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Propagative Phase Dynamics in Temporally Intermittent Systems.

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Abstract. – We study the spatial stability of systems at onset of the temporal intermittent transition to chaos. We show that the phase dynamics during laminar episodes, described as orbits near a saddle-node bifurcation of limit cycles, is *second order* in time and can lead to propagative behaviour.

The phenomenon of chaotic behaviour in deterministic nonlinear systems has attracted a great deal of interest in a wide range of disciplines [1]. Transition scenarios to erratic temporal regimes have been understood in the framework of dynamical system theory [2], and good experimental agreement has been found in confined systems where spatial degrees of freedom are quenched [3]. A question of interest is how the transition scenarios are modified in systems where spatial structures must be taken into account.

The intermittent transition to *chaos* is well documented experimentally [4-7] as well as theoretically [8, 9]. One of its basic mechanisms consists of a saddle-node bifurcation for a limit cycle, which is usually studied with the help of the Poincaré map [10]. In this letter we consider a spatially extended time-periodic flow undergoing such a saddle-node bifurcation and show that the coupling between the phase and the amplitude of the limit cycle may lead to propagative behaviour for long-wavelength phase modes. We first derive the nonlinear equation that governs the dynamics of these modes; we then exhibit a simple model in which propagative behaviour does indeed take place.

Let us first consider a real field governed by the evolution equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F(\nu, v) \,, \tag{1}$$

where F involves linear and nonlinear terms in v. We assume that a saddle-node bifurcation occurs at $\nu = 0$. For $\nu > 0$ eq. (1) has a family of solutions

$$v(t) = v_0(A, s) \,, \tag{2}$$

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of period 2π in $s = \omega(A)\,t$, which represent the limit cycle amplitudes of period $T(A) = 2\pi/\omega(A)$. A parametrizes both the stable and unstable limit cycles amplitudes, and the period T depends on it, as usual for nonlinear oscillations. From (1) and (2) we get

$$\omega(A)\frac{\partial v_0}{\partial s} = F(\nu(A),\, v_0(A,\, s)) \,. \tag{3}$$

To study the stability of the limit cycle solutions, we write

$$v(t) = v_0(A,\, s) + u(t) \,, \tag{4}$$

and linearize eq. (1) in u around $v_0(A, s)$. We get

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L(v_0(A,\, s))\, u \,, \tag{5}$$

where L is the Jacobian of F computed at v_0. We write, following Floquet analysis [11],

$$u(t) = V(s) \exp[\gamma t] \,,$$

where V is 2π periodic in s, and get from eq. (5)

$$\gamma V(s) = L\, V(s) - \omega(A)\frac{\mathrm{d}V}{\mathrm{d}s} = \mathscr{L} V \,. \tag{6}$$

An instability occurs when the Floquet exponent γ vanishes. Differentiating eq. (3) with respect to s, we get

$$\mathscr{L}\frac{\partial v_0}{\partial s} = 0 \,, \tag{7}$$

which shows that $\partial u_0/\partial s$ is a neutral mode. It corresponds to translational invariance in time of eq. (1). We differentiate (3) with respect to A, and find

$$\mathscr{L}\frac{\partial v_0}{\partial A} = -\nu'(A)\frac{\partial F}{\partial \nu} + \omega'(A)\frac{\partial v_0}{\partial s} \,. \tag{8}$$

At the bifurcation, $\nu(A_0) = 0$, using the fact that $\nu'(A_0) = 0$ for a saddle-node bifurcation gives

$$\mathscr{L}\frac{\partial v_0}{\partial A} = \omega'(A_0)\frac{\partial v_0}{\partial s} \,. \tag{9}$$

In the vicinity of the bifurcation we write

$$v(t) = v_0(A_0,\, s + \phi(\tau)) + \rho(\tau)\frac{\partial v_0}{\partial A}(A_0,\, s) + \mathscr{V}(s + \phi,\, \rho,\, \tau) \tag{10}$$

where τ is a slow time scale, and $\mathscr{V}$ stands for corrections orthogonal to the generalized eigenspace. For $v = 0$, we find from eqs. (5), (6) and (7) at the linear leading order

$$\frac{\partial \phi}{\partial \tau} = \omega'(A_0)\,\rho \,, \tag{11a}$$

$$\frac{\partial \rho}{\partial \tau} = 0 \,. \tag{11b}$$

Equations (11*a*), (11*b*) represent a codimension-two singularity [10]. One of the two zero-eigenvalues corresponds to the instability of the limit cycle amplitude. The other one comes from time translational invariance of eq. (1). Note that such situations always occur for any type of bifurcation of a limit cycle in an autonomous system [11]. In the case we have just treated the linear coupling between phase and amplitude is due to the linear variation of the limit cycle frequency with respect to its amplitude.

For $\nu \neq 0$, we have at leading order in ν, after averaging,

$$\frac{\partial \phi}{\partial \tau} = \omega'(A_0)\rho \,, \tag{12a}$$

$$\frac{\partial \rho}{\partial \tau} = \nu - \rho^2 \,. \tag{12b}$$

Note that eq. (12*b*) is the usual normal form for a saddle-node bifurcation [10] which gives the bifurcation diagram of fig. 1; the term linear in ρ has been eliminated by a translation, and $\nu\rho$ is of higher order in ν; the coefficients of ρ in (12*a*) and of ρ^2 in (12*b*) have been taken equal to unity with appropriate scalings of time and amplitude. Note that (12*a*) describes a renormalization of the frequency and is decoupled from (12*b*), it is thus usually ignored. This is no longer the case in space-dependent systems where the coupling between (12*a*) and (12*b*) will allow propagative behaviour (see below).

In spatially extended systems we expect that, in the vicinity of the saddle-node bifurcation, intermittent bursts and laminar episodes occur erraticly in time as for systems with a small number of degrees of freedom, but also inhomogeneously in space. For ϕ and ρ slowly varying in space, we write the perturbation $v(\boldsymbol{x}, \tau)$ under the form

$$v(\boldsymbol{x}, t) = v_0(A_0, s + \phi(\boldsymbol{x}, \tau)) + \rho(\boldsymbol{x}, \tau)\frac{\partial v_0}{\partial A} + \mathscr{V}(s + \phi, \rho, \boldsymbol{x}, \tau, \nabla) \tag{13}$$

and look for a gradient expansion of $\partial\phi/\partial\tau$ and $\partial\rho/\partial\tau$. At leading order in the gradient expansion, the form of the evolution equations for ϕ and ρ is determined by symmetry arguments:

Translational invariance in time (autonomous systems), which implies that the evolution equations do not depend explicitly on ϕ.

Space reflection symmetry, which implies that the number of $\boldsymbol{x}$-derivatives of ϕ and ρ is even.

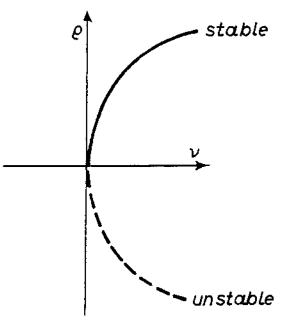

Fig. 1.

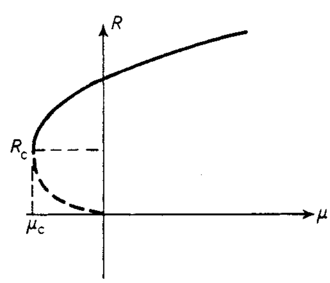

Fig. 2.

Fig. 1. – The diagram of a saddle-node bifurcation.

Fig. 2. – The diagram of a subcritical Hopf bifurcation.

We obtain at leading order

$$\frac{\partial\phi}{\partial\tau}=\rho\;,\tag{14a}$$

$$\frac{\partial\rho}{\partial\tau}=\nu-\rho^2+\nabla^2\phi+(\nabla\phi)^2.\tag{14b}$$

(Higher-order terms in (14a) can always be eliminated with a change of variables [10].) Note that in these equations, the slow space and time scales are of order $(\nu)^{-1/2}$; ρ is of order $\sqrt{\nu}$, whereas ϕ is of order one; however, the expansion is valid since ϕ only contributes through its slow space-derivatives.

We can compute easily equations similar to (14) when a small amplitude limit cycle undergoes a saddle-node bifurcation; this occurs for a subcritical Hopf bifurcation (see fig. 2). The evolution equation for the perturbation complex amplitude Z reads [12]

$$\frac{\partial Z}{\partial t}=\mu Z+(1+i\alpha)\frac{\partial^2 Z}{\partial x^2}+(\varepsilon+i\beta)\,|Z|^2Z-(\gamma+i\delta)\,|Z|^4Z\;,\tag{15}$$

where μ stands for the distance from criticality, and α, β, δ, ε and γ are real constants, with $\varepsilon>0$, $\gamma>0$. The solution

$$Z_0(t)=R_0\exp\,[i\Omega_0 t]\;,$$

with

$$R_0^2=\frac{\varepsilon\pm(\varepsilon^2+4\gamma\mu)^{1/2}}{2\gamma}\;,\qquad\qquad \Omega_0=\beta R_0^2-\delta R_0^4,$$

describes a spatially homogeneous limit cycle, which undergoes a saddle-node bifurcation for $\mu_c=-\varepsilon^2/4\gamma$ and $R_c^2=\varepsilon/2\gamma$ (see fig. 2). For $\mu>\mu_c$ the stability of the limit cycle with respect to inhomogeneous perturbations requires, $\alpha(\beta-2\delta R_0^2)+\varepsilon-2\gamma R_0^2<0$, and thus for $\mu\to\to\mu_c$,

$$\alpha\left(\beta-\frac{\varepsilon\delta}{\gamma}\right)<0\;\cdot\tag{16}$$

In the vicinity of the bifurcation we write

$$Z(x,\,t)=(R_c+\rho)\exp\,[i[\Omega_c t+\phi(x,\,\tau)]]\;,\qquad\qquad \mu=\mu_c+\nu/R_c$$

and get at leading order from (15),

$$\frac{\partial\phi}{\partial\tau}=2R_c(\beta-2\delta R_c^2)\,\rho+o(\sqrt{\nu})\;,\tag{17a}$$

$$\frac{\partial\rho}{\partial\tau}=\nu-2\varepsilon R_c\rho^2-\alpha R_c\frac{\partial^2\phi}{\partial x^2}-R_c\left(\frac{\partial\phi}{\partial x}\right)^2+o(\nu)\;.\tag{17b}$$

It follows from condition (16) that phase disturbances at the saddle-node bifurcation propagates with a velocity v_0,

$$v_0=\left[\frac{\alpha\varepsilon}{\gamma}\left(\frac{\delta\varepsilon}{\gamma}-\beta\right)\right]^{1/2}.\tag{18}$$

We note also that eqs. (17) admit a family of exact propagating solutions $\phi(x-ct)$, where ϕ' has a tgh profile; however, these solutions are unstable.

We have checked the propagative nature of the phase disturbances with a numerical integration of eq. (15) in the vicinity of the saddle-node bifurcation. We used a standard Fourier spectral method with 512 modes and Runge-Kutta time stepping [13]. The initial data is made periodic when necessary by multiplication with $(1+\cos(x/L))/2$, where $2\pi L$ is the periodicity of the Fourier series, taken large enough not to affect the local dynamics. Figure 3 demonstrates the propagative behaviour of a small phase gradient disturbance. Note that the velocity calculated from (18) is 9.7, while the velocity measured on fig. 3 is 10.4. This difference is a manifestation of dispersive effects (estimating the wave number of the initial perturbation $k \simeq 1/l$ gives the right order of magnitude for the difference). Figure 4 shows finite amplitude effects on nonlinear propagation.

We have shown that *second-order* phase dynamics is a generic feature of autonomous systems near a saddle-node bifurcation of limit cycles. Let us note, however, that when the

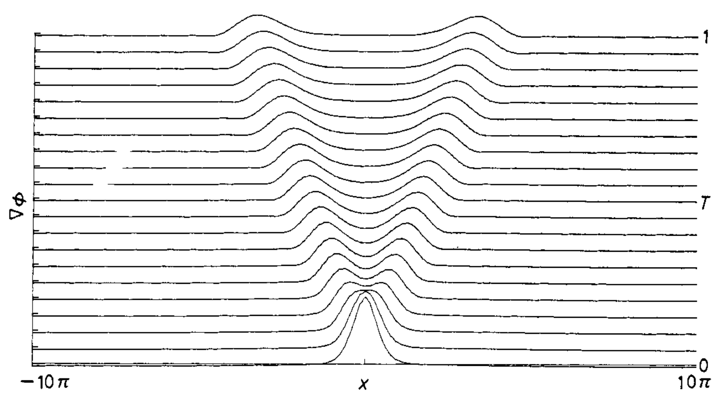

Fig. 3. – A plot of the temporal evolution of the phase gradient $\nabla\phi = \mathrm{Im}(\nabla Z/Z)$ for a subcritical Hopf bifurcation at the saddle-node of limit cycles. Here $\alpha = 4.00$, $\beta = -4.00$, $\gamma = 1.02$, $\delta = 4.00$, $\varepsilon = 2.02$, and $\mu = -\varepsilon^2/(4\gamma)$ (see eq. (15)), corresponding to a propagation at speed $v = 9.72$. The initial data was $\nabla\phi = a/\cosh^2(x/l)$ with $a = 0.02$ and $l = 1.47$. Outputs are every $t = 0.05$, and the curves have been shifted upwards after each output.

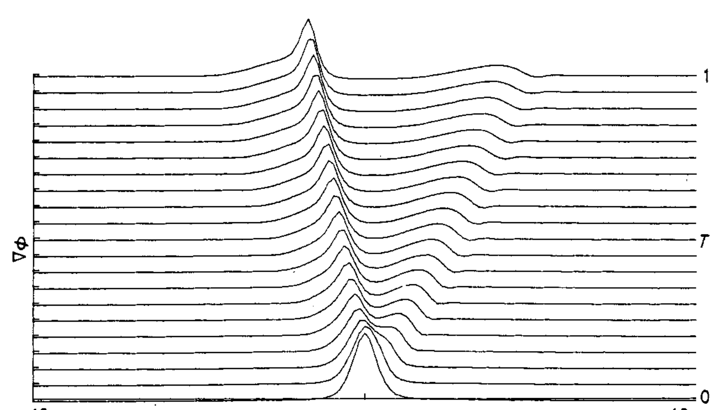

Fig. 4. – Same as fig. 3, but with $a = 1.02$. Note that, although phase gradients still propagate, nonlinear effects are now visible.

limit cycle is due to an external forcing, the temporal phase is quenched, and therefore is not involved in the dynamics. In systems with space reflection symmetry, a gradient expansion in the vicinity of the saddle-node bifurcation gives

$$\frac{\partial \rho}{\partial t} = \nu - \rho^2 + D\frac{\partial^2 \rho}{\partial x^2} .$$

In this equation the slow time scale is of order $(\nu)^{-1/2}$, whereas the space scale is of order $(\nu)^{-1/4}$, and spatial disturbances have a diffusive behaviour. We expect that this occurs with coupled logistic mappings near the transition to intermittency [14], since the phase of a limit cycle is obviously quenched when a Poincaré map is chosen. On the contrary, in autonomous systems, we predict a propagative behaviour for spatial disturbances, in the vicinity of an intermittent transition to chaos (¹).

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(¹) During the revision of this letter we became aware of ref. [15] where «propagating coherent oscillations» are observed in a partial differential equation proposed as a model for spatio-temporal intermittency.

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