

Testing field-theoretical methods on a classical cubic equation with stochastic driving

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Abstract. Approximation procedures for cubic stochastic processes are tested, in particular for their realisability properties. This is done on the algebraic cubic equation $\mu v(t) + \lambda v^3(t) = f(t)$ with zero-mean stationary gaussian driving $f(t)$. This equation is a limiting case for slowly varying driving of the Van der Pol, the Duffing and the cubic nonlinear Langevin equations. Exact solutions are compared with various field-theoretic approximation procedures generated by a variant of the Martin-Siggia-Rose formalism adapted to algebraic equations. The direct interaction approximation (DIA) loses realisability above a critical Reynolds number; for $\mu > 0$, $\lambda > 0$, it has a branch of spurious realisable self-excited solutions which subsist at zero driving. The renormalised vertex equations have no self-excited solution and a larger (but still finite) domain of realisability. In the quadratic DIA, obtained from an equivalent quadratic system for v and $x = v^2$, both pathologies disappear.

1. Introduction

It has often been suggested that incorporating vertex corrections or renormalisations may crucially improve the predictions of statistical theories in situations very far from gaussianity like intermittent turbulent flows (Martin *et al* 1973, henceforth referred to as MSR) or dynamical systems near bifurcations (King *et al* 1979). For a stochastic field with nonlinear dynamics, such theories then lead to a closed system of equations involving the complete two-point and vertex functions; in the absence of any particular simplification like a fluctuation-dissipation theorem, proliferation of indices and unknown functions make these equations difficult to handle. Even existence, uniqueness and realisability properties of the solutions have not been proved. Such properties are not ensured by the field-theoretical framework where such closures can be derived systematically (MSR, Phythian 1977, etc). The aim of this paper is to obtain definite answers to such questions on a simple model problem. We have tested two variants of Kraichnan's direct interaction approximation (DIA) and the first renormalised vertex approximation and made comparisons with exact results.

The simplest model appears to be an algebraic equation with low nonlinearity and stochasticity introduced by an inhomogeneous term. Requiring existence and uniqueness of a real solution leads us to exclude quadratic nonlinearity and to choose

$$\mu v(t) + \lambda v^3(t) = f(t), \quad (1.1)$$

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with positive damping μ and coupling constant λ . The force $f(t)$ is a stationary gaussian function with zero mean. This ensures that the statistics of v is invariant under reversal. White noise would be meaningless for a nonlinear algebraic equation. As in Frisch and Morf (1981), we assume here a band-limited force. Note that the algebraic model (1.1) can be derived from the Van der Pol oscillator, the Duffing equation and the cubic nonlinear Langevin equation in the limit of very slowly varying force. These equations have been used as model problem by different authors to test several approximations (Ziegler and Horner 1980, Pythian and Curtis 1980). Special attention was paid to the deviation from gaussianity and vertex renormalisation (Morton and Corrsin 1970, Bixon and Zwanzig 1971, Budgor *et al* 1976, King *et al* 1979), and to intermittency (Frisch and Morf 1981).

The outline of the paper is as follows. Section 2 formulates the problem. In § 3 we introduce a functional variant of the MSR formalism applicable to algebraic equations and we derive the Dyson equations of the cubic problem (1.1). In §§ 4 and 6, the DIA and the first-order renormalised vertex approximation are considered. In § 5, the DIA is applied to a quadratic system equivalent to the cubic problem. In § 7, these different closures are compared with the exact solution.

2. Formulation and exact solution

The algebraic stochastic model (1.1) can be used to investigate at least two different kinds of questions. First, there are the problems concerning the intermittent high-frequency behaviour (Frisch and Morf 1981). In the field-theoretic formulation this requires the use of four-point functions with all time arguments distinct, resulting in a rather untractable set of equations. Second, and this will be our aim, one can investigate questions of realisability and uniqueness on single-time (static) quantities. The fact that the single-time quantities decouple from the multiple-time quantities is clearly due to the absence of time derivative. Another consequence is that the response function $G(t, t')$ is of the form

$$G(t, t') = g\delta(t - t'). \quad (2.1)$$

Our main goal will be to evaluate the mean square velocity $\langle v^2 \rangle$ using various approximations. It is of interest to observe that there is only one dimensionless parameter in the problem, a Reynolds number

$$R = \lambda F / \mu^3, \quad (2.2)$$

where $F = \langle f^2 \rangle$. The exact mean square velocity is given by

$$\langle v^2 \rangle = (2\pi)^{-1/2} \int df v^2(f) \exp(-f^2/2). \quad (2.3)$$

Here $v(f)$ is the real root of (1.1), which has an explicit algebraic expression. Note that while such an explicit expression is restricted to equations with degree less than five, it is always possible to write an explicit integral representation of the generating function $\langle e^{i\zeta v} \rangle$. This is done as follows. Write that v satisfies an equation $\Lambda(v) + f = 0$. Replace this condition by a δ function (with a Jacobian $d\Lambda/dv$). Then exponentiate the δ

function. For gaussian f this yields

$$\langle e^{i\zeta v} \rangle = (2\pi)^{-1} \int dv dp (d\Lambda/dv) \exp(ip\Lambda - \frac{1}{2}p^2 + i\zeta v). \tag{2.4}$$

Moments are then obtained by differentiating with respect to ζ . This construction is parallel to field theory techniques (see next section); it implicitly assumes uniqueness of the solution of the ‘equation of motion’. Even so, uniqueness is not guaranteed in subsequent statistical approximations, a problem which we shall be faced with in § 4.

The exact mean square velocity (2.3) has been calculated numerically (the cubic equation is solved by Newton’s iteration; the integral is evaluated by Simpson’s method). For positive Reynolds number and $F = 1$, this yields the curve I of figure 1, with the asymptotic value 0.753 . . . at large Reynolds number.

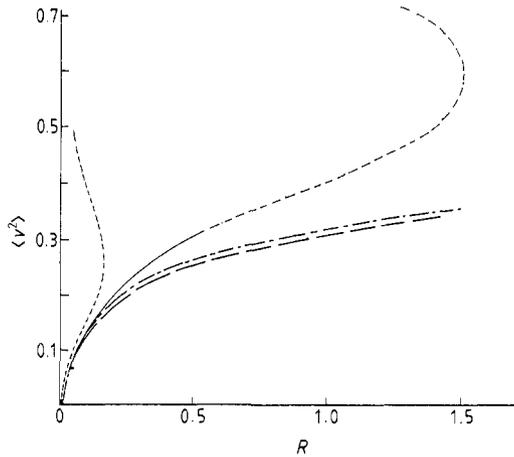


Figure 1. The velocity correlation $\langle v^2 \rangle$ plotted as a function of the Reynolds number R . --- exact result, curve I. cubic DIA, curve II. -.-.- quadratic DIA, curve III. — renormalised vertex approximation, curve IV.

3. Functional formalism and Dyson equations

As adapted to classical random systems by MSR (1973), Phythian (1977) and others, the field theory provides a natural framework to derive perturbative or renormalised expansions of the kind we want to test on our model, i.e. Kraichnan’s DIA and renormalised vertex closures. However, application of these techniques to an algebraic equation needs some care. The naive idea is to construct the generating functional for the process $v(t)$ using an infinite product of the generating functions (2.4) of the variable $v(t)$ for fixed t . This leads to difficulties because the absence of time derivative makes the usual exponentiation of the Jacobian inapplicable. A possibility is to introduce ghost fermionic fields (see e.g. Berezin 1966, Faddeev 1976, Lee 1976). Another possibility is to reintroduce a time derivative $m(dv/dt)$ in the equation and take the limit $m \downarrow 0$ on the final expressions. It is actually possible to deal directly with the algebraic equation using a judicious choice of the underlying discretisation of the functional formalism, namely (j denotes the discretisation index):

$$v_j + Rv_{j-1}^3 = f_{j-1}. \tag{3.1}$$

For smooth solutions, equation (3.1) is equivalent to equation (1.1) and the compatibility with the first procedure ($m \downarrow 0$) is ensured. The Jacobian is then equal to

$$\left| \det \left\| \frac{\partial}{\partial v_i} (v_j + R v_{j-1}^3) \right\| \right| = 1, \tag{3.2}$$

and locality and causality are consistently and naturally incorporated in the theory via a 'retarded bare vertex'. After gaussian average over the forces, the generating functional thus reads

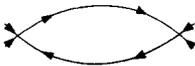
$$Z[\zeta, \zeta^*] = \int Dv Dp \exp(iA[v, p]) \exp\left(i \int d\tau (v\zeta + p\zeta^*)\right), \tag{3.3}$$

$$A[v, p] = \frac{i}{2} \int d\tau d\tau' p(\tau) F(\tau - \tau') p(\tau') + \int d\tau [p(\tau)v(\tau) + R p(\tau)v^3(\tau - \varepsilon)];$$

the two test fields ζ and ζ^* are used to generate the correlation and response functions. ε/Z is reminiscent of the discretisation (3.1) (see Feynman and Hibbs (1965) for an introduction to functional integrals, and Jouvét and Phythian (1979) and Langouche *et al* (1979a, b) for their connection with MSR and the role of the discretisation). If one separates the interaction and the gaussian parts, (3.3) can be rewritten as

$$Z[\zeta, \zeta^*] = \exp\left(iR \int d\tau p(\tau)v^3(\tau - \varepsilon)\right) \Big|_{\substack{p=i\delta/\delta\zeta^* \\ v=-i\delta/\delta\zeta}} Z_0[\zeta, \zeta^*], \tag{3.4}$$

$$Z_0[\zeta, \zeta^*] = \exp\left(-i \int d\tau \zeta(\tau)\zeta^*(\tau)\right) \exp\left(-\frac{1}{2} \int d\tau d\tau' \zeta(\tau)F(\tau - \tau')\zeta(\tau')\right).$$

This last form is well suited to define the usual Feynman rules, for the perturbation expansion; the retarded vertex implies that graphs with closed circuits of pv contractions like  or with tadpoles like  vanish; one can make contact order by order with the case where the equation contains a time derivative ($m > 0$) (Langouche *et al* 1979c): in the $m \downarrow 0$ limit, the influence of initial data disappears.

Using the definition of $Z[\zeta, \zeta^*]$, or its expression (3.3) together with a functional integration-by-parts lemma, one can derive two Schwinger equations. These two equations are rewritten as one 'spinorial' equation, and standard manipulations (MSR, Jona-Lasinio 1964, de Dominicis and Martin 1964) lead to the (matrix) Dyson equation:

$$\text{---} = \text{-----} + \text{---} \text{---} \text{---} \text{---} \text{---} \tag{3.5}$$

where ----- stands for the free matrix propagator and --- the renormalised one. The self-energy matrix $\text{---} \text{---} \text{---}$ is related to the full 1PI tensorial four-points function or vertex function $\Gamma_{\sigma_1\sigma_2\sigma_3\sigma_4}(t_1t_2t_3t_4) = \text{---} \text{---} \text{---}$ by

$$\text{---} \text{---} \text{---} = \frac{1}{3!} \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \tag{3.6}$$

where  represents the bare tensorial vertex $\gamma_{\sigma_1\sigma_2\sigma_3\sigma_4}(t_1t_2t_3t_4)$; the index σ takes the value + or - for respectively the field v or its conjugate p and is represented by arrows on the 'semi-dressed' diagrams of the next sections. For convenience γ has been

normalised according to

$$\gamma_{\sigma_1\sigma_2\sigma_3\sigma_4}(t_1t_2t_3t_4) = -3\lambda\delta(t_1-t_2-\varepsilon)\delta(t_1-t_3-\varepsilon)\delta(t_1-t_4-\varepsilon) \quad (3.7)$$

if $\sigma_1 = -, \sigma_2 = \sigma_3 = \sigma_4 = +$ (or circular permutations); the other components vanish. Causality properties of the renormalised function Γ are explicitly given in § 6.

4. Cubic direct interaction approximation (DIA)

Retaining only ‘direct interactions’ (Kraichnan 1959), one replaces the full vertex by the bare one in the self-energy (3.6). This yields a set of closed equations for the classical equivalent to two-point Green functions, namely the response and correlation functions. In graphical form the DIA equations read:

$$\text{---}\square\text{---} = \text{---}\square\text{---} + 18 \text{---}\text{---} + 3 \text{---}\text{---} \quad (4.1a)$$

$$\text{---}\square\text{---} = \text{---}\square\text{---} + 6 \text{---}\text{---} \quad (4.1b)$$

where we have used the following notation: \leftarrow is the response function $g = \langle \partial v / \partial f \rangle$; \longleftrightarrow is the correlation function $\langle v(t)v(t') \rangle = U(t-t')$. Free quantities are represented by broken lines; full lines represent renormalised quantities; \square stands for the correlation of forces $F(t-t') = \langle f(t)f(t') \rangle$ and $\leftarrow \begin{matrix} \nearrow \\ \leftarrow \\ \searrow \end{matrix}$ for the cubic vertex. When made explicit the DIA equations are rewritten

$$\mu g + 3\lambda U(0)g = 18\lambda^2 U^2(0)g^2 + 1, \quad (4.2a)$$

$$U(t) = g^2(F(t) + 6\lambda^2 U^3(t)). \quad (4.2b)$$

We recall that the DIA equations are an exact consequence of the stochastic model

$$\mu v_\alpha + \frac{\lambda}{N} \sum_{\beta} v_\beta v_\beta v_\alpha + \frac{\lambda}{N} \sum_{\beta\gamma\delta} \Phi_{\alpha\beta\gamma\delta} v_\beta v_\gamma v_\delta - f_\alpha = 0 \quad (4.3)$$

in the limit $N \rightarrow \infty$ (Kraichnan 1961). The $\Phi_{\alpha\beta\gamma\delta}$ ($\alpha, \beta, \gamma, \delta = 1, \dots, N$) are a set of zero mean and unit variance gaussian random variables completely symmetric in $\alpha, \beta, \gamma, \delta$ but otherwise independent. The f_α are N independent versions of the force. It is generally believed that the existence of a stochastic model automatically ensures realisability of the DIA equations. By realisability we mean positivity of the Fourier transform of $U(t)$ and, in particular, positivity of the mean energy $\frac{1}{2}U(0)$. Realisability of the DIA equations is actually ensured only if existence and uniqueness hold both for the DIA equations and the underlying statistical model (Frisch and Morf 1981). As we shall see, these properties does not necessarily hold for the DIA equations (we have not studied the corresponding question on the stochastic model).

We now concentrate on the uniqueness and realisability of the solution of the one-time closure obtained by specialising $t=0$ in equations (4.2). Introducing an auxiliary variable $y = \lambda g U(0)$, the system reduces to the equivalent polynomial equation

$$F_1(y) = (1/R)F_2(y), \quad F_1(y) \equiv (18y^2 - 3y + 1)^3, \quad F_2(y) \equiv y - 6y^3, \quad (4.4)$$

where R is the Reynolds number defined by equation (2.2). The two curves $F_1(y)$ and $(1/R)F_2(y)$ (figure 2) intersect only if $R \leq R_c \approx 0.153$. Above R_c there is no real solution and the DIA is not realisable; below R_c they are two intersections and we are faced with the problem of non-uniqueness. The two roots of equation (4.4) have been computed numerically; the two values of $U(0)$ are presented as functions of R in figure 1 (curve II). Note that the DIA introduces spurious solutions with non-zero kinetic energy when the forcing—or the Reynolds number—vanishes; the system (4.3) has indeed a solution

$$U(0) = \alpha\mu, \quad g = \beta/\mu \quad \text{with } \alpha, \beta > 0 \quad (4.5)$$

for $\mu > 0$, $\lambda = +1$ and $F(0) = 0$ (or $R = 0$). The above phenomena are due to the fact that DIA does not distinguish between positive ($+\lambda$) or negative ($-\lambda$) coupling constants (Rose 1979); the term $3\lambda U(0)g$ can indeed be absorbed in a redefinition of the damping μ . With the negative sign self-excited solutions are possible for the primitive problem, but they are incorporated in the DIA equations, whatever the sign of the coupling.

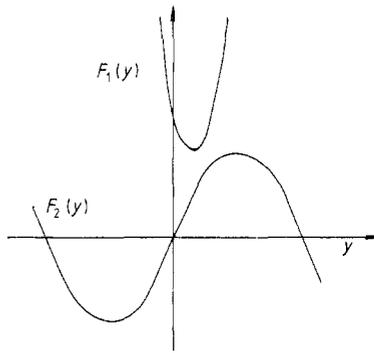


Figure 2. Geometrical representation of the loss of realisability and uniqueness of the one-time DIA. The quantity $y = \lambda g U(0)$ satisfies $F_1(y) = R^{-1}F_2(y)$ where R is the Reynolds number and $F_1(y) = (18y^2 - 3y + 1)^3$, $F_2(y) = y - 6y^3$.

5. Quadratic DIA

Spurious self-excited and non-realisable solutions are introduced by the DIA when applied to the cubic problem (1.1). It has been suggested in a somewhat similar context (the nonlinear Schrödinger equation) that self-excited solutions can be avoided by applying the DIA not directly to the cubic problem but to an equivalent quadratic problem (Rose 1979). Equation (1.1) becomes

$$\mu v(t) + \lambda x(t)v(t) = f(t), \quad (5.1a)$$

$$x(t) = v^2(t) + h(t). \quad (5.1b)$$

The random source term $h(t)$, which will eventually be set equal to zero, enables us to use the formalism of § 2.

The field $x(t)$ does not have zero mean. It is convenient to split it into mean and fluctuation,

$$x(t) = \bar{x}(t) + \tilde{x}(t). \quad (5.2)$$

The system (5.1) becomes

$$(\mu + \lambda \bar{x}(t))v(t) + \lambda v(t)\tilde{x}(t) = f(t), \tag{5.3a}$$

$$\tilde{x}(t) = v^2(t) - \bar{x}(t) + h(t). \tag{5.3b}$$

The mean value \bar{x} renormalises the damping μ . So, when the bare damping vanishes, the theory still has a non-zero damping.

The last system can be rewritten as

$$\psi = G^{(0)}f + G^{(0)}\left(\sum_{i=1,2} (\psi^+ T_i \psi) e_i - \bar{x} e_2\right) \tag{5.4}$$

where we have used the following notation.

$$\begin{aligned} \psi &= \begin{bmatrix} v \\ \tilde{x} \end{bmatrix}, & f &= \begin{bmatrix} f \\ h \end{bmatrix}, & G^{(0)} &= \left\| \begin{matrix} (\mu + \lambda \bar{x})^{-1} & 0 \\ 0 & 1 \end{matrix} \right\|, \\ e_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & e_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & T_1 &= \left\| \begin{matrix} 0 & -\lambda/2 \\ -\lambda/2 & 0 \end{matrix} \right\|, & T_2 &= \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\|, \end{aligned}$$

$G^{(0)}$ is the free propagator.

Assuming that the statistics of f is invariant under reflection ($f \rightarrow -f$), we notice that the solution of equations (5.3) is statistically invariant under $v(t) \rightarrow -v(t)$, $\tilde{x}(t) \rightarrow \tilde{x}(t)$ and $\bar{x}(t) \rightarrow \bar{x}(t)$. As a consequence,

$$\langle v(t)\tilde{x}(t) \rangle = 0, \quad \langle \partial \tilde{x} / \partial f \rangle = \langle \partial v / \partial h \rangle = 0; \tag{5.5}$$

therefore the DIA equations will contain only four independent quantities. In a graphical form these equations read:

$$U_1 = \leftarrow \rightarrow = \leftarrow \square \rightarrow + 4 \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow \tag{5.6a}$$

$$U_2 = \leftarrow \rightarrow = 2 \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow \tag{5.6b}$$

$$G_1 = \leftarrow \rightarrow = \leftarrow \rightarrow + 4 \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow + 4 \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow \tag{5.6c}$$

$$G_2 = \leftarrow \rightarrow = \leftarrow \rightarrow + 4 \leftarrow \rightarrow \leftarrow \rightarrow \leftarrow \rightarrow \tag{5.6d}$$

where we have used the following notation:

$$\begin{aligned} U_1 &= \langle v(t)v(0) \rangle = \leftarrow \rightarrow, & U_2 &= \langle \tilde{x}(t)\tilde{x}(0) \rangle = \leftarrow \rightarrow, \\ G_1 &= \left\langle \frac{\partial v}{\partial f} \right\rangle = \leftarrow \rightarrow, & G_2 &= \left\langle \frac{\partial \tilde{x}}{\partial h} \right\rangle = \leftarrow \rightarrow. \end{aligned}$$

In explicit form the quadratic DIA equations are

$$F(t)G_1^2 + \lambda^2 U_1(t)U_2(t)G_1^2 - U_1(t) = 0, \tag{5.7a}$$

$$2U_1(t)G_2^2 - U_2(t) = 0, \tag{5.7b}$$

$$(\mu + \lambda U(0))G_1 - \lambda^2 U_2(0)G_1^2 + 2\lambda U_1(0)G_1G_2 - 1 = 0, \tag{5.7c}$$

$$2\lambda U_1(0)G_1G_2 - 1 + G_2 = 0. \tag{5.7d}$$

We recall that μ is the damping coefficient, λ the vertex strength and $F(t)$ the force correlation. As in the cubic DIA, the two-point quadratic DIA can be reduced to a subsystem of one-point equations by putting $t = 0$ in (5.7). Let us first investigate the question of possible self-excited solutions. For this, we set $F(0) = 0$ and look for a realisable solution. With the auxiliary variable $y = \lambda U_1 G_1 G_2$, we solve system (5.7) exactly and obtain the trivial solution $U_1 = U_2 = 0$, $G_1 = \mu^{-1}$, $G_2 = 1$, together with a solution with negative energy. The absence of self-excited realisable solutions is not surprising because the quadratic DIA distinguishes between $+\lambda$ and $-\lambda$ coupling constants.

As for the cubic DIA, it is possible to reduce system (5.7) to a polynomial in a single variable $y = \lambda U_1 G_1 G_2$. Straightforward algebra leads to

$$(-4y^3 + 6y^2 - 5y + 1)^3 = (1/R)y(1 - 2y^2)(1 - 2y)^2; \tag{5.8}$$

U_1 , U_2 , G_1 and G_2 are given in terms of y by

$$U_1 = \frac{y}{\lambda(1-2y)} \left(\frac{F\lambda}{y} \frac{1-2y}{1-2y^2} \right)^{1/3}, \quad G_1 = \left(\frac{y}{F\lambda} \frac{1-2y^2}{1-2y} \right)^{1/3}, \tag{5.9}$$

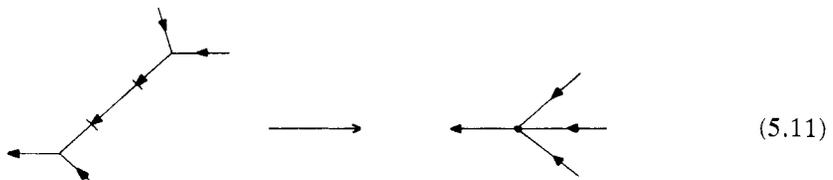
$$G_2 = 1 - 2y, \quad U_2 = 2U_1^2 G_2^2.$$

We have solved equation (5.8) numerically. The results for $U_1(0) = \langle v^2 \rangle$ given in figure 1, curve III, are in good agreement with the exact solution. The quadratic DIA has a single branch; it is also realisable at any Reynolds number. We can therefore investigate the infinite Reynolds number limit. On the primitive equation, the solution has a power law spectrum (Frisch and Morf 1981). In contrast, the spectrum of the quadratic DIA is exponential at any Reynolds number, even infinite. This discrepancy is probably due to an excessive renormalisation of the damping by the mean field \bar{x} .

Let us now make a remark about the quadratic DIA. When replacing the complete four-point vertex function by the bare one, we made an error of order λ^2 which produces an error of order λ^3 on the DIA propagator; the cubic DIA is thus exact up to order λ^2 . The transformation to a quadratic problem introduces a new vertex which does *not* carry the coupling constant λ . So the quadratic DIA is exact only to first order in λ ; the following diagram, for example, is of order λ^2 but is not incorporated in the quadratic DIA:



Consider the transformation



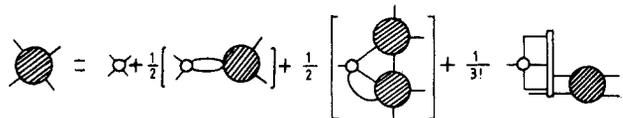
which corresponds in particle physics to suppression of the internal propagation associated with a heavy particle. Through this transformation the diagram (5.10) becomes


(5.12)

which is incorporated in the cubic DIA. The suppression of internal lines cannot create a vertex correction, and thus the diagram, incorporated in quadratic DIA are also contained in the cubic DIA. It follows that the cubic DIA is consistent with a primitive perturbation expansion up to a higher order than the quadratic DIA.

6. Renormalised vertex approximation

In this section we check existence and realisability properties of the first renormalised vertex approximation, together with, for one-point quantities, the quantitative agreement with exact results. The set of equations consists of the Dyson equations (3.5) and (3.6) supplemented by an equation for the four-point vertex function. The latter is obtained with the fourth derivative with respect to the spinorial field of the generating functional of the vertex functions. In a graphical form this equation reads:


(6.1)

where the square brackets indicate that the expressions are symmetrized. $\text{---}\text{---}$ denotes the tensorial bare vertex. The symbol $\frac{\delta}{\delta \text{---}}$ stands for the functional derivative of the renormalised vertex --- with respect to the (matrix) propagator. The causality properties of the vertex functions are given by the following relations:

$$\begin{aligned}
 \Gamma_{++++}^{(1\ 2\ 3\ 4)} &= 0, \\
 \Gamma_{++++}^{(1\ 2\ 3\ 4)} &= X_1 \delta(t_1 - t_2 - \epsilon) \delta(t_1 - t_3 - \epsilon) \delta(t_1 - t_4 - \epsilon), \\
 \Gamma_{-++\ +}^{(1\ 2\ 3\ 4)} &= X_2^a(t_1 - t_2) [\delta(t_1 - t_3 - \epsilon) \delta(t_2 - t_4 - \epsilon) + \delta(t_1 - t_4 - \epsilon) \delta(t_2 - t_3 - \epsilon)] \\
 &\quad + X_2^b(t_1 - t_2) [\delta(t_1 - t_3 - \epsilon) \delta(t_1 - t_4 - \epsilon) + \delta(t_2 - t_3 - \epsilon) \delta(t_2 - t_4 - \epsilon)], \\
 \Gamma_{----\ +}^{(1\ 2\ 3\ 4)} &= X_3(t_1 - t_2, t_1 - t_3) \delta(t_1 - t_4 - \epsilon) \\
 &\quad + X_3(t_2 - t_3, t_2 - t_1) \delta(t_2 - t_4 - \epsilon) + X_3(t_3 - t_1, t_3 - t_2) \delta(t_3 - t_4 - \epsilon), \\
 \Gamma_{----}^{(1\ 2\ 3\ 4)} &= X_4(t_1 - t_2; t_1 - t_3; t_1 - t_4).
 \end{aligned}$$
(6.2)

X_1 is a real number, X_2^a and X_2^b are functions of one variable, X_3 and X_4 are functions of two and three variables respectively.

The equations (3.5), (3.6) and (6.1) are not solvable exactly. The renormalised vertex approximation consists of expanding the bare vertex in powers of four-point vertex functions; we go to the second order.

where α is a functional coefficient which depends on the propagator \mathcal{G} ; it is obtained by identification in equation (6.1). The system then reads

$$(\mu - \Sigma_{-+}(0))g = 1, \tag{6.4a}$$

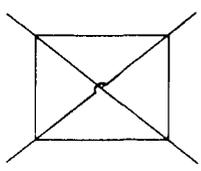
$$U(t) = g^2(R(t) + \Sigma_{--}(t)); \tag{6.4b}$$

$$\text{Diagram} = \text{Diagram} - \frac{1}{2} \left[\text{Diagram} \right] \tag{6.4c}$$

The system (6.4) splits up into four systems: one-, two-, three- and four-point subsystems. In this section we study the one-point subsystem which governs the realisability properties and existence of self-excited solutions:

$$\begin{aligned} U &= g^2[1 - U^2\lambda X_1 - 6gU^2\lambda X_2 - 9\lambda g^2UX_3 - g^3\lambda X_4], \\ 1 &= g[\mu + 3U\lambda + 3g\lambda U^2X_1 + 6\lambda g^2UX_2 + 3\lambda g^3X_3], \\ X_1 &= -6\lambda + 3gU^2X_1^2 + 6g^2X_1X_2, \\ X_2 &= X_2^a + X_2^b, \\ X_2 &= \frac{1}{2}U^2X_1 + 5gUX_1X_2 + 4g^2X_2^2 + 4g^2X_1X_3, \\ X_3 &= U^2X_1X_2 + 4gUX_2^2 + 3gUX_1X_3 + 8g^2X_2X_3 + \frac{1}{2}g^2X_1X_4, \\ X_4 &= 6U^2X_2^2 + 36gUX_1X_3 + 24g^2X_3^2 + 6g^2X_2X_4. \end{aligned} \tag{6.5}$$

This system has been solved numerically on a computer HP 9845B. The solution is calculated analytically for small Reynolds number R . For increasing R the physical solution is followed by continuity using an iterative procedure (Newton's method) initialised by the solution at a slightly smaller Reynolds number. In this way we have found a realisable branch of solutions; figure 1, curve IV, shows the correlation U versus R . No other realisable branch was found exploring the system (6.5) with random initialisation. The curves IV and II (cubic DIA) are qualitatively similar; however, the reality of the vertex function X_2 limits the realisability range of the solution to Reynolds numbers less than 0.5271. Consequently, this branch does not contain the analogue of the realisable self-excited solutions of curve II; this is related to the fact that the renormalised vertex approximation (system (6.5)) distinguishes between a coupling constant and its negative. Let us close this section by a remark about the validity of the renormalised vertex approximation. This approximation is exact up to the third order in the perturbative expansion in powers of the coupling constant. Indeed, the first graph (in an expansion in powers of λ) which is not taken into account by the renormalised vertex approximation is the non-planar graph



which is of order λ^4 . This shows that the renormalised vertex approximation is exact up to third order in λ .

7. Concluding remarks

To compare the different approximations studied in this paper, we summarise their qualitative properties in table 1. Quantitative comparison with exact results is presented in figure 1.

Table 1.

	Cubic DIA	Quadratic DIA	Vertex renormalised
Primitive expansion	λ^2	λ	λ^3
Realisability	$R < 0.153$	Any R	$R < 0.527$
Self-excited solutions at positive Reynolds number	Yes	No	Not found

Let us emphasise that resummation procedures should not be ranked according to the order up to which they agree with the primitive expansion. Indeed, the quadratic DIA, which is exact only up to first order in λ , is in very good quantitative agreement ($\approx 5\%$) with the exact solution, both for small Reynolds number and in the limit $R \rightarrow \infty$. Structural properties preventing, for example, the existence of self-excited solutions appear to be more relevant.

In addition, arguments for the realisability of the closure's solutions based on the existence of an associated stochastic model should be handled with care. This argument is valid only if existence and uniqueness of the solution of both model and closure are ensured (Frisch and Morf 1981). This is clearly not the case for the cubic DIA, which has two real solutions for $R < R_c$ and only complex solutions for $R > R_c$; similarly the realisability of the iterative solution of DIA equations (Morton and Corrsin 1970) requires convergence of the iteration procedure; this question is easily examined on our algebraic model, and the procedure is found to be non-convergent for $R > R_c$.

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References

- Berezin F A 1966 *The Method of Second Quantization* (New York: Academic) p 49
 Bixon M and Zwanzig E 1971 *J. Stat. Phys.* **3** 245–60
 Budgor A B, Lindenberg K and Shuler H E 1976 *J. Stat. Phys.* **15** 375–91
 de Dominicis C 1963 *J. Math. Phys.* **4** 255–65
 de Dominicis C and Martin P C 1964 *J. Math. Phys.* **5** 14
 Faddeev L D 1976 in *Methods in Field Theory Les Houches 1975* ed R Balian and J Zinn-Justin (Amsterdam: North-Holland) p 28
 Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
 Frisch U and Morf R 1981 *Phys. Rev. A* **23** 2673–705
 Jona-Lasinio G 1964 *Nuovo Cimento* **34** 1790

- Jouvet B and Phythian R 1979 *Phys. Rev. A* **19** 1350
King H, Decker U and Haake F 1979 *Z. Phys. B* **36** 205
Kraichnan R H 1959 *J. Fluid. Mech.* **5** 497
— 1961 *J. Math. Phys.* **2** 124; erratum **3** 205 (1962)
Langouche F, Roekaerts D and Tirapegui E 1979a *Physica* **95A** 252–74
— 1979b *Phys. Rev. D* **20** 419–32
— 1979c *Prog. Theor. Phys.* 1617–28
Lee B W 1976 in *Methods in Field Theory Les Houches 1975* ed R Balian and J Zinn-Justin (Amsterdam: North-Holland) p 88
Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev. A* **8** 423–37
Morton J B and Corrsin S 1970 *J. Stat. Phys.* **2** 153–94
Phythian R 1977 *J. Phys. A: Math. Gen.* **10** 777–89
Phythian R and Curtis W D 1980 *J. Phys. A: Math. Gen.* **13** 1575–86
Rose H A 1974 *PhD Thesis* Harvard University
— 1979 *Private communication*
Ziegler K and Horner H 1980 *Z. Phys. B* **37** 339–41