AN APPROXIMATE REPRESENTATION OF SU(2) ORDERED EXPONENTIALS IN THE STOCHASTIC LIMIT

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A method of approximating an SU(2) ordered exponential $U(t|E)$, defined in terms of a two-dimensional input vector $E(t) = \hat{E}(t)E(t), E = +(E^2)^{1/2}$, is suggested for the stochastic limit of strongly varying $\hat{E}$, where $|d\hat{E}/dt| \gg E$. In the context of an "averaged" representation, where the high frequency fluctuations normally superimposed upon a relatively slowly-varying background are suppressed, $U(t|E)$ is given as a functional of $E(t)$, suitable for use as an integrand in a functional integral with gaussian white-noise weighting.

Ordered exponentials (OEs) are typically found, or may be employed in every branch of physics which deals with the causal time development of systems containing more than one degree of freedom. In very general quantized systems, for example, one may ask for the time development of a wave function [1,2]. In a modern, atomic physics context of Stark line-broadening where random electric fields produced by a background plasma act on the emitting atom, the latter's wave function, and hence the form of the emitted frequency spectrum, may be writen [‡1] in terms of an OE which is then subjected to fluctuations of the random electric fields. In abelian and (especially) non-abelian quantum field theories, one has relevant expressions [‡2] for exact Green functions and objects constructed from them (such as closed fermion loops) given in terms of exponentials ordered with respect to a proper-time parameter; for abelian gauge theories in greater than two space–time dimensions, or in non-abelian gauge theories of any dimension, all formal manipulations reach their limit of practical application – even when other dynamical approximations are employed [‡3] – precisely at the point where one requires an explicit expression for an OE, and then an estimate of a functional integral over fluctuations of the fields that appear in that OE. A variety of examples drawn from classical physics may also be given [‡4].

The time-honored way of proceeding, as well as the simplest method imaginable, has been to resort to a perturbative development. However, in strong-coupling (SC) problems this is impossible. For some years it has been known [1,2] how to approximate OEs in an "adiabatic" or "quasi-static" SC situation;

[‡1] An exact treatment of a model "scalar" radiation has been given by Frisch and Brissaud [3]. The realistic situation involves dipole radiation, and is expressable in terms of an ordered, rather than an ordinary exponential.

[‡2] A most useful treatment, which has application to a wide variety of fields, has been given by Fradkin [4]. The original idea stems from Schwinger [5].

[‡3] This can be clearly seen in the context of recent infrared models: ref. [6], where the SC analysis performed for scalar $(QED)_4$ would fail for ordinary $(QED)_4$; and in ref. [7], where a generalization to the closed fermion loops of $(QCD)_2$ would meet an OE.

[‡4] For example, an application to Navier–Stokes fluid flow and the onset of turbulence, using an SU(3) formalism, may be found in ref. [8].

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but this is not suitable for rapidly fluctuating input fields, as discussed below. The purpose of these remarks is to exhibit an approximation to a finite-dimensional OE in a "stochastic" or wildly fluctuating" situation; here, the output of the method, an approximate OE ≡ $\overline{\mathrm{OE}}$, is given as a functional of the wildly fluctuating input fields that generate the OE. A typical application of our result would use an $\overline{\mathrm{OE}}$ as (part of) the integrand of a functional integral whose field fluctuations are defined by gaussian stochastic, or white-noise weighting. We here state our results for the simplest case of SU(2), using as input a time-dependent vector $\boldsymbol{E}(t)$ in the (x, y) plane; and we then compare the result of an exactly soluble, white-noise functional integration over the SU(2) OE with the same integration, which can also be exactly performed, over our $\overline{\mathrm{OE}}$. Generalizations, derivations, computer displays, and all details will appear elsewhere.

The ordered exponential in question is given as the solution to

$$\partial U/\partial t = \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{E}(t)U, \quad U|_{t=0} = 1 ; \tag{1}$$

that is

$$U(t|E) = \left[\exp\left(\mathrm{i}\int_0^t \mathrm{d}t'\boldsymbol{\sigma}\cdot\boldsymbol{E}(t')\right)\right]_+ , \tag{2}$$

where the σ_i denote the Pauli matrices. Writing $U = F_0 + \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{F}$, substitution into (1) generates a set of coupled, non-linear equations for (F_0, F_i), with the unitarity restriction $F_0^2 + F^2 = 1$. For a particular choice of $\boldsymbol{E}(t) = \hat{\boldsymbol{E}}(t)E(t)$, $E = +(E^2)^{1/2}$, this form is convenient for the direct numerical integration of (1).

We are concerned with the approximate representation of the $F_{0,i}$ as functionals of $\boldsymbol{E}(t)$ for SC, $\int_0^t \mathrm{d}t'\, E(t') \gg 1$, and in the "stochastic" limit $(|\mathrm{d}\hat{\boldsymbol{E}}/\mathrm{d}t|/E) \equiv \rho(t) \gg 1$. The other SC limit, the "adiabatic" situation where $\rho \ll 1$, has been discussed and applied in various physical contexts [1,2,8,9]. One can, in fact, specify an algorithm for closely approximating $U(t|E)$ in this simpler region where variations of the unit vector $\hat{\boldsymbol{E}}(t)$ are small. In the stochastic limit, dealing with a wildly fluctuating $\hat{\boldsymbol{E}}$, our approximate results will be denoted by $\bar{F}_{0,i}(t|E)$ and should be understood as follows. For any input $\boldsymbol{E}$ with $\rho \gg 1$ the exact (numerically integrated) $F_{0,i}$ have the form of rapid fluctuations superimposed upon a more slowly-varying background of frequency $\sim E/2\pi\rho$. Our $\bar{F}_{0,i}$ are "averaged" in the sense that they reproduce the slowly varying background but not the rapid fluctuations. They are obtained by an analysis which replaces the exact differential equations of this system by approximate equations valid over time intervals larger than the very small ones associated with the high-frequency (ρE) oscillations; our results are

$$\bar{F}_0 = \cos G , \quad \bar{F}_3 [(1 - \xi^2)/\rho\xi] \sin G , \tag{3}$$

$$\bar{F}_1 = \xi \sin G \cos L + [(1 - \xi^2)/\rho] \cos G \sin L , \tag{4a}$$

$$\bar{F}_2 = \xi \sin G \sin L - [(1 - \xi^2)/\rho] \cos G \cos L , \tag{4b}$$

with

$$\xi = (1 + \tfrac{1}{2}\rho^2)^{1/2}\{1 - [1 - (1 + \tfrac{1}{2}\rho^2)^{-2}]^{1/2}\}^{1/2} ,$$

$$G = \int_0^t \mathrm{d}t'\, E\xi , \quad L = \int_0^t \mathrm{d}t'\, E\rho .$$

Limiting forms for $\xi(\rho)$ are: $\xi|_{\rho\gg 1} \sim (1/\rho) + ...$, $\xi|_{\rho\ll 1} \sim 1 - (\rho/2) + ...$.

For $\rho > 5$ the accuracy of (3) is surprisingly good, with typical errors no worse than a few percent, and frequently considerably less. For various input situations, but not all, a small phase lag may appear between $\bar{F}_0$ and F_0, and between $\bar{F}_3$ and F_3; but this should have little effect on subsequent functional integration over $\bar{U}$ if the result of the integration is exponentially damped, as in eqs. (5). For the $\bar{F}_{1,2}$ of (4) the situation is less satisfactory, since they do not display some of the low frequency behavior of the correct $F_{1,2}$; however, in the stochastic limit, they are small and unitarity is preserved:

$$\bar{F}_0^2 + \sum_{i=1}^{3} \bar{F}_i^2 = 1 + \mathrm{O}(1/\rho^2) .$$

What is really quite pleasing is the ability of the $\bar{F}_{0,3}$ to track in phase with, while providing averaged values of their numerically integrated counterparts, for a variety of input $\boldsymbol{E}(t)$, even for situations where $\rho \sim \mathrm{O}(1)$.

That these approximations can be useful when calculating a functional integral over $U(t|E)$ with gaussian white-noise weighting is suggested by comparing the familiar, soluble problem of such integration over U with the same functional integral over $\bar{F}_0$, which is also exactly obtainable. These calculations are performed by breaking up the region of integration into n intervals labeled by t_i, of width $\Delta t = t/n$; in the integration over each ith region E_i scales as $(\Delta t)^{-1/2}$, and hence the ρ_i in $\xi(\rho)$ scale as $(\Delta t)^{-1/2}$, so that the $\rho \gg 1$ stochastic limit is appropriate. With this technique, the exact functional integral over the exact OE,

$$N \int \mathrm{d}[E] \exp\left(-\frac{1}{2c} \int_0^t \mathrm{d}t' \, E^2(t')\right) F_0(t|E) = \mathrm{e}^{-tc} \quad (5a)$$

may be compared with the exact functional integral over our $\overline{\mathrm{OE}}$,

$$N \int \mathrm{d}[E] \exp\left(-\frac{1}{2c} \int_0^t \mathrm{d}t' \, E^2(t')\right) \bar{F}_0(t|E) = \mathrm{e}^{-tc} \cos[2tc \ln(2/c\Delta t)] \,, \quad (5b)$$

where in both cases the normalization is given by

$$N^{-1} = \int \mathrm{d}[E] \exp\left(-\frac{1}{2c} \int_0^t \mathrm{d}t' \, E^2(t')\right) .$$

The difference between these results is the spurious oscillation of (5b), which requires little insight to understand and remove: An infinitesimal error is made in each t_i interval, which, unless discarded, will grow larger with each interval's contribution. The error may be removed by first calculating the real part of each interval's contribution and then summing over all t_i,

$$[\mathrm{Re}\{1 - tc/n + (2itc/n\pi)\ln(2/c\Delta t)\}]^n|_{n\to\infty} \,,$$

rather than following the procedure that leads to the result of (5b), calculating the sum of all t_i contributions and then taking the real part of that quantity,

$$\mathrm{Re}\{[1 - tc/n + (2itc/n\pi)\ln(2/c\Delta t)]^n|_{n\to\infty}\} \,.$$

In this elementary example a simple "renormalization" correction can be adopted to obtain the same results for a white-noise functional integral over an OE and its corresponding $\overline{\mathrm{OE}}$. Hence we expect $\bar{U}(t|E)$ to be useful in the estimation of more complicated functional integrals as long as the restriction to white noise averaging is kept [‡5], or for any similar restriction to $\rho \gg 1$.

Generalization of the stochastic construction to a three-dimensional input $E(t)$, and to arbitrary SU(N) for the adiabatic limit, together with the derivation of the results quoted above, will be given elsewhere.

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[‡5] For non-white-noise gaussian weighting, $\rho_i \sim O(1)$ so that one requires a decent representation of $\bar{U}$ for all ρ. However, in various SC problems E is proportional to a dimensionless coupling g, so that for $g \gg 1$, one has in effect $\rho \ll 1$; precisely this adiabatic limit was treated some years ago – without justification – on a particle physics context [9].

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