BRIEF COMMUNICATIONS

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Numerical characterization of localized solutions in plane Poiseuille flow

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Stable localized wave packets in two-dimensional incompressible Poiseuille flow are characterized by direct simulation of the Navier–Stokes equations. These asymmetric wave packets depend only on the Reynolds number when the channel periodicity length is sufficiently large. They are created through a saddle-node bifurcation at Re ~2330, well below the critical Reynolds number for finite-amplitude Tollmien–Schlichting waves. Pairs of wave packets are found to interact repulsively, and no evidence for locking has been observed. Finally, an amplitude-equation model is suggested which incorporates the spatial asymmetry of the localized solutions.

Wall-bounded flows such as pipe flows or Blasius boundary-layer flows are known to undergo discontinuous transitions from the basic laminar state.[1] Such behavior cannot be accounted for in terms of a supercritical bifurcation since the system instead jumps abruptly to another solution branch. The relevant feature seems to be the coexistence of a stable laminar branch and at least one nontrivial branch corresponding to a more complex stable flow, and a subcritical model is appropriate.[2] Wall-bounded flows also display other characteristics; (i) small-scale three-dimensional turbulence and (ii) spatial localization of the nontrivial regions (e.g., spots in the Blasius layer and slugs in pipe flows).

Our purpose here is to investigate the possibility of obtaining localized regions of nontrivial flow in a purely two-dimensional context. Plane Poiseuille flow is perhaps the archetypal example of a flow with two stable coexisting branches over a range of Reynolds numbers, the nontrivial branch corresponding to a periodic train of finite-amplitude Tollmien–Schlichting waves.[3] This flow is governed by the Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\frac{1}{\mathrm{Re}}\nabla^2\mathbf{u},$$

$$\nabla\cdot\mathbf{u}=0,$$

with the no-slip boundary conditions $\mathbf{u}(x,\pm 1)=0$.

In order to solve these equations numerically, we have chosen to use pseudospectral methods because of their precision and ease of implementation. We use a standard Fourier/Chebyshev spectral decomposition[4] and our flow is thus x periodic such that $\mathbf{u}(x+L,y)=\mathbf{u}(x,y)$. The boundary conditions are imposed directly on the vorticity as an integral condition (thus obviating the need for Green's functions) and the total flux through the channel is held constant at 4/3. Our code has been checked against the predictions of linear Orr–Sommerfeld theory[5] and against the results from previous studies of Tollmien–Schlichting solutions.[6] One simulation time unit requires 1.0 sec of CPU time, running on a Cray 2 at a resolution of (200×33) with a time step of 0.025.

The rest of this Brief Communication is organized as follows; first, we check that the localized solutions are stable in the limit of large L and thus depend only on the Reynolds number. We then consider the disappearance of these solutions as the Reynolds number is decreased, establishing a new threshold for the stability of plane Poiseuille flow, lower than those previously known. Next, we study the interaction between two adjacent localized solutions, and we conclude by suggesting an amplitude-equation model that exhibits the asymmetry characteristic of these wave packets.

In a previous paper,[7] Jimenez has studied in detail the transition to turbulence in two-dimensional Poiseuille flow for small to moderate values of the periodicity length L ($L=8\pi$ and $L=16\pi$, about 4 and 8 Tollmien–Schlichting wavelengths, respectively). He was able to identify solutions consisting of spatially localized wave packets and investigated their stability for increasing Reynolds numbers. He observed a splitting process at Re ~ 5000 and proposed this as a mechanism for the transition to chaos, and verified this by measuring the Lyapunov exponents of the flow.

The localized wave packets were observed by Jimenez[7] at Reynolds numbers of 3000 and 4000 with a periodicity $L=16\pi$. It may be argued that, in the limit of large L when most of the channel flow is essentially laminar, the form of the localized solution should be governed only by the Reynolds number, independent of L. We have run simulations with values of L up to 80π at various numerical resolutions, and our results indicate that the behavior of

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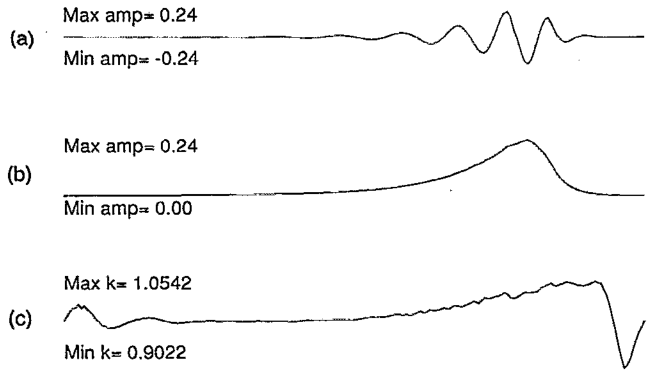

FIG. 1. (a) Midchannel vorticity plotted as a function of space for a stable wave packet at Re=2400 with a channel periodicity length $L=20\pi$ (200×33). (b) Modulus of the complexified vorticity. (c) Wave number of the complexified vorticity. The ripples at the left-hand end are spurious, arising from the vanishingly small amplitude of the vorticity.

the wave packet stabilizes above $L=20\pi$. A resolution of (200×33) was found to be sufficient for spectral convergence in the range $2000<\mathrm{Re}<3000$ for $L=20\pi$.

A typical example of a localized wave packet is shown in Fig. 1(a) where we have plotted the midchannel vorticity $\omega(x,0)$ for Re=2400, $L=20\pi$. We can model this instantaneous vorticity distribution by $\omega(x,0) = \mathrm{Real}[A \exp(ikx)]$, where A and k are functions of x. The amplitude envelope $A(x)$ is plotted in Fig. 1(b); much of the channel remains laminar while the wave packet is seen to be strongly asymmetric with a sharp leading edge and a gentler trailing edge. The wave packet propagates downstream (from left to right) at a constant velocity ≈ 0.7 times the laminar midchanel velocity. The constituent waves seen in Fig. 1(a) progress with about half the wave packet velocity, and thus move backward in the frame of reference moving with the packet. Furthermore, the wave number $k(x)\approx 1$ decreases slightly from the front to the back of the wave group, as shown in Fig. 1(c).

Our calculations are performed at constant flux rather than constant pressure gradient, so a different pressure drop would be expected for a localized solution and a purely laminar flow. We find that the presence of a wave packet increases the pressure drop between any two points bracketing the packet by $\Delta p=2.05\times 10^{-3}$ at Re=2400 (for laminar flow at this Reynolds number, $dp/dx=0.833\times 10^{-3}$).

Given the uniqueness of bounded flow at sufficiently small Reynolds numbers and the presence of the trivial laminar solution down to Re=0, the localized solution branch must disappear as the Reynolds number is decreased past a critical threshold. We show in Fig. 2 the variation of amplitude and velocity of the wave-packet envelope with Reynolds number; the data were obtained by reducing the Reynolds number in discrete steps and using the previous converged solution as the new initial flow. Our results indicate that the nontrivial solution branch disappears at a saddle-node bifurcation, and at $\mathrm{Re}_{\mathrm{crit}}=2330$ we find only the laminar solution.

Apart from the single-wave-packet solution, Jimenez[7]

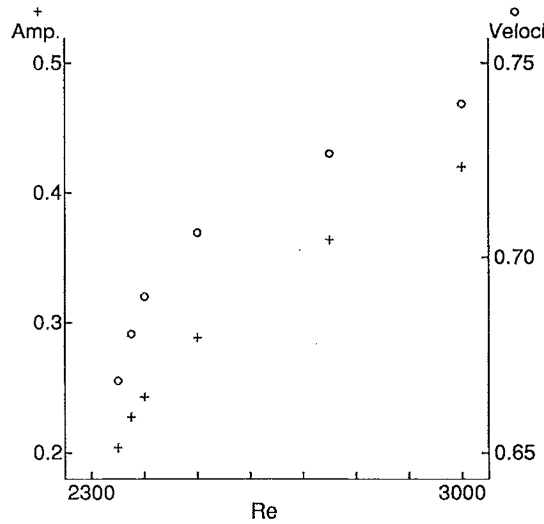

FIG. 2. Amplitude and velocity dependence of wave packets close to the critical Reynolds number, in the limit of large channel length ($L=20\pi$). Our data indicate a saddle-node bifurcation at Re=2330.

found an alternative stable solution at Re=4000, $L=16\pi$, consisting of two back-to-back wave packets of different amplitudes. However, in this instance the channel periodicity was too short for there to be any region of purely laminar flow, and in our studies at larger values of L we have observed no such locked solutions. Indeed, the interaction between two wave packets is always found to be repulsive, as illustrated in Fig. 3 ($L=40\pi$, Re=2400) where the amplitude envelope is plotted at intervals of 20 time units (time increases along the vertical axis). The equilibrium solution consists of wave packets spaced apart by the maximum possible distance (within the constraint of channel periodicity).

Stable localized wave-packet solutions have been found in the vicinity of a subcritical bifurcation in an amplitude-equation model proposed by Thual and Fauve.[8] These solutions are spatially symmetric, but an asymmetry may be introduced by adding to the model equation a term in the first derivative of the complex amplitude W with respect to x. The modified equation is as follows:

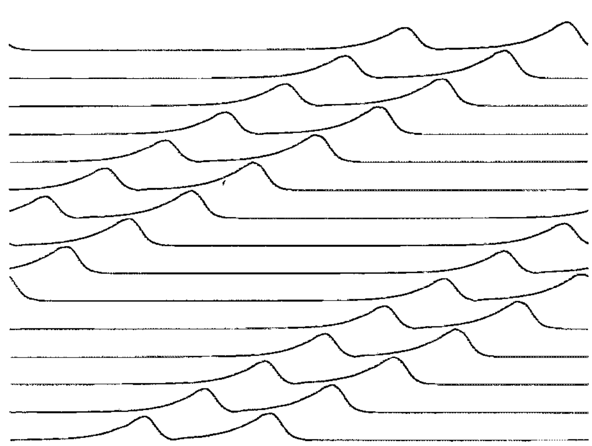

FIG. 3. Repulsive interaction of a pair of wave packets. The modulus of the complexified midchannel vorticity is plotted at intervals of 20 time units, for Re=2400 with channel periodicity length $L=40\pi$ (400×33).

$$\frac{\partial W}{\partial t} = -W + (u+iv)\frac{\partial W}{\partial x} + (1+i\alpha)\frac{\partial^2 W}{\partial x^2} + \text{h.o.t.}$$

The imaginary "group velocity" term is sufficient to break the symmetry in the wave number at $\pm\infty$, and nonlinear terms will couple this to a corresponding asymmetry of amplitude.

In conclusion, we have shown that there is a stable branch of localized wave-packet solutions existing in an otherwise laminar plane Poiseuille flow. These solutions disappear in a saddle-node bifurcation at Re $\sim$2330, in contrast to the minimum Reynolds number of 2900 for the existence of periodic trains of finite-amplitude Tollmien–Schlichting waves. However, although previous studies[9] suggest a rapid growth of perturbations in the third dimension in the case of unmodulated wave trains, the question of the three-dimensional stability of localized solutions remains open.

ACKNOWLEDGMENTS

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[1] D. J. Tritton, *Physical Fluid Dynamics,* 2nd ed. (Clarendon, Oxford, 1988).

[2] J. T. Stuart, "On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows," J. Fluid Mech. **9**, 353 (1960).

[3] J. P. Zahn, J. Toomre, E. A. Spiegel, and D. O. Gough, "Nonlinear cellular motions in Poiseuille channel flow," J. Fluid Mech. **64**, 319 (1974).

[4] D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods* (SIAM, Philadelphia, PA, 1977).

[5] S. A. Orszag, "Accurate solution of the Orr–Sommerfeld stability equation," J. Fluid Mech. **50**, 689 (1971).

[6] J. Jimenez, "Bifurcations and bursting in two-dimensional Poiseuille flow," Phys. Fluids **30**, 3644 (1987).

[7] J. Jimenez, "Transition to turbulence in two-dimensional Poiseuille flow," J. Fluid Mech. **218**, 265 (1990).

[8] O. Thual and S. Fauve, "Localized structures generated by subcritical instabilities," J. Phys. (Paris) **49**, 1829 (1988).

[9] S. A. Orszag and A. T. Patera, "Secondary instability of wall-bounded shear flows," J. Fluid Mech. **128**, 347 (1983).