Eur. J. Mech., B/Fluids, **12**, n° 5, 683-693, 1993

Linear hydrodynamic instability of circular jets with thin shear layers

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ABSTRACT. — A new asymptotic expansion well adapted to the linear stability study of compressible jets with thin shear layers is presented. The validity of the approximation is established by making numerical comparisons with solutions of the full cylindrical Rayleigh equation. In particular helical modes are shown to be asymptotically equivalent to three-dimensional disturbances of planar shear layers. When the Mach number is zero Squire's theorem is recovered in the limit of vanishing shear layer thickness, thereby establishing that the axisymmetric mode is necessarily the most unstable. Finally the asymptotic expansion is shown to converge on both the $-\alpha_i^+$ and $-\alpha_i^-$ branches of the absolute convective transition that takes place when a small back flow is added to the jet.

1. Introduction

The objective of this paper is to present an asymptotic approximation that is well suited to study the linear stability characteristics of circular jets with velocity profile $u(r)$ and density profile $\rho(r)$ varying significantly only inside a thin circular layer of radius R and thickness θ. Thus we consider axisymmetric basic profiles of the form

$$u = u((r-\mathrm{R})/\theta), \tag{1}$$

$$\rho = \rho((r-\mathrm{R})/\theta), \tag{2}$$

where r is the radial distance from the axis of the jet. The fast variable in the asymptotic expansion is defined as $\xi = (r-\mathrm{R})/\theta$, and the small parameter is the ratio of length scales $\varepsilon = \theta/\mathrm{R}$.

The paper is organized as follows: in Section 2 the linear stability problem in cylindrical geometry is demonstrated to be approximated at leading order in ε by the usual Rayleigh equation for planar shear layers. Section 3 is devoted to a numerical validation and discussion of the results. The classical hyperbolic tangent velocity profile studied by Michalke is used as a test problem. The linear stability characteristics of circular jets with thin shear layers are shown to be well approximated by the eigenvalues of the

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0997-7546/93/05/$ 4.00/

corresponding planar problem for a wide range of values of ε. Section 4 contains our concluding remarks.

2. Asymptotic expansion

As discussed for instance by [Michalke, 1984], the cylindrical Rayleigh equation that governs the linear instability of axisymmetric jets is given by

$$\mathrm{L}\,\mathrm{P}(r)=0, \tag{3}$$

where the L operator is defined as

$$\mathrm{L}=\frac{d^2}{dr^2}+\left[\frac{1}{r}-2\,\frac{(du/dr)}{(u-c)}-\frac{(d\rho/dr)}{\rho}\right]\frac{d}{dr}-\left[\alpha^2\,(1-\mathrm{M}^2\,(u-c)^2\,\rho)+\left(\frac{m}{r}\right)^2\right], \tag{4}$$

with the boundary conditions

$$\lim_{r\to 0}\ \mathrm{P}(r)=\lim_{r\to 0}\ \mathrm{I}_m(\alpha\sqrt{1-\mathrm{M}^2\,(u-c)^2\,\rho}\,r), \tag{5}$$

$$\lim_{r\to 0}\ \mathrm{P}(r)=\lim_{r\to 0}\ \mathrm{K}_m(\alpha\sqrt{1-\mathrm{M}^2\,(u-c)^2\,\rho}\,r), \tag{6}$$

where I_m and K_m are the usual modified Bessel functions.

In the above formulation, the function $\mathrm{P}(r)$ represents the pressure eigenfunction, $u(r)$ and $\rho(r)$ respectively denote the mean velocity and density profiles, α is the wavenumber of the disturbance in the basic flow direction, ω is the frequency and $c=\omega/\alpha$ is the complex phase velocity. The integer m defines the azimuthal wavenumber and M the Mach number.

The boundary conditions (5) and (6) are obtained by noting that both for $r\to 0$ and $r\to\infty$ the velocity and density profiles are constant and equation (3) reduces to the Helmholtz equation, the solutions of which are given by (5) and (6).

This eigenvalue problem with complex eigenfunctions $\mathrm{P}(r)$ can be formulated in two classically distinct ways as shown for example by [Huerre & Monkewitz, 1990]. In the spatial stability framework, the frequency of the perturbation ω and the azimuthal wavenumber m are given real, *i.e.* the perburbation is harmonic in time, and the spatial wavenumber α is a complex unknown, thus possibly giving rise to spatial growth. In the temporal stability analysis, both components α and m of the wavenumber are given real, *i.e.* the perturbation is spatially periodic, and the frequency ω is complex, thus leading to possible temporal growth.

Let us introduce the change of variables

$$\alpha=\alpha_0/\varepsilon,\qquad m=\beta_0\,\mathrm{R}/\varepsilon,\qquad \omega=\omega_0/\varepsilon. \tag{7}$$

This scaling is motivated by dimensional analysis: if U_0 denotes the centerline velocity, we want to keep R and U_0 constant while letting $\varepsilon=\theta/\mathrm{R}\to 0$. Wavenumber and frequency should scale as θ^{-1} and U_0/θ respectively. The scaling for m is obtained by noting that m/R should scale as a wavenumber component, in the same manner as α.

The operator L and the eigenfunction P can then be expanded as

$$L = L_0 + O(\varepsilon), \tag{8}$$

$$P(\xi, \varepsilon) = P_0 + O(\varepsilon). \tag{9}$$

At leading order in ε one then obtains the following equation:

$$L_0 P_0 = 0, \tag{10}$$

where the operator L_0 reads

$$L_0 = \frac{d^2}{d\xi^2} - \left[2\frac{(du/d\xi)}{(u-c_0)} + \frac{(d\rho/d\xi)}{\rho}\right]\frac{d}{d\xi} - [\alpha_0^2(1 - M^2(u-c_0)^2\rho) + \beta_0^2], \tag{11}$$

and the boundary conditions (5) and (6), in the case $m \neq 0$, reduce to

$$\lim_{\xi \to -\infty} P_0(\xi) = \lim_{\xi \to +\infty} P_0(\xi) = 0. \tag{12}$$

In the case $m = 0$, (5) and (6) become

$$\lim_{\xi \to +\infty} P_0(\xi) = 0 \quad \text{and} \quad \lim_{\xi \to -\infty} P_0(\xi) = 1. \tag{13}$$

Note that the boundary condition for $\xi \to -\infty$ should, strictly speaking, be applied at $\xi = -1/\varepsilon$. In the region $\xi \ll -1/\varepsilon$, P_0 vanishes like $\exp[\sqrt{\alpha_0^2(1 - M^2(u-c_0)^2\rho) + \beta_0^2}\,\xi]$ and thus we can equivalently use equation (12).

At leading order in the asymptotic expansion, the equation described by the operator L_0 is identical to the Rayleigh equation corresponding to a plane mixing layer [Djordjevic *et al.*, 1989] with a profile given by the jet profile expressed in the scaled variables. It is

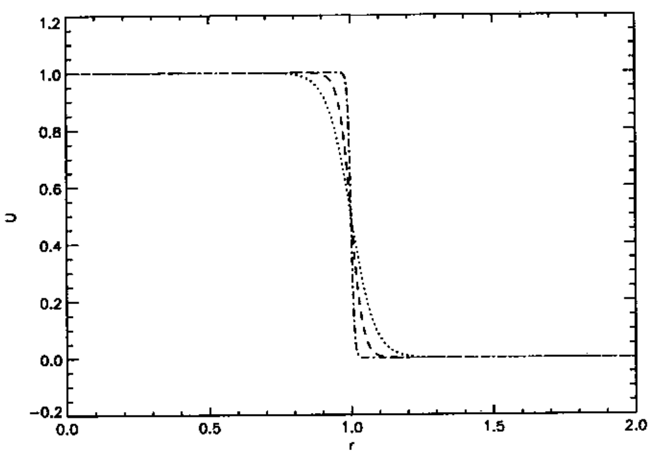

Fig. 1. — A plot of the tanh velocity profile (equation (19) with R = 1): dotted $\varepsilon = 0.08$, dashed $\varepsilon = 0.04$ and dotted dashed $\varepsilon = 0.01$.

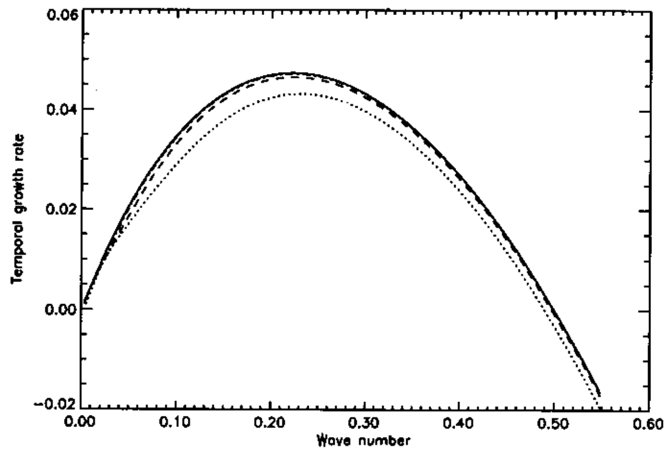

Fig. 2 *a*. — Temporal growth rates ω_i versus wavenumber α for the $m=0$ axisymmetric mode of the constant density, $M=0$, jet. Dotted $\varepsilon=0.08$, dashed $\varepsilon=0.04$, dotted dashed $\varepsilon=0.01$. The solid curve is the leading order asymptotic approximation.

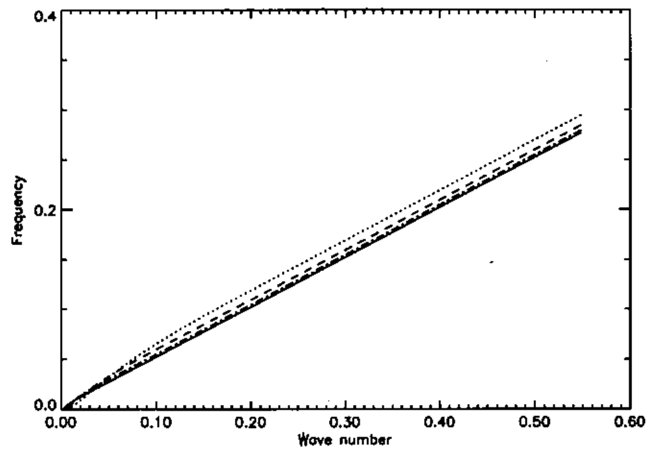

Fig. 2 *b*. — Same as Figure 2 *a* but frequency ω_r versus wave number α.

well known that for incompressible $(M=0)$ mixing layers, in the case of temporal instability, Squire's theorem holds and equation (11) can be written:

$$\tilde{\mathfrak{L}}\tilde{P}=0, \tag{14}$$

with

$$\tilde{\mathfrak{L}}=\frac{d^2}{d\xi^2}-\left[2\,\frac{(du/d\xi)}{(u-\tilde{\omega}/\tilde{\alpha})}+\frac{(d\rho/d\xi)}{\rho}\right]\frac{d}{d\xi}-\tilde{\alpha}^2, \tag{15}$$

and

$$\tilde{\alpha}=(\alpha_0^2+\beta_0^2)^{1/2}, \qquad \tilde{\omega}=(\alpha_0^2+\beta_0^2)^{1/2}\,\omega_0/\alpha_0. \tag{16}$$

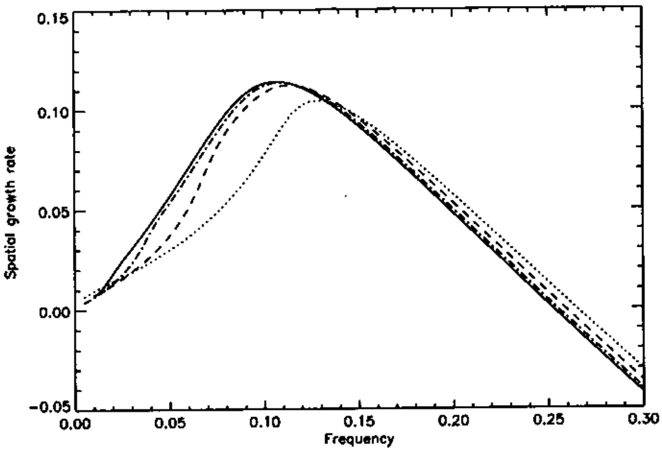

Fig. 3 *a*. — Spatical growth rate $-\alpha_i$ versus frequency ω for the same conditions as in Figure 2 *a*).

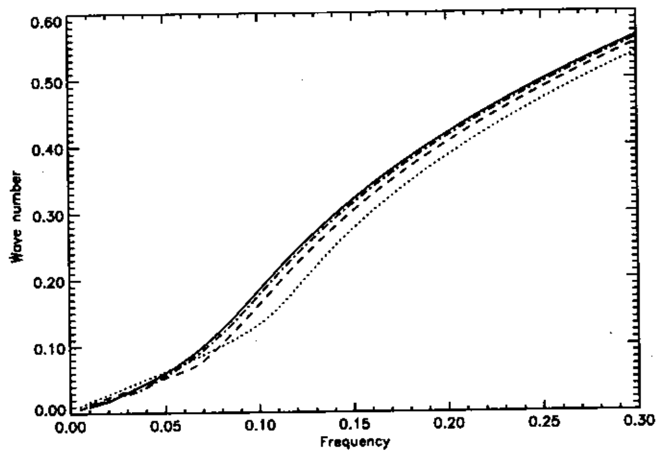

Fig. 3 *b*. — Same as Figure 3 *a*) but wavenumber α_r versus frequency ω.

This transformation shows that the most unstable mode is always two-dimensional. Equation (11) thus implies that, in the limit of vanishing shear layer thickness for incompressible jets, Squire's theorem will be recovered, thereby establishing that the axisymmetric jet mode, *i.e.* $m=0$, is then necessarily the most unstable. Note that in general there is no known analogue of Squire's theorem for axisymmetric jets [Batchelor & Gill, 1962].

Let us close this section by indicating how the expansion could be continued to higher orders. At order one we obtain:

$$L_0 P_1 = -L_1 P_0. \tag{17}$$

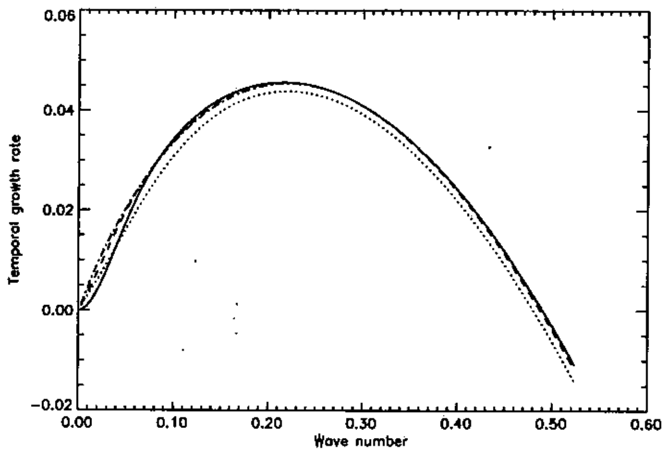

Fig. 4*a*. — Temporal instability for variable density jets (equation (20) with $R_d=0.5$): ω_i versus α, $M=0$, $m=0$. Dotted $\varepsilon=0.08$, dashed $\varepsilon=0.04$, dotted dashed $\varepsilon=0.01$. The solid line is the leading-order asymptotic approximation.

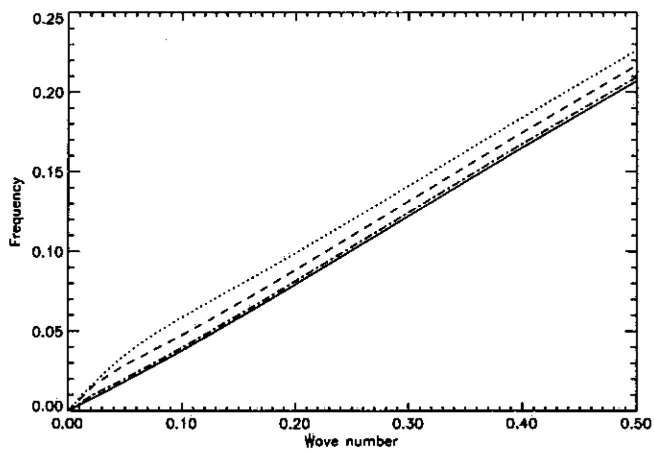

Fig. 4*b*. — Same as Figure 4*a* but ω_r versus α.

The procedure to solve this equation is to use the solvability condition (Fredholm alternative):

$$\langle L_1 P_0, P_0^\dagger \rangle \equiv \int_{-\infty}^{+\infty} (L_1 P_0) P_0^\dagger \, d\xi = 0, \tag{18}$$

where the adjoint eigenfunction $P_0^\dagger$ is defined by $L_0^\dagger P_0^\dagger = 0$ and $L_0^\dagger$ is the adjoint operator of L_0.

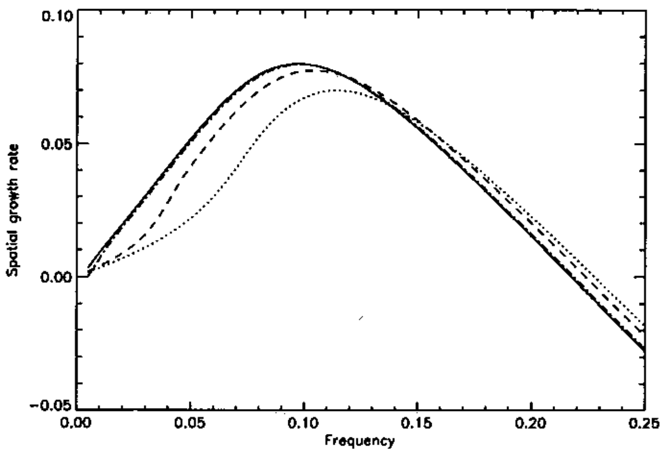

Fig. 5 *a*. — Spatial growth rate for constant density sonic M = 1 jets: $-\alpha_i$ versus ω.
Dotted $\varepsilon = 0.08$, dashed $\varepsilon = 0.04$ and dotted dashed $\varepsilon = 0.01$.
The solid line is the leading order asymptotic approximation.

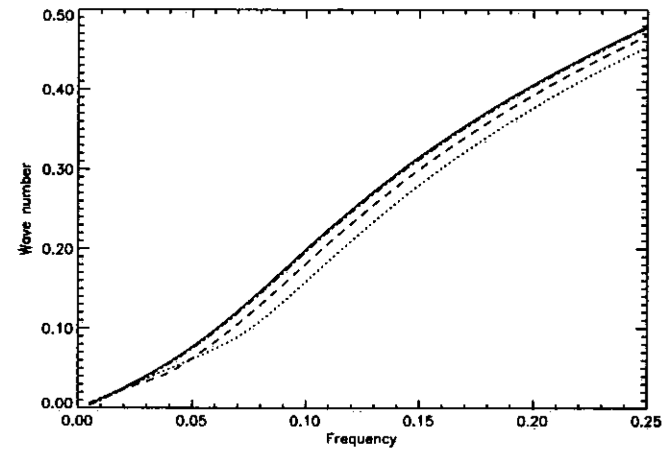

Fig. 5 *b*. — Same as Figure 5 *a* but α_r versus ω.

3. Numerical results

In this section the range of validity of the above approximation is examined for the classical family of hyperbolic tangent velocity profiles first introduced by [Michalke, 1984]:

(19) $$u(r) = (U_0/2)[1 + \tanh((R - r)/2\theta)],$$

(20) $$\rho_0/\rho(r) = R_d + (1 - R_d)\, u(r)/U_0,$$

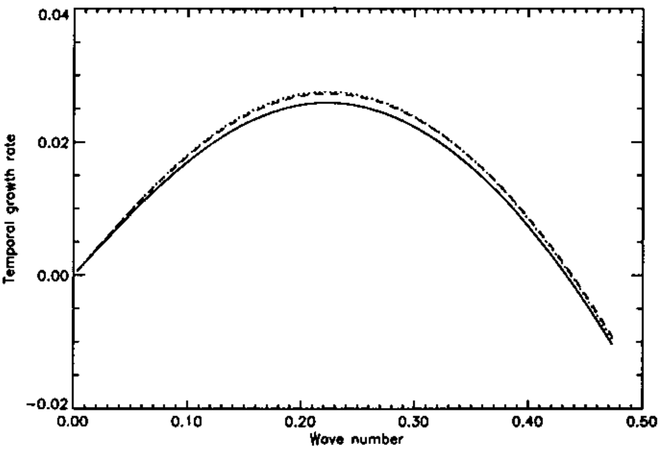

Fig. 6*a*. – Temporal growth rate in the same conditions as Figure 2*a* for non axisymmetric modes. Dotted $m=3$, $\varepsilon=0.08$, dashed $m=6$, $\varepsilon=0.04$, dotted dashed $m=25$, $\varepsilon=0.01$. The solid line is the leading order asymptotic approximation with $\beta_0=0.25$.

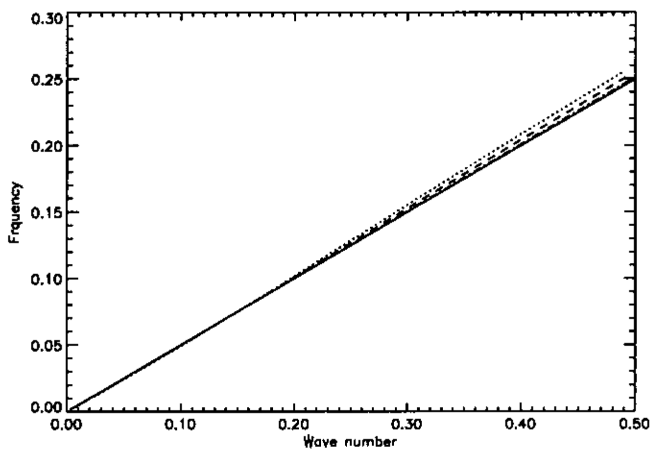

Fig. 6*b*. – Same as Figure 6*a* but ω_r versus α.

where U_0 and ρ_0 are the centerline velocity and density, ρ_∞ is the density at $r \to +\infty$ and

$$R_d = \rho_0/\rho_\infty. \tag{21}$$

In Figure 1 the velocity profile is represented for various values of θ/R.

The cylindrical equations (3), (4), (5) and (6), and the planar equations (10), (11) and (12) for the Rayleigh stability problems are solved numerically by using a shooting method in the complex α or ω plane. In order to obtain eigenvalues that are inviscid limits of corresponding viscous solutions of the Orr-Sommerfeld equation, we deform the integration path in the upper half complex r or ξ plane in the usual appropriate way as described in [Drazin & Reid, 1981]. In this manner one avoids the singularity associated

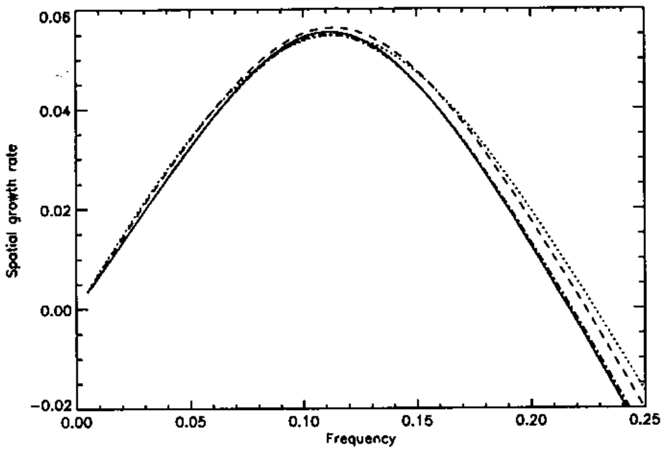

Fig. 7 *a*. – Spatial growth rate $-\alpha_i$ versus ω in the same conditions as in Figure 6 *a*.

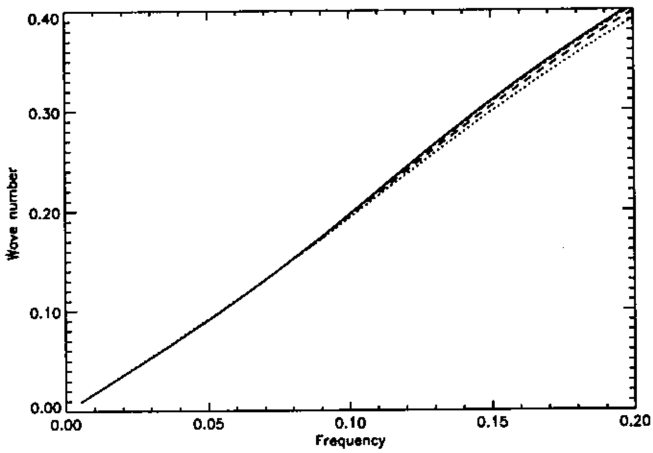

Fig. 7 *b*. – Same as Figure 7 *a* but α_r versus ω.

with the critical point. Our codes were validated by checking against the values published in the literature.

As a first numerical test we have compared the solution of the complete cylindrical Rayleigh equation at $m=0$, $M=0$, and constant density with that of leading order in the asymptotic expansion. The results are displayed in Figure 2 for the temporal case and Figure 3 for the spatial problem. Note that the overall features of the plots are already in qualitative agreement for $\varepsilon=0.08$ but the quantitative values are quite different. At $\varepsilon=0.04$ the quantitative agreement is already much better. At $\varepsilon=0.01$ the curves are indistinguishable.

In order to further test the asymptotic expansion, we checked that the quantitative agreement with the exact results remains very good when the density is non constant (*Fig.* 4) or when the Mach number is non zero (*Fig.* 5).

To check the asymptotic expansion for non axisymmetric disturbances we have kept β_0 constant by changing the azimuthal wave number together with ε (*see* equation (7)).

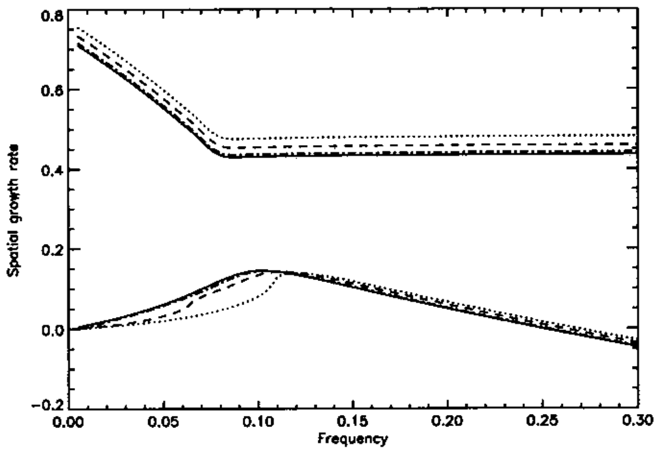

Fig. 8 *a*. − Spatial growth rate versus frequency for the two branches $-\alpha_i^+$ and $-\alpha_i^-$ at $R_u = 1.15$ before the absolute convective transition ($M=0$, $m=0$, $\rho=$cte). Dotted $\varepsilon=0.08$, dashed $\varepsilon=0.04$ and dotted dashed $\varepsilon=0.01$. The solid line is the leading order asymptotic approximation.

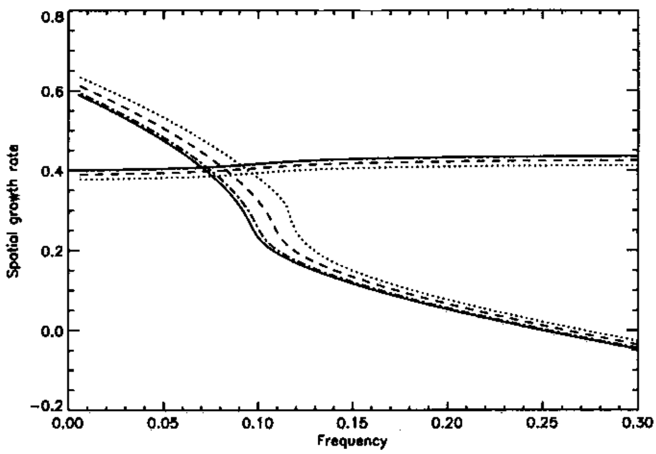

Fig. 8 *b*. − Same as Figure 8 *a* but at $R_u = 1.35$, after the absolute convective transition.

The results are shown in Figures 6 and 7 for the temporal and spatial case respectively. The agreement is again seen to be very good.

It is well known that linear flow instabilities can have a convective or absolute character, the spatial stability theory being well suited in the former case and the temporal theory in the latter. In a given flow that depends on some parameters, branch switching in spatial stability calculations can be used effectively as an indication of a qualitative change of the instability from convective to absolute [Huerre & Monkewitz, 1990]. As a last test of the asymptotic expansion presented in this paper we have checked that it was converging on both branches that are interchanged at the transition obtained when some amount of back flow is added to the velocity of the jet. Replacing equation (19) by

$$u(r) = (U_0/2)\,[1 + R_u \tanh((R - r)/(2\theta))], \quad (22)$$

where R_u denotes the velocity ratio $(U_0 - U_\infty)/(U_0 + U_\infty)$ with U_∞ being the outer flow free stream velocity, one obtains a profile that reduces, in the asymptotic approximation defined by equation (7), to the free shear layer profile studied in [Huerre & Monkewitz, 1985]. This profile is known to undergo an absolute convective transition at $R_u = 1.315$. The convergence of the cylindrical Rayleigh solutions to both free shear layer branches $-\alpha_i^+$ and $-\alpha_i^-$ is shown in the convective case in Figure 8 *a* and in the absolute case in Figure 8 *b*.

4. Conclusion

The simplified asymptotic linear stability theory that we have derived has been numerically shown to work well quantitatively in simple test cases at values of ε in excess of 10^{-2}. For the temporal instability of zero Mach number jets the asymptotic expansion establishes that the most unstable mode is axisymmetric in the limit of vanishing shear layer thickness. Furthermore when a little back flow is added to the mean jet velocity profile, the asymptotic expansion is shown to work well before and after the absolute convective transition.

Finally let us remark that there is an intuitive physical interpretation of the form of our leading order linear operator L_0. Indeed when one makes the change of variable $\xi = (r - R)/\theta$ when $\theta/R \to 0$, the ξ coordinate of the central axis of the circular jet goes to minus infinity. From the point of view of the ξ coordinate the curvature of the shear layer vanishes. One is thus effectively dealing in this asymptotic approximation with a plane shear layer.

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(Manuscript received November 11, 1992;
accepted June 2, 1993.)