - \frac{1}{2}\mathrm{trace} E_\alpha I \tag{29}$$

because the flow is not necessarily incompressible. The stationarity of Ξ with respect to the velocity fields indeed yields the Stokes equation (5) and the boundary condition for the stress tensor (9).

We look for a minimum of $\Xi(n)$. We replace the velocity fields (23–24) in the generalized dissipation Ξ:

$$\Xi - \Xi(\epsilon_n = 0) = \epsilon_n^2 \frac{\pi Q_\mathrm{i} \gamma}{2R_0} \left[\frac{3(\mu_\mathrm{i}+\mu_\mathrm{o}) + 4(\mu_\mathrm{i}-\mu_\mathrm{o})n}{(\mu_\mathrm{i}+\mu_\mathrm{o})} \frac{(\mu_\mathrm{o}-\mu_\mathrm{i})Q_\mathrm{i}}{\gamma R_0} - 2 + 3n^2 \right]. \tag{30}$$

The functional has a single minimum for $n = n_\mathrm{D}$,

$$n_\mathrm{D} = \frac{8(\mu_\mathrm{i}-\mu_\mathrm{o})^2 Q_\mathrm{i}}{3\gamma(\mu_\mathrm{i}+\mu_\mathrm{o})R_0}, \tag{31}$$

inside the band of unstable wavenumbers ($\omega_{n_\mathrm{D}} > 0$). Here a contrast in viscosities can lead to the selection of a wavenumber $n_\mathrm{D} > n_\mathrm{G}$.

4 Discussion

We now discuss the physical effects contained in the dispersion relations (16,26). The first term, which is independent on viscosities, $\omega_n^{\rm I} = Q_{\rm i}/R_0^2$ was given for a constant suction $q_{\rm i} = q$ strictly inside and no suction $q_{\rm o} = 0$ outside. For a general and *smooth* suction $q(r)$, similar computations show that this term must be replaced by

$$\omega_n^{\rm I} = \frac{2}{R_0^2} \int_0^{R_0} r q(r)\,{\rm d}r - q(R_0). \tag{32}$$

It is of purely kinematic origin: if $\omega_n^{\rm I} > 0$ then the inward radial velocity increases from the interface to the center, so that any inward bump of the interface would be amplified. This term was previously derived in two special cases. For a Stokes flow with a point sink [19], $q = Q\delta(r)/r$, so that $\omega_n^{\rm I} = 2Q/R_0^2$; for a Hele-Shaw cell with a lifted plate [8,6], the rate of lifiting is uniform $q_{\rm i} = q_{\rm o} = q$, so that $\omega_n^{\rm I} = 0$ and suction has no kinematic effect. This kinematic effect is the only destabilising mechanism for Stokes flows as studied here (see eq. (26)). For Darcy flows, it is well-known that pushing the more viscous fluid is destabilising, as can be seen in the second term of (16). Finally, surface tension is always stabilizing.

Once the system is found to be unstable, the standard procedure to determine the selected modes is to maximize the growth rate ω_n. However, this criterion is rigorous only near the threshold of an instability [1]. Moreover, it does not agree with experimental observations on Hele-Shaw cells with a lifted plate [6] or lipid rafts in vesicles [7]. As a consequence, we propose to use another extremum principle instead – the minimization of the generalized dissipation. We introduced the functionals (18,28), the stationarity of which yields Darcy's and Stokes equations respectively. The minimization of the generalized dissipations leads to the selection of modes which are different from the ones with maximum growth, and which seem to be closer to experiments.

An experimental test of the best extremum principle could be done by pumping at the center of a Hele-Shaw cell with two fluids having the same viscosity. The destabilization would be due to the kinematic mechanism (as given by Eq. (32)); maximum growth rate would lead to a mode $n = 2$ (Eq. (16)) whereas minimum dissipation would lead to $n > 2$ in general (Eq. (18)). Another experimental challenge would be to devise a two-dimensional Stokes suction flow that is more controlled than lipid membranes.

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