

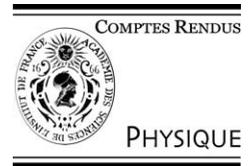


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Bose–Einstein condensates: recent advances in collective effects/Avancées récentes sur les effets collectifs dans les condensats de Bose–Einstein

Boundary layers in Gross–Pitaevskii superflow around a disk

Chi-Tuong Pham^a, Caroline Nore^b, Marc-Étienne Brachet^{a,*}

^a *Laboratoire de physique statistique, École normale supérieure, 24, rue Lhomond, 75231 Paris cedex 05, France*

^b *LIMSI-CNRS, BP 130, 91403 Orsay cedex, France*

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Abstract

Simple explicit expressions describing stationary superflows around a circular disk are found by solving the Gross–Pitaevskii equation at low Mach numbers in the boundary layer approximation. The expression for the velocity potential presents both a short-range layer and a long-range term that is interpreted as stemming from a renormalization of the size of the disk. **To cite this article:** *C.-T. Pham et al., C. R. Physique 5 (2004).*

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Résumé

Couches limites dans un écoulement superfluide autour d'un disque décrit par l'équation de Gross–Pitaevskii. Les solutions stationnaires décrivant les couches limites d'un écoulement superfluide autour d'un disque sont explicitement déduites de l'équation de Gross–Pitaevskii à faibles nombres de Mach. L'expression du potentiel des vitesses présente un terme de couche limite à courte portée et un terme à longue portée qui est interprété physiquement comme une renormalisation de la taille du disque. **Pour citer cet article :** *C.-T. Pham et al., C. R. Physique 5 (2004).*

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Mots-clés : Mécanique des fluides ; Superfluidité ; Couches limites et obstacles

1. Introduction

Since the first experimental observations of Bose–Einstein condensation in trapped dilute gases [1–3], the field is in rapid evolution. Recent results include the production and detection of an isolated quantized vortex [4,5], the nucleation of several vortices [6] and details of vortex dynamics [7]. The dynamics of these compressible nonlinear quantum fluids is accurately described by the Gross–Pitaevskii equation (GPE) [8–10] allowing direct quantitative comparison between theory and experiment [11]. While the nucleation of a vortex in a rotating Bose–Einstein condensate is quite well understood [12–14], the mechanism for vortex production behind a detuned laser [15] is still an open problem. This experiment corresponds to a superflow around a disk.

A mathematical model of superfluid ⁴He, valid at temperatures low enough for the normal fluid to be negligible, is the GPE, also called the nonlinear Schrödinger equation [16]. A recent experiment [17] in ⁴He performed at temperatures less than 130 mK has used an oscillating grid to probe the flow properties. It was observed that, with increasing oscillation amplitude, the superflow abruptly changes: the frequency response becomes nonlinear and is shifted towards a decreasing resonance peak. A further increase yields a transition to turbulence.

* Corresponding author.

E-mail addresses: pham@lps.ens.fr (C.-T. Pham), nore@limsi.fr (C. Nore), brachet@lps.ens.fr (M.-É. Brachet).

The superflow around a 2D cylinder using numerical simulations of the GPE was first studied by Frisch, Pomeau and Rica [18]. They observed a transition to a dissipative regime characterized by vortex nucleation that they interpreted in terms of a saddle-node bifurcation of the stationary solutions of the GPE. Using numerical continuation techniques, Huepe and Brachet [19,20] obtained the complete bifurcation diagram in which the stable and unstable branches are connected through a saddle-node bifurcation. The dynamics around an obstacle is governed by two non-dimensional parameters: the Mach number $\mathcal{M} = |\mathbf{v}|/c$ (where \mathbf{v} is the flow velocity and c the sound speed) and the ratio of the healing length ξ to the diameter of the disk D . The healing length determines the vortex core radius and also the thickness of the healing layer connecting the solution at infinity with the vanishing solution on the obstacle. Huepe and Brachet [20] have varied ξ/D and found that the critical Mach number \mathcal{M}^c converges for $\xi/D \rightarrow 0$ to an Eulerian value that Rica [21] computed using an asymptotic expansion in Mach number. Note that this Eulerian computation did not take into account dispersive effects due to the so-called quantum pressure.

The main purpose of this paper is to present stationary solutions at low Mach numbers including quantum pressure dispersive effects. The boundary layer approximations are obtained from a closed-form integral representation of the solutions, expressed in the Madelung variables: the fluid density and the velocity potential. We show that Eulerian solutions are recovered for $\xi/D = 0$ and that asymptotic solutions for $\xi/D \rightarrow 0$ consist in two parts: an internal boundary layer structure that matches smoothly to an external mainstream flow. Our results can be interpreted in terms of a renormalization of the obstacle radius. The outline of the paper is organized as follows: in Section 2, the general problem is formulated, along with the Madelung transformation; in Section 3, the integral representation of the stationary solutions is derived and the boundary layer limits are obtained. Section 4 is our conclusion.

2. Governing equations

In this section, we present the hydrodynamic form of the Gross–Pitaevskii equation (GPE) that models the effect of a disk of radius $r_0 = 1$ ($D = 2$ is its diameter), moving at constant speed $\mathbf{v} = v\mathbf{e}_x$ in a two-dimensional superfluid at rest. In the frame of the disk, the system is equivalent to a superflow around a disk, with constant speed $-\mathbf{v}$ at infinity. The system can be described with the following action functional

$$\mathcal{A}[\psi, \bar{\psi}] = \int dt \left\{ \sqrt{2}c\xi \int_{\Omega} d^2x \frac{i}{2} [\bar{\psi} \partial_t \psi - \psi \partial_t \bar{\psi}] - \mathcal{F} \right\}, \quad (1)$$

where ψ is a complex field, $\bar{\psi}$ its conjugate. The speed of sound c and the so-called healing length ξ are the physical parameters of the system. \mathcal{F} is the energy of the system that can be written

$$\mathcal{F}[\psi, \bar{\psi}] = \mathcal{E} - \mathbf{v} \cdot \mathcal{P} \quad (2)$$

with

$$\mathcal{E}[\psi, \bar{\psi}] = c^2 \int_{\Omega} d^2x \left[\xi^2 |\nabla \psi|^2 + \frac{1}{2} (|\psi|^2 - 1)^2 \right], \quad (3)$$

$$\mathcal{P}[\psi, \bar{\psi}] = \sqrt{2}c\xi \int_{\Omega} d^2x \frac{i}{2} [(\psi - 1)\nabla \bar{\psi} - (\bar{\psi} - 1)\nabla \psi]. \quad (4)$$

The Euler–Lagrange equation corresponding to (1) provides the GPE

$$i\partial_t \psi = \frac{c}{\sqrt{2}\xi} [-\xi^2 \Delta \psi - \psi + |\psi|^2 \psi] + i\mathbf{v} \cdot \nabla \psi. \quad (5)$$

Boundary conditions are $\psi = 0$ at the border of the disk.

This equation can be mapped into two hydrodynamical equations by applying Madelung’s transformation [16]

$$\psi = \sqrt{\rho} \exp\left(\frac{i\phi}{\sqrt{2}c\xi}\right). \quad (6)$$

The real and imaginary parts of GPE give for a fluid of density ρ and velocity

$$\mathbf{U} = \nabla \phi - \mathbf{v} \quad (7)$$

the following equations of motion

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{U}) = 0, \quad (8)$$

$$\partial_t \phi = -\frac{1}{2}(\nabla \phi)^2 + c^2(1 - \rho) + c^2 \xi^2 \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} + \mathbf{v} \cdot \nabla \phi. \quad (9)$$

These equations correspond to the continuity equation and the Bernoulli equation with a supplementary quantum pressure term for a barotropic compressible and irrotational flow. Note that in the limit $\xi/D \rightarrow 0$, the quantum pressure term vanishes and we recover the system of equations describing an Eulerian flow.

We now proceed to another change of variables, defining (if the speed \mathbf{v} is nonzero)

$$\mathcal{M} = |\mathbf{v}|/c, \quad (10)$$

$$\varphi = -(\phi - vr \cos \theta)/v. \quad (11)$$

The Bernoulli and continuity equations then can be written

$$0 = \xi^2 \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - \rho + 1 + \frac{\mathcal{M}^2}{2} [1 - (\nabla \varphi)^2], \quad (12)$$

$$0 = \Delta \varphi + \nabla \rho \cdot \nabla \varphi. \quad (13)$$

The boundary conditions on the disk become

$$\rho|_{r=r_0} = 0,$$

$$\partial_r \varphi|_{r=r_0} = 0.$$

We now study the case of finite but small Mach number and expand ρ and φ as

$$\rho = \rho^{(0)} + \mathcal{M}^2 \rho^{(1)} + \dots + \mathcal{M}^{2k} \rho^{(k)} + \dots, \quad (14)$$

$$\varphi = \varphi^{(0)} + \mathcal{M}^2 \varphi^{(1)} + \dots + \mathcal{M}^{2k} \varphi^{(k)} + \dots. \quad (15)$$

Note that if one knows φ at order \mathcal{M}^{2k} , one can formally deduce ρ at order $\mathcal{M}^{2(k+1)}$ by solving (12). Potential φ can then be computed at order $\mathcal{M}^{2(k+1)}$ by solving (13).

3. Boundary layer solutions

We now proceed to the computation of the boundary layer stationary solutions, using polar coordinate $x = r \cos \theta$ and $y = r \sin \theta$.

3.1. Density boundary layer

First, note that when the Mach number is zero, $\varphi = 0$ is solution of the stationary equations and ρ satisfies

$$\xi^2 \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - \rho + 1 = 0. \quad (16)$$

Writing $\rho(r, \theta) = R^2(r)$ yields the equation

$$\xi^2 \Delta R + R - R^3 = \xi^2 \left(\partial_{rr} + \frac{1}{r} \partial_r \right) R + R - R^3 = 0 \quad (17)$$

with boundary conditions $R(r_0 = 1) = 0$. An approximation for the solution of this equation, valid up to order ξ , is to neglect the term $(\xi^2/r) \partial_r \rho$, for small values of ξ , so that the approximate ρ can be identified with the stationary solution in the one-dimensional case

$$\rho^{(0)} = \tanh^2 \left(\frac{r-1}{\sqrt{2}\xi} \right). \quad (18)$$

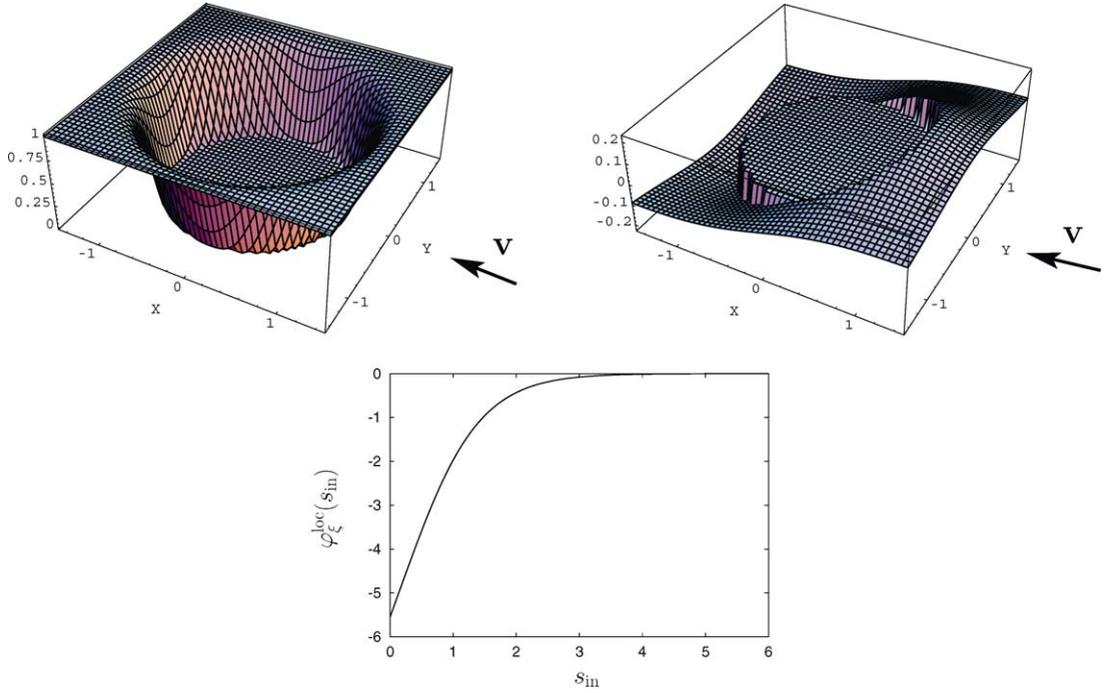


Fig. 1. Upperleft part: density $\rho^{(0)}$ (see Eq. (18)). Upperright part: $\tilde{\varphi}^{(0)}$ (see Eq. (23)). Lower part: $\varphi_{\xi}^{\text{loc}}(s_{\text{in}})$ (see Eq. (30)). All the figures are plotted with $\xi = 0.1$.

3.2. Velocity potential boundary layer

The velocity potential $\varphi^{(0)}$ satisfies

$$\Delta\varphi^{(0)} = -\nabla\rho^{(0)} \cdot \nabla\varphi^{(0)}. \quad (19)$$

We write $\varphi = \varphi_{\text{Euler}}^{(0)} + \tilde{\varphi}^{(0)}$ where $\varphi_{\text{Euler}}^{(0)} = (r + 1/r)\cos\theta$ is the solution at order 0 in \mathcal{M}^2 of the Eulerian flow. Eq. (19) gives the following equation for $\tilde{\varphi}^{(0)}$

$$\Delta\tilde{\varphi}^{(0)} = -\nabla\rho^{(0)} \cdot \nabla(\varphi_{\text{Euler}}^{(0)} + \tilde{\varphi}^{(0)}). \quad (20)$$

In the right-hand side of equation (20), one can keep at order of our computations $\nabla\varphi_{\text{Euler}}^{(0)}$ and drop the term $\nabla\tilde{\varphi}^{(0)}$. We will check the validity of this approximation in the discussion (see below). We now have to solve

$$\Delta\tilde{\varphi}^{(0)} = -\nabla\rho^{(0)} \cdot \nabla\varphi_{\text{Euler}}^{(0)} = \text{RHS} \quad (21)$$

with

$$\text{RHS} = -\frac{\sqrt{2}(r^2 - 1)\cos\theta \operatorname{sech}^2((r-1)/(\sqrt{2}\xi)) \tanh((r-1)/(\sqrt{2}\xi))}{\xi r^2}. \quad (22)$$

Eq. (21) can be explicitly solved using the method of parameter variation, so that the solution $\tilde{\varphi}^{(0)}$ can be written as

$$\tilde{\varphi}^{(0)} = \varphi_{\xi} \cos\theta, \quad (23)$$

where φ_{ξ} is the sum of two terms

$$\varphi_{\xi}(r) = \frac{1}{r}\varphi_{\xi}^{\text{far}} + \varphi_{\xi}^{\text{loc}}(r) \quad (24)$$

with

$$\varphi_{\xi}^{\text{far}} = \frac{1}{2\sqrt{2\xi}} \int_1^{+\infty} [f_1(x) - f_2(x)] dx, \quad (25)$$

$$\varphi_{\xi}^{\text{loc}}(r) = \frac{1}{2\sqrt{2\xi}} \left\{ -\frac{1}{r} \int_r^{+\infty} f_1(x) dx - r \int_r^{+\infty} f_2(x) dx \right\} \quad (26)$$

and

$$f_1(x) = 2(x^2 - 1) \operatorname{sech}^2 \frac{x-1}{\sqrt{2\xi}} \tanh \frac{x-1}{\sqrt{2\xi}}, \quad (27)$$

$$f_2(x) = 2 \left(\frac{1}{x^2} - 1 \right) \operatorname{sech}^2 \frac{x-1}{\sqrt{2\xi}} \tanh \frac{x-1}{\sqrt{2\xi}}. \quad (28)$$

In the limit $\xi/D \rightarrow 0$, after successive integrations by parts and letting r be $r = 1 + \sqrt{2\xi}s_{\text{in}}$, one finds at order ξ^2

$$\varphi_{\xi}^{\text{far}} = 2\sqrt{2\xi} - 4\xi^2 \log 2 + \mathcal{O}(\xi^3), \quad (29)$$

$$\varphi_{\xi}^{\text{loc}}(s_{\text{in}}) = 4\xi^2 s_{\text{in}} [\tanh s_{\text{in}} - 1] - 8\xi^2 [\log 2 + \log \cosh s_{\text{in}} - s_{\text{in}}] + \mathcal{O}(\xi^3). \quad (30)$$

4. Discussion and conclusions

The standard inner and outer solutions of boundary layer theory [22] are defined by:

$$\varphi_{\xi}^{\text{in}}(s_{\text{in}}) = \varphi_{\xi}(1 + \sqrt{2\xi}s_{\text{in}}), \quad (31)$$

$$\varphi_{\xi}^{\text{out}}(r) = \varphi_{\xi}(r). \quad (32)$$

Recasting our main result (24) in this form, together with the expressions (29) and (30), yields

$$\varphi_{\xi}^{\text{in}}(s_{\text{in}}) = 2\sqrt{2\xi} + \xi^2 [-12 \log 2 + 4s_{\text{in}} \tanh s_{\text{in}} - 8 \log \cosh s_{\text{in}}] \quad (33)$$

and

$$\varphi_{\xi}^{\text{out}}(r) = \frac{2\sqrt{2\xi} - 4\xi^2 \log 2}{r}. \quad (34)$$

Note that the outer expression stems only from the long-range term. In contrast, the inner expression (33) mixes contributions from the local and also the long-range term. Similar results were obtained directly, using matched expansions, for a spherical obstacle in [23]. This reference also includes the governing matched expansion equations for the case of a 2D disk; however, the authors did not give the solution to these equations.

In order to check out global validity of our solution, we have estimated the difference between RHS and $\Delta \tilde{\varphi}^{(0)}$ using the expressions (29) and (30). The relative error on the right-hand side of (20) is found to be of order ξ^2 which is consistent with the order of the $\rho^{(0)}$ approximation.

The long-range term in the velocity potential can be physically interpreted as a renormalization of the diameter of the disk. Indeed, the compressible Eulerian flow around a disk of radius r_0 admits at order zero in \mathcal{M}^2 the following solution:

$$\varphi_{\text{Euler},r_0}^{(0)} = r \cos \theta + \frac{r_0^2 \cos \theta}{r} \quad (35)$$

in order to satisfy the boundary condition $\partial_r \varphi_{\text{Euler},r_0}^{(0)}|_{r=r_0} = 0$. For small \mathcal{M} and ξ/D , the velocity potential of our superflow (Eq. (23)) at first order in ξ is therefore equivalent at large distances to that of an Eulerian flow around a disk of radius r_{eff} given by $r_{\text{eff}}^2 = 1 + 2\sqrt{2\xi}$. In term of the obstacle radius r_0 , this equation can be written:

$$\left(\frac{r_{\text{eff}}}{r_0} \right)^2 = 1 + 2\sqrt{2} \frac{\xi}{r_0}. \quad (36)$$

At this order, the approximation made in Eq. (21) is then valid, since the RHS is of order ξ^{-1} and $\nabla \rho^{(0)} \cdot \nabla \tilde{\varphi}^{(0)}$ is of order ξ^0 . The expression we find for $\tilde{\varphi}^{(0)}$ is then the first term of a perturbative development in ξ of the solution of Eq. (19).

Note that, at this order, this renormalized radius does not depend on the Mach number $\mathcal{M} \ll 1$ at which no drag is experienced by the disk. Similar results based on matched expansions were obtained for a spherical obstacle [23]. In the context

of Bose–Einstein condensates, the numerical computation of the pressure drag on a cylindrical obstacle shows a screening effect due to an effective renormalization of the obstacle radius [24]. However this result was obtained in a nonstationary regime at which the Mach number is supersonic. Moreover no quantitative law of this effective radius with respect to \mathcal{M} was derived.

The main consequence of our analytical result is that, at low ξ/D and low Mach number \mathcal{M} , the renormalization of the obstacle radius is of order ξ . This renormalization may seem natural, however, one can find a renormalized radius of order ξ^2 when imposing other boundary conditions [25].

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