Radiation and vortex dynamics in the nonlinear Schrödinger equation

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Sound emission produced by the interaction of several vortices in a two-dimensional homogeneous system obeying the nonlinear Schrödinger (NLS) equation is considered. The radiation effect is explicitly computed in terms of assumed vortex motion. The results are applied to a simple test case of two corotating vortices. The prediction is compared to the result of numerical simulations of the NLS equation. The numerical data give support to the estimate of radiation.

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I. INTRODUCTION

Strong turbulent effective dissipation has been observed to take place in inviscid and conservative systems, in the context of (compressible) low-temperature superfluid turbulence [1,2]. Vortices are thus subject to some significant dynamical dissipation mechanism. It has been suggested that sound emission from the vortices is the major decay process [3–5]. Detailed mechanisms are fully three dimensional (3D). They involve initial vortex reconnection followed by secondary excitation of long-wavelength helical waves, known as Kelvin waves, along the vortex line and their subsequent decay into sound waves [6]. It appears that evaluating these complicated 3D effects from first principles is a formidable task at the present time.

The purpose of the present paper is to compute the simpler analogous problem in two dimensions. We thus consider sound emission produced by the interaction of several vortices in a 2D homogenous system obeying the nonlinear Schrödinger (NLS) equation.

Our main result is that the far field, and thus the radiation effect can be directly computed in terms of an assumed vortex motion [see Eq. (17)]. These main formulas are then applied to the simple test case of two corotating vortices, reproducing theoretical estimates of the same test case [7,8], and the prediction is compared to the result of numerical integrations of the NLS equation.

The paper is organized as follows. In Sec. II we establish the basic proprieties of the NLS equation and recall the general expression for the field produced by moving vortices. Section III is devoted to the derivation of explicit trajectory-dependent expressions for the radiative contribution to the far field and the radiated energy flux. Section IV contains the determination of vortex trajectories by numerical solutions of the NLS and the comparison with theoretical predictions. Discussion and conclusions are finally given in Sec. V.

II. NONLINEAR SCHRÖDINGER EQUATION

We consider the nonlinear Schrodinger equation (NLSE) written with the physically relevant parameters: the coherence length ξ and the sound velocity c,

$$i\frac{\partial\psi}{\partial t} = \frac{c}{\sqrt{2}\xi}(-\xi^2\Delta\psi - \psi + |\psi|^2\psi). \qquad (1)$$

This equation has Galilean invariance with the transformation $\psi(x,t)\to\psi(x-vt,t)e^{i(v\cdot x-v^2t/2)}$ and it also has a Lagrangian structure from which we can calculate an energy-momentum tensor and the conserved quantities corresponding to space-time translations [4].

We can map the NLSE to hydrodynamics equations using the Madelung transformation defined by

$$\psi(x,t) = \sqrt{\rho(x,t)}\exp\left(i\frac{\phi(x,t)}{\sqrt{2}c\xi}\right). \qquad (2)$$

Replacing Eq. (2) in the NLSE (1) and separating real and imaginary parts we get

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\nabla\phi) = 0, \qquad (3)$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = c^2(1-\rho) + c^2\xi^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \qquad (4)$$

We recognize here the continuity equation (3) for a fluid of density ρ and velocity $\mathbf{v}=\nabla\phi$ and the Bernoulli equation (4), except for the last term which is usually called quantum pressure since it has no analog in standard fluid mechanics (it is proportional to $\hbar^2$ in the superfluidity context and it can be neglected when the semiclassical limit is taken).

We note that, if the function ψ has a zero, the density ρ is well defined but the phase ϕ is undefined. The existence of a zero requires the real and the imaginary parts of ψ to vanish simultaneously and consequently these kind of singularities generically appear as curves in 3D and points in 2D. These topological defects have the property that their circulation is a multiple of $4\pi\alpha$ ($\alpha=c\xi/\sqrt{2}$), and for this reason they are called *quantum vortices* in the context of superfluidity. In 2D a stationary vortex solution centered at the origin can be constructed in polar coordinates (ρ,θ) using the ansatz $\rho(r,\theta)=\rho_0(r)^2$ and $\phi(r,\theta)=2\alpha m\theta$, with $m\in\mathbb{Z}$ the vortex charge. The function $\rho_0(r)$ satisfies the equation

$$\frac{\partial^2\rho_0}{\partial r^2}+\frac{1}{r}\frac{\partial\rho_0}{\partial r}+\frac{1}{\xi^2}\left(1-\rho_0^2-\frac{\xi^2m^2}{r^2}\right)\rho_0=0$$

with boundary conditions $\rho_0(0)=0$, $\rho_0(\infty)=1$. This equation can be solved numerically and one finds that ρ_0 is an increasing function in $[0,\infty]$. The region where the function is much smaller than 1, the core of the vortex, increases with m for fixed ξ and c.

Replacing ρ from (4) into (3) we obtain the exact equation

$$\frac{\partial^2\phi}{\partial t^2}-c^2\Delta\phi=-\frac{\partial\phi}{\partial t}\Delta\phi-\frac{\partial(\nabla\phi)^2}{\partial t}-\nabla\cdot\left(\frac{(\nabla\phi)^2}{2}\nabla\phi\right)+\frac{\partial\Gamma}{\partial t}+\nabla\cdot(\Gamma\nabla\phi)$$

where $\Gamma(\rho)=(c^2\xi^2/2)\Delta\sqrt{\rho}/\sqrt{\rho}$. Far from the vortex, the field is almost constant and we can perturb it by putting $\rho=1+2s$. If we consider long-wavelength propagative disturbances the appropriates scalings are

$$\phi=o(1),\quad s=o(\epsilon),$$

$$\frac{\partial}{\partial x_i}=o(\epsilon),\quad \frac{\partial}{\partial t}=o(\epsilon),$$

with ϵ a small parameter. We obtain:

$$\Gamma(s)=\frac{c^2\xi^2}{2}[\Delta s-s\Delta s-(\nabla s)^2+o(\epsilon^5)] \tag{5}$$

and then

$$\frac{\partial\Gamma}{\partial t}+\nabla\cdot(\Gamma\nabla\phi)=\frac{c^2\xi^2}{2}\Delta\frac{\partial s}{\partial t}+o(\epsilon^5)=-\frac{c^2\xi^2}{2}\Delta^2\phi+o(\epsilon^5).$$

Finally we find in the leading order

$$\frac{\partial^2\phi}{\partial t^2}-c^2\Delta\phi=-\frac{\partial\phi}{\partial t}\Delta\phi-\frac{\partial(\nabla\phi)^2}{\partial t}-\nabla\cdot\left(\frac{(\nabla\phi)^2}{2}\nabla\phi\right)-\frac{c^2\xi^2}{2}\Delta^2\phi. \tag{6}$$

Equation (6) was derived in [9] and it is invariant under Galilean transformations.

Near the vortex [7] and considering the asymptotic proprieties of ρ_0^2 in the core it is possible to show that

$$\frac{\partial\Gamma}{\partial t}+\nabla\cdot(\Gamma\nabla\phi)=o(\epsilon^3). \tag{7}$$

Hence, to the lowest order, the phase satisfies a wave equation with a boundary condition in its circulation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\Delta\phi, \tag{8}$$

$$\int_{\mathcal{C}}\nabla\phi\cdot d\mathbf{l}=4\pi m\alpha, \tag{9}$$

where m is an integer, $\alpha=c\xi/\sqrt{2}$, and $\mathcal{C}$ is a circuit around the center of the vortex, i.e., the point in 2D where the field vanishes.

In the case of a moving vortex of charge m, trajectory $R(t)$ and velocity $\dot{R}(t)$ it is possible to give a formal expression of the time derivative of the solution of Eqs. (8) and (9) [10]. Introducing a branch of discontinuity and using the Green function of the wave equation, this expression reads

$$\dot{\phi}(x,t)=4\pi\alpha m\epsilon_{ij}\int_{-\infty}^{\infty}dt'\dot{R}_i(t')\frac{\partial G}{\partial x^j}[x-R(t'),t-t'], \tag{10}$$

where ϵ_{ij} is the Levi-Cività symbol ($\epsilon_{12}=1=-\epsilon_{21}$, $\epsilon_{11}=\epsilon_{22}=0$) and G is the two-dimensional Green function,

$$G(x-x',t-t')=-\frac{c\theta(c(t-t')-|x-x'|)}{2\pi\sqrt{c^2(t-t')^2-|x-x'|^2}}. \tag{11}$$

We remark the well-known fact (Huygens principle) that in even dimensions the Green function of the wave equation does not have a local support as it is the case in odd dimensions; this implies from formula (10) that the value of $\dot{\phi}$ at a given time depends on all the past history of the vortex. Because of this we can expect nonlocal expressions in the far field.

III. FAR FIELD

We now turn to the derivation of an expression for the field far away from the center of the moving vortex (far field). This expression will allow us to characterize the radiation emitted by a vortex describing an arbitrary trajectory. Our calculation will be done in the limit of small velocities, $v\ll c$, and we shall assume that the vortex is constrained to move in a bounded domain in which case all divergence in the integrals are avoided. As we have pointed out the NLS equation can be derived from a Lagrangian from which we can construct an energy-momentum tensor and a Poynting vector S in order to calculate the energy flux. The radiation can be expressed by

$$J=\lim_{r\to\infty}\int_0^{2\pi}S\cdot\hat{\mathbf{n}}r\,d\theta. \tag{12}$$

where S at the leading order reads $S=(\partial\phi/\partial t)\nabla\phi$ [4]. In the far-field approximation $\nabla\phi\cdot\hat{\mathbf{n}}$ can be replaced by $\dot{\phi}/c$, and therefore the only nonvanishing contributions come from terms of order $O(1/\sqrt{r})$ in the time derivative of the phase.

In order to express Eq. (10) as a function of the trajectory we can formally write

$$\dot{\phi}(x,t)=4\pi\alpha m\epsilon_{ij}\frac{\partial}{\partial x^j}\int_{-\infty}^{\infty}dt'd^2y\dot{R}_i(t')\times G(x-y,t-t')\delta(y-R(t')) \tag{13}$$

and perform the δ expansion

$$\delta(y-R)=\delta(y)-R_k\frac{\partial}{\partial y_k}\delta(y)+\frac{1}{2}R_kR_l\frac{\partial}{\partial y_k\partial y_l}\delta(y)\times\frac{1}{3!}R_kR_lR_m\frac{\partial}{\partial y_k\partial y_l\partial y_m}\delta(y)+\cdots. \quad (14)$$

To calculate the vortex-trajectory-dependent integral in Eq. (10) we define for an arbitrary function $W:\mathbb{C}\to\mathbb{C}$ and a function $f:\mathbb{R}\to\mathbb{R}^n$ the W derivative of f by

$$W\left(\beta\frac{d}{dt}\right)f(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}W(\beta i\omega)\hat{f}(\omega)e^{i\omega t} \quad (15)$$

where $\hat{f}$ is the Fourier transform of f and β an arbitrary parameter.

Replacing Eq. (14) in Eq. (13) and using the definition (15) we obtain (see Appendix A for details)

$$\dot{\phi}(x,t)=2\alpha m\epsilon_{ij}\left[\partial_j\left(K_0\left(\frac{r}{c}\frac{d}{dt}\right)\dot{R}_i\right)-\partial_{jk}\left(K_0\left(\frac{r}{c}\frac{d}{dt}\right)\dot{R}_iR_k\right)\right]+2\alpha m\epsilon_{ij}\left[\frac{1}{2}\partial_{jkl}\left(K_0\left(\frac{r}{c}\frac{d}{dt}\right)\dot{R}_iR_kR_l\right)-\frac{1}{6}\partial_{jklm}\left(K_0\left(\frac{r}{c}\frac{d}{dt}\right)\dot{R}_iR_kR_lR_m\right)\right]+\cdots, \quad (16)$$

where all trajectories are evaluated at t, $\partial_{i_1i_2\cdots i_n}=\partial^n/\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}$ and $K_n(z)$ is the modified Bessel function of the second kind.

When $z\to\infty$, $K_0(z)\approx\sqrt{\pi/2z}e^{-z}$ the K_0 derivative of the first term of Eq. (16) can be evaluated as

$$K_0\left(\frac{r}{c}\frac{d}{dt}\right)\dot{R}_i(t)\approx\sqrt{\frac{\pi}{2}\frac{c}{r}}\partial_{t^{-1/2}}e^{-(r/c)d/dt}\dot{R}_i(t)+o(r^{-3/2})=\sqrt{\frac{\pi}{2}\frac{c}{r}}\partial_{t^{-1/2}}\dot{R}_i(t_r)+o(r^{-3/2})$$

where $t_r=t-r/c$ is the retarded time. Thus the radiative contribution of the two lowest orders of Eq. (16) is explicitly given by

$$\dot{\phi}(x,t)=-\alpha m\sqrt{\frac{2\pi}{r}}\epsilon_{ij}\;c^{1/2}\hat{n}_j\partial_{t^{-1/2}}\left(\frac{R_i}{c}\right)\bigg|_{t=t_r}-\alpha m\sqrt{\frac{2\pi}{r}}\epsilon_{ij}\;c^{1/2}\hat{n}_j\hat{n}_k\partial_{t^{3/2}}\left(\frac{\dot{R}_iR_k}{c^2}\right)\bigg|_{t=t_r}. \quad (17)$$

We remark the important difference between this formula (17) and the radiation formulae for moving electrons in three dimensions [11]. Here one finds that in the lowest order the field depends of the $\frac{3}{2}$-*derivative* of the vortex trajectory instead of the 2-*derivative* which appears in the 3D case [12]. We also note that all the functions are evaluated at the retarded time and the fractional derivatives that appear here put in evidence the nonlocality of the 2D Green function, i.e., one must know the whole trajectory of the vortex from $t=-\infty$ to $t=t_r$ in order to calculate Eq. (17). Now using (17) together with the expression of the flux of energy (12) yields after a straightforward calculation

$$J=\frac{2m^2\alpha^2\pi^2}{c^2}|\partial_{t^{3/2}}R|^2+\frac{2m^2\alpha^2\pi^2}{c^4}\partial_{t^{3/2}}(\dot{R}_iR_j)\partial_{t^{3/2}}(\dot{R}_kR_l)N_{ijkl}, \quad (18)$$

where $N_{ijkl}=(1/\pi)\int_0^{2\pi}\epsilon^{i\gamma}\epsilon^{k\mu}n_\gamma n_j n_\mu n_l d\theta$. This explicit formula for the radiation of a moving vortex in terms of its trajectory is one of our main results.

In the case of a single uniformly rotating vortex, $R(t)=ae^{i\omega t}$, the formula can be simplified. The term $|\partial_{t^{3/2}}R|^2$ is given by

$$|\partial_{t^{3/2}}R|^2=\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}(i\omega_1)^{3/2}(-i\omega_2)^{3/2}\times\hat{R}(\omega_1)\hat{R}^*(\omega_2)e^{it_r(\omega_1-\omega_2)}=\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}(i\omega_1)^{3/2}(-i\omega_2)^{3/2}(a2\pi)^2e^{it_r(\omega_1-\omega_2)}\times\delta(\omega_1-\omega)\delta(\omega_2-\omega)=a^2\omega^3,$$

and therefore the lower-order term of Eq. (18) reads

$$J=\frac{2m^2\alpha^2\pi^2}{c^2}a^2\omega^3=m^2\pi^2c^2\xi^2M^2\omega, \quad (19)$$

where we have reintroduced c and ξ and defined the Mach number $M=a\omega/c$.

We shall apply now this result to the case of two rotating vortices, separated by a distance $2a$, with the same unitary charge ($m=1$). It is well known that in the incompressible approximation the vortices will rotate in a perfect circle one around the other with a frequency $\omega=\alpha/a^2$. For two vortices the total far field produced is just the superposition of the field produced by each one, taking into account their charges which here are equal. In the case of two rotating vortices their trajectories are symmetric and then the odd power of R in Eq. (16) vanishes. Hence we get to the lowest order

$$\dot{\phi}(x,t)=-4\alpha\epsilon_{ij}\partial_{jk}\left(K_0\left(\frac{r}{c}\frac{d}{dt}\right)\dot{R}_iR_k\right)=2\alpha a^2\omega\mathrm{Re}\left\{e^{i2\omega(t-\theta)}\left[\frac{4\omega^2}{c^2}K_0''\left(i\frac{2\omega}{c}r\right)+i\frac{2\omega}{cr}K_0'\left(i\frac{2\omega}{c}r\right)\right]\right\}. \quad (20)$$

Note that the wavelength $\lambda=2\pi c/\omega$ of waves emitted by the vortex appears explicitly in formula (20) and the incompressible limit $a\ll r\ll\lambda$ is easily obtained using the asymptotic of $K_0(z)$ for small z (see Appendix A). It reads

$$\dot{\phi}=-8\alpha\omega\frac{a^2}{r^2}\cos 2(\omega t-\theta) \quad (21)$$

which is the well-known first order of the multipolar expansion [7,8]. The next orders can be obtained similarly.

The radiative far field is obtained in the limit $a\ll\lambda\ll r$ and yields, using Eq. (12), the energy lost due to radiation,

$$J = 16\pi^2\alpha^2\frac{a^4\omega^4}{c^4}\omega = 8\pi^2c^2\xi^2M^4\omega. \quad (22)$$

Note that this energy flux is very small. If we now make an energy balance between the radiated energy J and the variation of energy due to a change in the distance between the vortex, we can obtain a simple equation for the radius a. At lowest order the energy is simply $\mathcal{E}=\frac{1}{2}\int dx(\nabla\phi)^2$. For two vortices separated a distance $2a$ the interaction part of the energy, after some algebra, reads [7,8]

$$\mathcal{E}_{\mathrm{int}} = -4\pi c^2\xi^2 \ln 2a. \quad (23)$$

Thus $d\mathcal{E}_{\mathrm{int}}/dt=-J$ leads to an evolution equation for the radius:

$$\frac{da}{dt} = \frac{\pi c\xi^5}{2\sqrt{2}}\frac{1}{a^5}. \quad (24)$$

From this equation we obtain the law $a(t)=[a_0^6+(3\pi/\sqrt{2})\xi^5 ct]^{1/6}$ which shows that the vortex distance increases, but very slowly. This result was obtained in 1966 by Klyatskin [8] using a matching between a compressible, but vortex independent far field and an incompressible near field. Note that there are some misprints in the literature [7,8] which lead to erroneous values of prefactor in formula (24); see [15] for details.

Because of the very simple form of the terms in development (16) one can get $\dot{\phi}$ explicitly at any order in M. This explicit series reads (see Appendix B):

$$\dot{\phi}(x,t) = \sum_{n=1}^{\infty} \dot{\phi}_{M^{2n}}M^{2n} \quad (25)$$

with

$$\dot{\phi}_{M^{2n}}(x,t) = -4\alpha\sqrt{\frac{\pi c\omega}{r}}\frac{1}{(2n-1)!}\sum_{l=0}^{n}\binom{2n}{l}(n-l)^{2n-1/2} \times\cos\left[2(n-l)(\theta-\omega t_r)+\frac{\pi}{2}\left(2n-\frac{1}{2}\right)\right] \quad (26)$$

and it follows that the total radiated power can be expanded as

$$J = 16\pi^2\alpha^2\omega\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-M^2)^{n+m}}{(2n-1)!(2m-1)!}\sum_{l=0}^{n}\binom{2n}{l}\binom{2m}{l+m-n} \times(n-l)^{2(n+m)-1} \quad (27)$$

$$=16\pi^2\alpha^2M^4\omega\left(1-\frac{4M^2}{3}+\frac{17M^4}{4}-\frac{157M^6}{18}+\frac{91783M^8}{4320}-\cdots\right). \quad (28)$$

This series has a finite radius of convergence equal to $M_c=0.667$; hence we can expect that radiation effects in the NLSE will be well described by (28) only for lower values of M.

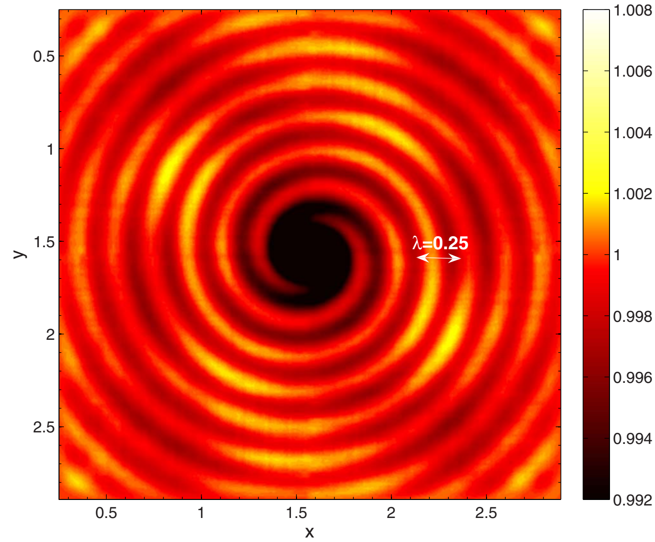

FIG. 1. (Color online) Radiation of a vortex pair with an initial separation of $2a=2.53\xi$ ($M=0.56$, $\lambda=0.2493$). Run of NLSE with absorbing boundaries (see below) using 512^2 Fourier modes.

IV. NUMERICAL DETERMINATION OF RADIATION AND VORTEX TRAJECTORIES

To numerically the integrate NLSE, we will use using standard [13] pseudospectral methods. In order to work with complex periodic fields (which must have zero total topological charge) we place the two rotating vortices within a box of mirror symmetries. Equation (1) is solved numerically using pseudospectral methods that were specially tailored to the mirror symmetries of the initial data in order to gain on both computer time and central memory storage. The corresponding Fourier pseudospectral algorithms are described in detail in Ref. [4].

The 2π-periodic fields are symmetric by reflection on the lines $x=0,\pi$ and $y=0,\pi$ that constitute the boundaries of the so-called impermeable box. We prepare an initial condition by letting a charge-2 vortex, situated at the center of the impermeable box, evolve under the Ginzburg-Landau real (GLR) dynamics [which can be easily obtained from Eq. (1) by performing a Wick rotation $\tau=it$]. The vortex then splits into two single-charge vortices and the GLR dynamics is continued until they reach a distance $2a$ (see Ref. [4] for details). These initial data are then evolved under NLS dynamics (1). The physical parameters are $c=0.0625$, $\xi=0.0177$, 0.0089, 0.0044, 0.0022, and $512^2, 1024^2, 2048^2, 4096^2$ Fourier modes are used, respectively.

A typical result with 512^2 Fourier modes is shown on Fig. 1, where we plot the density $\rho=|\psi|^2$ and emitted waves can be clearly observed. Note that the wavelength $\lambda=c2\pi/\omega=0.2493$ is illustrated by a double arrow.

The vortex trajectories are determined by the following procedure. First a rough location is found by seeking the grid point with minimum density ρ. In a second step Newton iterations are used in order to determine the precise location of the vortex by solving the equation $\psi(x)=0$. A typical vortex trajectory obtained in this way, with an initial separation

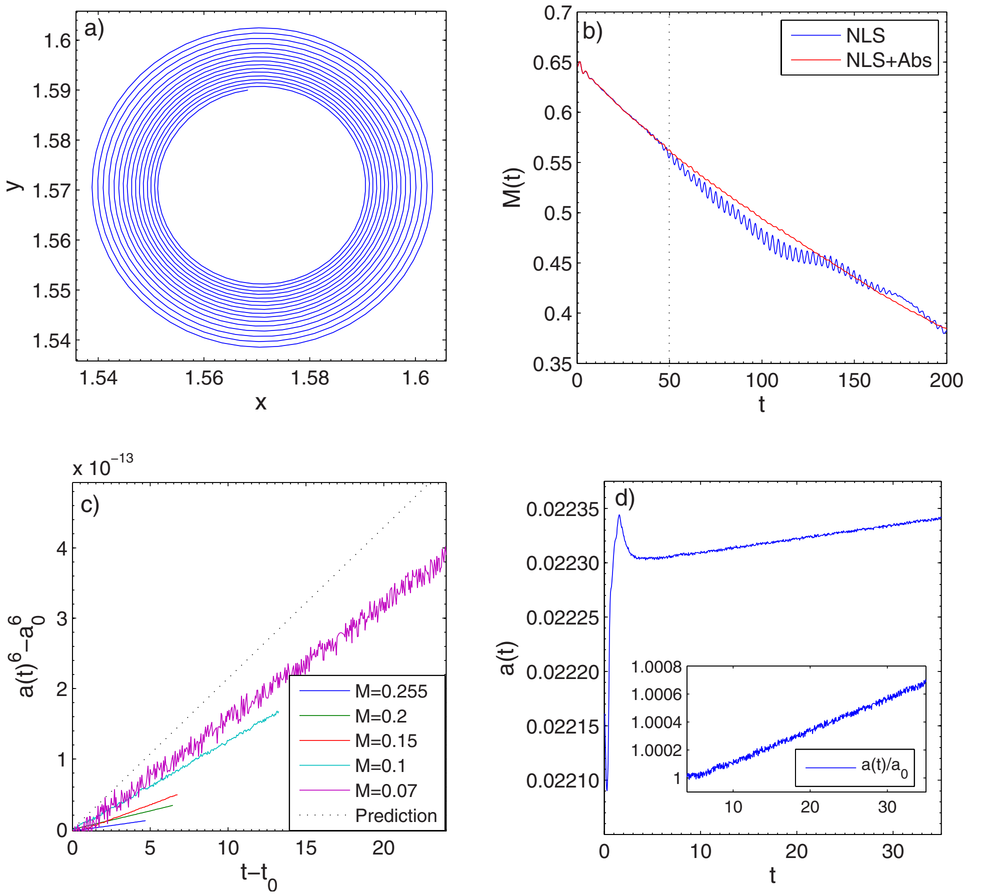

FIG. 2. (Color online) (a) Vortex trajectory with an initial separation of $2a=2.18\xi$. (b) Temporal evolution of Mach number with and without absorbing boundaries. Initial separation of $2a=2.18\xi$. (c) Temporal evolution of $a(t)^6-a_0$ for different Mach numbers. (d) Temporal evolution of $a(t)$ with Mach number $M=0.077$. Inset: Temporal evolution of $a(t)/a_0$ after the transient. (a) and (b) use 512^2 Fourier modes and (d) and (c) 4096^2.

of $2a=2.18\xi$, is plotted on Fig. 2(a). It is apparent on the figure that the vortex follows a spiral trajectory, with a very weak increase of its radius at each turn.

There are two natural time scales in the problem. The first one corresponds to the transient adaptation to the initial condition and it is of the order of the time for the wave coming from one vortex to travel to the other. Thus, an inferior bound is $t_{\rm tr}=2a/c$ depending on the original separation $2a$. The second time scale corresponds to the time taken by the waves produced by the vortices placed in neighbor impermeable cells to arrive at another cell. It can easily be estimated as $t_w=\pi/c$. In the present numerical calculations these values are $t_{\rm tr}=0.5-5$ and $t_w=50$. The temporal evolution of the Mach number $M=a\omega/c$ is plotted in Fig. 2(b); note that the transient time $t_{\rm tr}$ and the arrival of waves at $t_w=50$ (vertical dashed line) are clearly observed.

During the transient time the vortices emit a large amount of sound that significantly perturbs the vortex trajectories when it arrives at the neighbor cells, as can be seen in Fig. 2(b). In order to isolate the vortex we added absorption in the border of the cell, performing a modified GLR step between each NLS step. The GLR modified step consists in the GLR equation with the right-hand side modulated by a function that is null almost everywhere except near the border. The results of absorption are also shown in Fig. 2(b). Note that oscillations are effectively reduced without significant modification of the vortex trajectories before t_w.

In Fig. 2(c) the radius as a function of time is displayed for different initial conditions. Note that the slope increases when the Mach number M diminishes, as well as the oscillations.

Figure 2(d) displays the temporal evolution of the radius corresponding to the run with the smallest Mach number. Note that the total increase of the radius after the transient time is less than 0.08% [see inset in Fig. 2(d)]. This explains the large fluctuation in the corresponding curve in Fig. 2(c).

We now turn to the comparison of the results of the numerical integrations of the NLSE with the theoretical prediction (24). To wit, we measure the relative variation of M in a fraction β of turns of the vortex pair for each trajectory (obtained with different ξ and resolutions). This turnover time is simply $T=2\pi/\omega=2\pi\sqrt{2}a^2/c\xi$ and from the theoretical prediction (24) it follows that

$$M(\beta T) = M_0[1 + 24\beta\pi^2 M_0^4]^{-1/6} \approx M_0(1 - 4\beta\pi^2 M_0^4) \tag{29}$$

for M small. Note that the Mach numbers can be expressed directly using ξ and a by $M=\xi/\sqrt{2}a$.

Each initial condition obtained using GLR dynamics is evolved under the NLSE for more than a turn of the vortices. The mean value of $[1-M(\beta T)/M_0]/(4\beta\pi^2 M_0^4)$ (over the NLS evolution) for different values of Mach number and resolutions is plotted in Fig. 3 The value of β is always between 1/8 and 1. It is chosen in each case in order to obtain clear data.

The error bars shown for small Mach number in Fig. 3 are produced by the oscillation of the trajectories [see Fig. 2(c)].

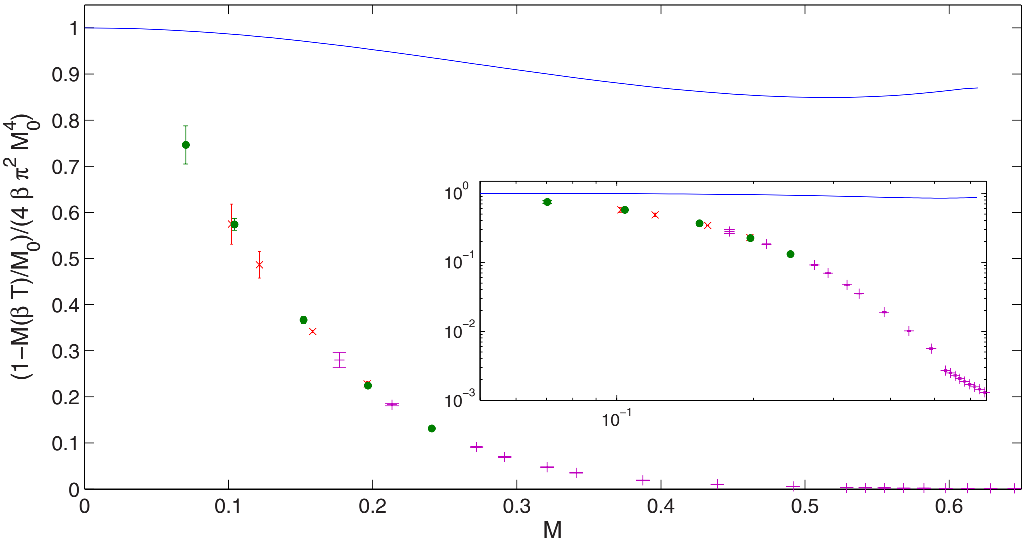

FIG. 3. (Color online) Relative variation of $[1-M(\beta T)/M_0]/(4\beta\pi^2 M_0^4)$ as a function of M. Solid line is the theoretical prediction (28), with limit 1 for $M\to 0$. Inset: same plot in log-log scale. Runs with 4096^2 (●), 2048^2, (×), and 1024^2 (+) Fourier modes.

These errors can be reduced by taking lower values of ξ at constant M, but the resolution must then be increased in order to well resolve the vortex. Thus, with a fixed resolution it is impossible to go to arbitrarily lows values of the Mach number. However, using up to 4096^2 Fourier modes, the values seem to approach well the theoretical prediction as the Mach number decreases, as is apparent in the inset of Fig. 3. On the other hand, for the computed intermediate values of M, the data are not in good agreement, even when using the high orders in M of the series (28). It is clear that at these intermediate Mach numbers the dispersive effects of the NLSE, which were not taken into account in our computation, become relevant.

V. CONCLUSION

Our main result for the radiation far field directly expressed in terms of vortex motion [Eqs. (25)–(28)] was validated by comparison with the result of numerical simulations of the NLS equation in a simple test case. The numerical data showed that the relative variation of $\Delta M/M^4$ in a fraction of turns is below the theoretical prediction at intermediate values of M. However, the data also display a clear tendency for $\Delta M/M^4$ to increase in the small Mach number limit, in a way that seems consistent with reaching the theoretical value.

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APPENDIX A: FAR-FIELD CALCULATION

Formulas (13) and (14) are concerned with integrals of the type

$$\int_{-\infty}^{\infty} dt' f(t')G(x,t-t') = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\hat{f}(\omega)e^{i\omega t'}G(x,t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\hat{f}(\omega)\frac{e^{i\omega t}}{2\pi}K_0\left(i\frac{r\omega}{c}\right),$$

where $K_n(z)$ is the modified Bessel function of the second kind [14]. We can use the K_0 derivative in order to give a formal expression; hence

$$\int_{-\infty}^{\infty} dt' f(t')G(x,t-t') = \frac{1}{2\pi}K_0\left(\frac{r}{c}\frac{d}{dt}\right)f(t). \tag{A1}$$

In the last expression, the function K_0 is understood as a functional operator applied in Fourier space. We now use the series expansion around $z=\infty$ of the function K_0,

$$K_0(z) = \sqrt{\frac{\pi}{2}}e^{-z}\left(z^{-1/2} - \frac{1}{8}z^{-3/2} + \frac{9}{128}z^{-5/2} + o(z^{-7/2})\right),$$

and the fractional derivative Eq. (A1), becomes

$$K_0\left(\frac{r}{c}\frac{d}{dt}\right)f(t) = \sqrt{\frac{\pi}{2}}\left[\left(\frac{r}{c}\right)^{-1/2}\partial_{s^{-1/2}}f - \frac{1}{8}\left(\frac{r}{c}\right)^{-3/2}\partial_{s^{-3/2}}f + \frac{9}{128}\left(\frac{r}{c}\right)^{-5/2}\partial_{s^{-5/2}}f + \cdots\right]\Bigg|_{t_r}, \tag{A2}$$

where $t_r = t - r/c$. Formula (16) is directly obtained by replacing Eq. (14) in Eq. (13), using Eqs. (A1) and (A2) and noting that $\partial G/\partial y_k = -\partial G/\partial x_k$.

In order to get the multipolar expansion we write the series expansion of $K_0(z)$ for $z\approx 0$:

$$K_0(z) \approx -\ln(z) + \ln(2) - \gamma_E + \frac{1}{4}[-\ln(z) + \ln(2) - \gamma_E + 1]z^2 + o(z^4) \quad \text{(A3)}$$

with γ_E=0.572 215 6… the Euler-Mascheroni constant. The multipolar expression (21) follows directly from Eqs. (20) and (A3).

APPENDIX B: SERIES EXPANSION

In this appendix we compute the series (25). Let n be an even number; from Eq. (13) it is apparent that the contribution of order M^n to the series is coming from the term

$$\dot{\phi}_{M^n}M^n = \frac{4\alpha\epsilon_{j_0j_1}}{(n-1)!}\partial_{j_0j_2\cdots j_n}\left(K_0\left(\frac{r}{c}\frac{d}{dt}\right)\dot{R}_{j_1}R_{j_2}\cdots R_{j_n}\right) = \frac{4\alpha}{c^n(n-1)!}\sqrt{\frac{\pi c}{2r}}\epsilon_{j_0j_1}\hat{n}_{j_0}\hat{n}_{j_2}\cdots\hat{n}_{j_n} \times \partial_{t^{n-1/2}}(\dot{R}_{j_1}R_{j_2}\cdots R_{j_n}), \quad \text{(B1)}$$

where we have used that, in the radiative limit,

$$K_0\left(\frac{r}{c}\frac{d}{dt}\right)f(t) = \sqrt{\frac{\pi c}{2r}}\,\partial_{s^{-1/2}}f(s)|_{s=t_r}$$

and thus $\partial_{j_k} = -\hat{n}_{j_k}c^{-1}\partial_t$.

Let $\Gamma_k \in \mathbb{C}$ be such that $R_k = \mathrm{Re}(\Gamma_k e^{i\omega t})$; then

$$\dot{R}_{j_1}R_{j_2}\cdots R_{j_n} = \frac{1}{2^n}(i\omega\Gamma_{j_1}e^{i\omega t} - i\omega\bar{\Gamma}_{j_1}e^{-i\omega t})(\Gamma_{j_2}e^{i\omega t} + \bar{\Gamma}_{j_2}e^{-i\omega t})\cdots(\Gamma_{j_n}e^{i\omega t} + \bar{\Gamma}_{j_n}e^{-i\omega t}),$$

$$\begin{aligned}\dot{R}_{j_1}R_{j_2}\cdots R_{j_n} = {}& \frac{\omega}{2^n}(i\Gamma_{j_1}\Gamma_{j_2}\cdots\Gamma_{j_n}e^{in\omega t} + \text{c.c.}) \\ &+ \frac{\omega}{2^n}(-i\bar{\Gamma}_{j_1}\Gamma_{j_2}\cdots\Gamma_{j_n}e^{i(n-2)\omega t} + i\Gamma_{j_1}\bar{\Gamma}_{j_2}\cdots\Gamma_{j_n}e^{i(n-2)\omega t} + \cdots + i\Gamma_{j_1}\Gamma_{j_2}\cdots\bar{\Gamma}_{j_n}e^{i(n-2)\omega t} + \text{c.c.}) \\ &+ \frac{\omega}{2^n}(-i\bar{\Gamma}_{j_1}\bar{\Gamma}_{j_2}\cdots\Gamma_{j_n}e^{i(n-4)\omega t} + i\Gamma_{j_1}\bar{\Gamma}_{j_2}\bar{\Gamma}_{j_3}\cdots\Gamma_{j_n}e^{i(n-4)\omega t} + \cdots + i\Gamma_{j_1}\Gamma_{j_2}\cdots\bar{\Gamma}_{j_{n-1}}\bar{\Gamma}_{j_n}e^{i(n-4)\omega t} + \text{c.c.}) + \cdots \\ &+ \frac{\omega}{2^n}(-i\bar{\Gamma}_{j_1}\bar{\Gamma}_{j_2}\cdots\bar{\Gamma}_{j_{n/2}}\Gamma_{j_{n/2+1}}\cdots\Gamma_{j_n}e^{i(n-2n/2)\omega t} + \cdots + \text{c.c.}).\end{aligned} \quad \text{(B2)}$$

We can directly check that

$$-\epsilon_{j_0j_1}\hat{n}_{j_0}\bar{\Gamma}_{j_1}\hat{n}_{j_2}\Gamma_{j_2} = \epsilon_{j_0j_1}\hat{n}_{j_0}\Gamma_{j_1}\hat{n}_{j_2}\bar{\Gamma}_{j_2},$$

$$\epsilon_{j_0j_1}\hat{n}_{j_0}\Gamma_{j_1} = iae^{-i\theta},$$

$$\hat{n}_j\Gamma_j = ae^{-i\theta},$$

$$\hat{n}_j\bar{\Gamma}_j = ae^{i\theta},$$

and thus each term inside each parenthesis in the right-hand side of Eq. (B2) has the same value. We see in (B2) that the contribution with the frequency $(n-2l)\omega$ has l terms (of the total of n) with Γ conjugate and thus $\binom{n}{l}$ ways of choosing them. Hence

$$\epsilon_{j_0j_1}\hat{n}_{j_0}\cdots\hat{n}_{j_n}\dot{R}_{j_1}R_{j_2}\cdots R_{j_n} = -\frac{\omega a^n}{2^{n-1}}\mathrm{Re}\left[\binom{n}{0}e^{in(\omega t_r-\theta)} + \binom{n}{1}e^{i(n-2)(\omega t_r-\theta)} + \cdots + \binom{n}{\frac{n}{2}}e^{i(n-2n/2)(\omega t_r-\theta)}\right] \quad \text{(B3)}$$

$$= -\frac{\omega a^n}{2^{n-1}}\sum_{l=0}^{n/2}\binom{n}{l}\mathrm{Re}(e^{i(n-2l)(\omega t_r-\theta)}). \quad \text{(B4)}$$

From (B1) we obtain

$$\dot{\phi}_{M^n}M^n = 4\alpha\sqrt{\frac{\pi c}{2r}}\frac{\omega a^n}{2^{n-1}(n-1)!}\sum_{l=0}^{n/2}\binom{n}{l}\partial_{t^{n-1/2}}\mathrm{Re}(e^{i(n-2l)(\omega t_r-\theta)})$$
$$= 4\alpha\sqrt{\frac{\pi\omega c}{2r}}\left(\frac{a\omega}{c}\right)^n\frac{1}{2^{n-1}(n-1)!}\sum_{l=0}^{n/2}\binom{n}{l}(n-2l)^{n-1/2}\mathrm{Re}(i^{n-1/2}e^{i(n-2l)(\omega t_r-\theta)}) \tag{B5}$$

and finally formula (25) follows directly.

The total radiated power (28) is easily obtained by noting that

$$\int_0^{2\theta} d\theta \cos\left[(n-2l)(\omega t_r-\theta)+\frac{\pi}{2}\left(n-\frac{1}{2}\right)\right]\cos\left[(m-2k)(\omega t_r-\theta)+\frac{\pi}{2}\left(m-\frac{1}{2}\right)\right] = \pi\delta_{n-m,2l-2k}\cos\frac{\pi}{2}(n-m)$$

for all n,m,l,k integer.

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[15] There are several misprints in the principal references of the literature [7,8] that need to be clarified in order to check that our prefactor in formula (24) is indeed correct. First, in Klyatskin [8], formula (2.11) uses the same definition of the energy as we do: $\mathcal{E}=(1/2)\int dx(\nabla\phi)^2$, but, in formula (2.12), the vortex interaction energy is two times greater than in our formula (23). This leads to a difference in a factor $1/2$ from our Eq. (24) for the radius a. Second, in Pismen [7], Chap. 4, formula (4.26), the definition of the total energy is $\mathcal{E}=(1/8)\int dx(\nabla\phi)^2$, leading to a vortex interaction energy $\mathcal{E}_{\mathrm{int}}=-\pi c^2\xi^2\ln 2a$ that is consistent with our formula (23). However, the energy flux used in [7], Chap. 4, formula (4.48) is not consistent with this definition of the interaction energy [furthermore, a factor π is also missing in formula (4.50) for J] leading to a difference of a factor $\pi/8$ from our Eq. (24) for the radius a.