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Relativistic hydrodynamics of semiclassical quantum fluids

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Abstract

The description in terms of hydrodynamical variables of a relativistic superfluid, modeled by a semiclassical wave equation, is given using a generalized Madelung transformation. The Galilean limit is shown (for both wavefunction and fluid variables) to be the well known Landau–Pitaevski model of superflows at $T = 0$ K. The special relativistic elementary classical acoustic and vortex excitations are explicited. A model for a relativistic self-gravitating superfluid is obtained by minimally coupling the wave equation to Einstein's gravity. The equations corresponding to a static star (using fluid variables) and an isotropic cosmology are derived.

1. Introduction

Recently, much work has been devoted to the study of the hydrodynamics of a Galilean superfluid described in a semi-classical context by the non-linear Schrödinger equation (NLSE) [1–3]. In this approach, the so-called Madelung transformation maps the wave dynamics into a fluid dynamical description. The physical phenomena contained in this picture are irrotational ideal fluid dynamics, acoustics and line vortices corresponding to the nodes of the complex wave function. A special relativistic generalization of the NLSE dynamics, using the nonlinear Klein–Gordon equation (NLKGE), has been studied by J.C. Neu [4,5]) with emphasis on the derivation of equations of motion for vortices, without taking into account the acoustic sector of the dynamics. In a general relativistic framework, static solutions of this wave equation describing boson stars have already been considered by various authors [6,7], but without a Madelung-like correspondence to usual hydrodynamics. In cosmology, unidimensional topological defects of general relativistic complex scalar fields have been interpreted as cosmic strings and proposed as a possible ingredient for large-scale structure formation [8,9]. General relativistic models of superfluidity as a generalization of the Galilean two fluids model [10–12] have been considered, among others, by Israel [13], but, to the best of our knowledge, a complete quantum derivation of the equations of motion for the relativistic perfect fluid (at $T = 0$ K) has not yet been carried out.

The aim of this paper is to propose a unifying point of view for these different subjects through (relativistic) hydrodynamics and to explore some

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related interesting points which have not been submitted yet to fully scrutiny in the literature. In Section 2 we study carefully the relations between the special relativistic NLKGE and the standard relativistic fluid dynamics, using a generalized Madelung transformation. The non-trivial and rather involved Galilean limit is characterized, both in the Lagrangian description of the equations of motion and with the conserved currents. The dispersion relation corresponding to special relativistic acoustics is obtained. The special relativistic generalization of Ginzburg–Pitaevski quantum vortices is discussed in terms of hydrodynamical variables.

In Section 3, we give a complete hydrodynamical description of semi-classical boson star models in general relativity, exhibit a natural generalization of the classical Tolman–Oppenheimer–Volkov equation for these objects and its Newtonian limit. In Section 4, we derive the cosmological equations describing a "toy universe" where the only form of matter is the NLKG superfluid and relate it to more standard cosmological models.

Finally, in Section 5, we conclude by indicating a few problems left open for further study.

Notation. Throughout the paper, the units will be chosen in order to insure $\hbar = 1$, and the signature of the Lorenzian metric will be chosen to be negative. In Section 2, c will *not* be set equal to 1 and the space–time coordinates x^μ will be written $(ct, \boldsymbol{x})$, corresponding to the Minkovskian metric η given by $\eta = \text{diag}\ (1, -1, -1, -1)$.

2. The special relativistic Bose-condensate at $T = 0$ K

2.1. General presentation of the special relativistic equations

Let $\Psi(x^\mu)$ be the relativistic wave function of the condensate in its fundamental state. Ψ will satisfy the equation of motion which derives from the so-called non-linear Klein–Gordon (NLKG) Lagrangian, which, after convenient scaling for the wave functions, takes the form:

$$L(\Psi^*_\mu, \Psi_\mu, \Psi^*, \Psi) = \Psi^*_\mu \Psi^\mu - m_b^2 c^2 |\Psi|^2 - \frac{1}{\xi^2}(|\Psi|^2 - 1)^2, \quad (2.1.1)$$

where m_b is the boson mass and ξ some characteristic coherence length. Having already set $\hbar = 1$, the only units we are still free to fix are the unit for length and for time. Choosing ξ as unit length and ξ/c as the unit of time, L can be recast in the somewhat simpler form:

$$L(\Psi^*_\mu, \Psi_\mu, \Psi^*, \Psi) = \Psi^*_\mu \Psi^\mu - m^2 c^2 |\Psi|^2 - (|\Psi|^2 - 1)^2 \quad (2.1.2)$$

with $m = m_b \xi^2$ denoting the mass of the bosons in the new system of units. Expression (2.1.2) will be retained in all the paragraphs of this section. Another choice of units will be adopted in Section 3.

As expected, the equations of motion derived from L are equivalent to the usual NLKG equation:

$$\Psi^\mu{}_\mu + m^2 c^2 \Psi + 2\Psi(|\Psi|^2 - 1) = 0\,. \quad (2.1.3)$$

Replacing the complex wave function Ψ by its modulus r and its phase θ, one can rewrite the Lagrangian density as:

$$L(r_\mu, \theta_\mu, r, \theta) = r_\mu r^\mu + r^2 \theta_\mu \theta^\mu - m^2 c^2 r^2 - (r^2 - 1)^2\,. \quad (2.1.4)$$

The Lagrange equation associated with r is:

$$r^\mu{}_\mu - r\theta_\mu \theta^\mu + m^2 c^2 r + 2r(r^2 - 1) = 0 \quad (2.1.5)$$

and that associated with θ reads:

$$2r_\mu \theta^\mu + r\theta^\mu{}_\mu = 0\,. \quad (2.1.6)$$

These equations taken together are obviously equivalent to (2.1.3).

As is well-known, the conserved current associated with the $U(1)$ invariance of L, j_μ, may be chosen to be:

$$j_\mu = -\frac{1}{m}\,\mathrm{Im}(\Psi_\mu \Psi^*) = -\frac{1}{m} r^2 \theta_\mu \qquad (2.1.7)$$

and the canonical energy–momentum tensor, conveniently normalized, is given by:

$$T_{\mu\nu} = \frac{1}{2m}\left[\Psi^*_\mu \Psi_\nu + \Psi_\mu \Psi^*_\nu - L\eta_{\mu\nu}\right]$$
$$= \frac{1}{m}\left[r_\mu r_\nu + r^2 \theta_\mu \theta_\nu - \tfrac{1}{2} L \eta_{\mu\nu}\right]. \qquad (2.1.8)$$

Two remarks should be made at this point. First, (2.1.6) clearly implies the conservation of j. Second, T, like the Lagrangian density L, contains a constant term independent of the fields, which makes T non-zero (and equal to $(1/2m)\eta$), even when the wave function identically vanishes. This constant, somewhat unphysical term is harmless as long as we restrict ourselves to special relativity and we will retain it in all of Section 2. It will be naturally discarded in Section 3.

2.2. The Galilean limit

To study the Galilean limit, let us fix some 4-vector K with norm mc and introduce a new function Φ defined by:

$$\Phi = \mathrm{e}^{\mathrm{i}K\cdot x}\psi = r\,\mathrm{e}^{\mathrm{i}\phi}. \qquad (2.2.1)$$

In terms of Φ, L reads:

$$L(\Phi^*_\mu, \Phi_\mu, \Phi^*, \Phi) = \Phi^*_\mu \Phi^\mu + 2K^\mu\,\mathrm{Im}(\Phi^*_\mu \Phi)$$
$$-(|\Phi|^2 - 1)^2 \qquad (2.2.2)$$

or, using the phase and modulus of Φ:

$$L(r_\mu, \phi_\mu, r, \phi) = r_\mu r^\mu + r^2 \phi_\mu \phi^\mu - 2r^2 K_\mu \phi^\mu$$
$$-(r^2 - 1)^2 \qquad (2.2.3)$$

and Eq. (2.1.3) becomes:

$$\Phi^\mu{}_\mu - 2\mathrm{i}K^\mu \Phi_\mu + 2\Phi(|\Phi|^2 - 1) = 0. \qquad (2.2.4)$$

The equations equivalent to (2.1.5) and (2.1.6) read:

$$r^\mu{}_\mu - r\phi_\mu \phi^\mu + 2rK_\mu \phi^\mu + 2r(r^2 - 1) = 0, \qquad (2.2.5)$$

$$2r_\mu(\phi^\mu - K^\mu) + r\phi^\mu{}_\mu = 0. \qquad (2.2.6)$$

As for the conserved currents, one has:

$$j_\mu = \frac{1}{m}\left[-\mathrm{Im}(\Phi_\mu \Phi^*) + |\Phi|^2 K_\mu\right]$$
$$= \frac{1}{m}\left[-r^2 \phi_\mu + r^2 K_\mu\right] \qquad (2.2.7)$$

and

$$T_{\mu\nu} = \frac{1}{2m}\left[\Phi^*_\mu \Phi_\nu + \Phi_\mu \Phi^*_\nu - L\eta_{\mu\nu}\right]$$
$$+ \frac{K_\mu}{m}\,\mathrm{Im}(\Phi^*_\nu \Phi)$$
$$+ \frac{K_\nu}{m}\,\mathrm{Im}(\Phi^*_\mu \Phi) + \frac{K_\mu K_\nu}{m}|\Phi|^2$$
$$= \frac{r_\mu r_\nu}{m} + \frac{r^2}{m}\phi_\mu \phi_\nu - \frac{r^2}{m}(K_\mu \phi_\nu + \phi_\mu K_\nu)$$
$$+ r^2 \frac{K_\mu K_\nu}{m} - \frac{L}{2m}\eta_{\mu\nu}. \qquad (2.2.8)$$

To recover from (2.2.4) and (2.2.2) the usual NLS equation and the classical Lagrangian density from which it derives, one has only to set the contravariant components of K equal to $(mc, \mathbf{0})$ and let c tend to infinity. Moreover, if one defines the Galilean particle density j_0^{G} and the associated current density 3-vector $\boldsymbol{j}^{\mathrm{G}}$ by:

$$j_0^{\mathrm{G}} = \lim_{c\to\infty} \frac{j_0}{c}, \quad \boldsymbol{j}^{\mathrm{G}} = \lim_{c\to\infty} \boldsymbol{j}. \qquad (2.2.9)$$

Then, choosing the same value of K as before, one gets:

$$j_0^{\mathrm{G}} = r^2, \quad \boldsymbol{j}^{\mathrm{G}} = \frac{1}{m} r^2 \nabla\phi \qquad (2.2.10)$$

and (2.2.6) reduces to the Galilean continuity equation associated to NLS as c tends to infinity.

Since the conservation equation:

$$\partial^\mu T_{\mu\nu} = \partial^\mu T_{\nu\mu} = 0 \qquad (2.2.11)$$

may be understood as a direct consequence of the equations of motion, it has also, by the procedure outlined above, a limit which is identical to the energy–momentum balance derived from the NLS equation. However, obtaining the associated asymptotic form of the energy–momentum tensor is a little more involved. To understand how it works, let us review briefly how one solves the corresponding problem for point particles, i.e. what one must do to obtain the Galilean (kinetic) energy and momentum from the special relativistic energy–impulse 4-vector p. As is well known, the trick consists in first subtracting from p the product of the mass m, the constancy of which is associated to matter conservation, by some 4-vector K normalized to c^2. One has then to choose the components of K to be $(c^2, \mathbf{0})$ and let the velocity of the particle, v, be much less than c, retaining only the higher order terms (in $1/c$) in the expression obtained. This strongly suggest the following generalization for the case of the continuum we are dealing with: First, subtract from T the (tensorial) product of j by the vector K used in (2.2.1) in order to obtain a new second order tensor $\tilde{T}$. Then, set the components of K equal to $(mc, \mathbf{0})$, as before and, finally, obtain the correct asymptotic expression for $\tilde{T}$ as c tends to infinity. According to this procedure, one has:

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - j_\mu K_\nu = \frac{1}{m}\left[r_\mu r_\nu + r^2\phi_\mu\phi_\nu - r^2 K_\mu\phi_\nu - \tfrac{1}{2}L\eta_{\mu\nu}\right]. \tag{2.2.12}$$

Note that $\tilde{T}$, unlike T, is not symmetric so that the conservation equations (2.2.11) only imply:

$$\partial^\mu \tilde{T}_{\mu\nu} = 0 \tag{2.2.13}$$

and not:

$$\partial^\mu \tilde{T}_{\nu\mu} = 0 . \tag{2.2.14}$$

Let us then define the Galilean energy density $\tilde{T}^{G}_{00}$, the associated current density vector $\tilde{T}^{G}_{i0}$, the 3-momentum density $\tilde{T}^{G}_{0i}$ and the corresponding stress tensor $\tilde{T}^{G}_{ij}$ by:

$$\tilde{T}^{G}_{00} = \lim_{c\to\infty} \tilde{T}_{00} ,$$

$$\tilde{T}^{G}_{i0} = \lim_{c\to\infty} c\tilde{T}_{i0} ,$$

$$\tilde{T}^{G}_{0i} = \lim_{c\to\infty} \frac{\tilde{T}_{0i}}{c} ,$$

$$\tilde{T}^{G}_{ij} = \lim_{c\to\infty} \tilde{T}_{ij} . \tag{2.2.15}$$

These different quantities, for $K = (mc, \mathbf{0})$, turn out to read:

$$\tilde{T}^{G}_{00} = \frac{1}{2m}\left[(\nabla r)^2 + r^2(\nabla\phi)^2 + (r^2-1)^2\right],$$

$$\tilde{T}^{G}_{i0} = \frac{1}{m}(r_t r_i + r^2\phi_i\phi_t),$$

$$\tilde{T}^{G}_{0i} = -r^2\phi_i ,$$

$$\tilde{T}^{G}_{ij} = \frac{1}{m}\left[r_i r_j - \tfrac{1}{2}(\nabla r)^2\delta_{ij}\right] + \frac{r^2}{m}\left[\phi_i\phi_j - \tfrac{1}{2}(\nabla\phi)^2\delta_{ij}\right] - \frac{1}{2m}\left[2r^2 m\phi_t + (r^2-1)^2\right]\delta_{ij} \tag{2.2.16}$$

and (2.2.13) gives back the standard energy and momentum conservation equations associated to the NLS equation. Obviously, the use of indices $\in \{0, 1, 2, 3\}$ in (2.2.16) is purely formal and does not mean that $\tilde{T}^{G}$ is a Lorentzian 4-tensor. Let us close this discussion by remarking that further insights concerning the meaning of the various terms in (2.2.8) and (2.2.16) will be gained by the hydrodynamical presentation of the following paragraph.

2.3. Hydrodynamical presentation

We will now show that the special-relativistic Bose-condensate admits a description in terms of hydrodynamical variables, and that this description allows one to get, in the Galilean limit, the usual hydrodynamical description of the NLS equation.

To begin with, let us identify the 4-velocity u and the scalar particle density n of the equivalent fluid. By (2.1.7), one has to set:

$$u_\mu = -\theta_\mu(\theta^\alpha\theta_\alpha)^{-1/2} \tag{2.3.1}$$

and

$$n = \frac{r^2}{m}(\theta^\mu\theta_\mu)^{1/2} \tag{2.3.2}$$

so that the particle current density might take the usual form:

$$j_\mu = n u_\mu \,. \tag{2.3.3.}$$

Relations (2.3.1) and (2.3.2) are the special relativistic generalization of the Madelung transformation. At this point, we note that the possibility of defining n and u by (2.3.2) and (2.3.1) is correlated to the assumption that $\theta_\mu \theta^\mu$ remains positive, i.e. the 4-vector j remains timelike in the interesting region of space–time; this condition will be automatically satisfied everywhere if we restrict our study to not too important perturbations around the homogeneous equilibrium state of the condensate, described by $j = (m, \mathbf{0})$. However, a representation of a generic solution of the NLKGE in hydrodynamical terms may not be possible everywhere. A simple and interesting example is provided with the straight vortex solution, described in 2.5.

The following step is to identify the thermodynamical functions of the condensate. As in the Galilean case, the 'superflow' of the special relativistic fluid associated to it should always be irrotational. We will consequently choose the scalar enthalpy density, w, to be:

$$w = \frac{r^2}{m} \theta^\mu \theta_\mu \tag{2.3.4}$$

in order to obtain, from (2.3.1) and (2.3.2):

$$\theta_\mu = -\frac{w}{n} u_\mu \tag{2.3.5}$$

which is the usual special relativistic condition for potential flow [11].

Using the three definitions (2.3.1), (2.3.2) and (2.3.4), one can write the energy-momentum tensor in the form:

$$T_{\mu\nu} = \left(\frac{n}{\sqrt{w}}\right)_\mu \left(\frac{n}{\sqrt{w}}\right)_\nu + w u_\mu u_\nu - \frac{L}{2m} \eta_{\mu\nu} \tag{2.3.6}$$

which strongly suggests, by comparison with the expression of the same tensor for a perfect special relativistic fluid, that the 'pressure' p has some link with the Lagrangian density L. We will retain for p the following definition:

$$\begin{aligned} p &= \frac{L}{2m} - \frac{1}{2}\left[\frac{1}{m}\left(\frac{n}{\sqrt{w}}\left(\frac{n}{\sqrt{w}}\right)_\mu\right)^\mu\right] + \frac{1}{2m} \\ &= \frac{L}{2m} + \frac{1}{2m} + q \end{aligned} \tag{2.3.7}$$

so that T can be written:

$$\begin{aligned} T_{\mu\nu} &= w u_\mu u_\nu - p\eta_{\mu\nu} + \left(\frac{n}{\sqrt{w}}\right)_\mu \left(\frac{n}{\sqrt{w}}\right)_\nu \\ &\quad + q\eta_{\mu\nu} + \frac{1}{2m}\eta_{\mu\nu} \,. \end{aligned} \tag{2.3.8}$$

The first two terms in the preceding expression form the ideal fluid part of T. The next two terms stem from the dispersive nature of the fluid; the quantity q can be appropriately called 'the quantum pressure' of the fluid, keeping in mind that what is loosely called 'quantum pressure effects' is a manifestation of both q and the other dispersive term in T. The last term in (2.3.8) is the constant alluded to at the end of 2.2. Moreover, using (2.3.2), (2.3.4), (2.1.4), (2.1.5) and (2.1.6), it is easy to check that, for all possible motions of the condensate, the value of p takes the much simpler form:

$$p = mn^4/(2w^2) \,. \tag{2.3.9}$$

As for the internal energy density ε, we will conserve the classical expression $\varepsilon = w - p$. Since we describe the quantum liquid at $T = 0$ K, it is natural to suppose that its entropy vanishes identically. Therefore, (2.3.9) can be viewed as the equation of state of the superfluid at $T = 0$ K. The chemical potential μ of the Bose-condensate can then be expressed, by standard thermodynamical arguments, as the specific enthalpy (per particle), namely $\mu = w/n$.

To check that the preceding identifications are meaningful, let us derive from them the fluid variables associated with the Galilean condensate (at $T = 0$ K). In accordance with (2.3.3), it is natural to define the Galilean particle density n^{G} and the Galilean fluid velocity $\boldsymbol{v}^{\mathrm{G}}$ by:

$$n^{\mathrm{G}} = \lim_{c\to\infty} \frac{n u_0}{c} = \lim_{c\to\infty} \frac{n}{c} \tag{2.3.10}$$

and

$$\boldsymbol{u} = \frac{1}{c}\frac{\boldsymbol{v}^{\mathrm{G}}}{\sqrt{1-(\boldsymbol{v}^{\mathrm{G}}/c)^2}}\,. \tag{2.3.11}$$

From (2.3.2), (2.3.4) and (2.3.10), one deduces that, in the Galilean limit, $n/\sqrt{w}$ is equivalent to $\sqrt{n^{\mathrm{G}}/m}$; this fact, together with (2.3.10), implies that w is equivalent to $n^{\mathrm{G}}mc^2$ as c tends to infinity. Using this, (2.3.5) reads, in the Galilean limit:

$$\nabla\theta = m\boldsymbol{v}^{\mathrm{G}} \tag{2.3.12}$$

insuring that the flow defined by $\boldsymbol{v}^{\mathrm{G}}$ is indeed potential in the Galilean sense. On the other hand, relation (2.3.9) implies immediately that the pressure p, for all possible motions, is identically equal to $(n^{\mathrm{G}})^2/2m$, which is the correct Galilean equation of state. In the same way, the definition of the quantum pressure q. (2.3.8), implies that this quantity, in the Galilean limit, is equivalent to $1/4m\,\Delta n^{\mathrm{G}}$. Moreover, choosing again $K = (\mathrm{mc}, \mathbf{0})$, (2.2.12) leads to:

$$T_{00} = \tilde{T}_{00} + mcj_0\,. \tag{2.3.13}$$

From this relation, by use of (2.2.16) and the definition of n^{G}, one obtains directly the asymptotic expansion of ε in the Galilean limit:

$$\varepsilon \approx n^{\mathrm{G}}mc^2 + \frac{1}{2m}n^{\mathrm{G}}(n^{\mathrm{G}}-2) \tag{2.3.14}$$

which leads us to the right expression of the Galilean internal energy density ε^{G}:

$$\varepsilon^{\mathrm{G}} = \frac{1}{2m}n^{\mathrm{G}}(n^{\mathrm{G}}-2)\,. \tag{2.3.15}$$

The Galilean enthalpy density w^{G} and chemical potential $\mu^{\mathrm{G}} = w^{\mathrm{G}}/n^{\mathrm{G}}$ are then obviously given by:

$$w^{\mathrm{G}} = \frac{1}{m}n^{\mathrm{G}}(n^{\mathrm{G}}-1)\,, \tag{2.3.16}$$

$$\mu^{\mathrm{G}} = n^{\mathrm{G}} - 1 \tag{2.3.17}$$

and are related to the asymptotic form of the corresponding special relativistic quantities by:

$$w \approx n^{\mathrm{G}}mc^2 + w^{\mathrm{G}} \tag{2.3.18}$$

and

$$\mu \approx mc + \frac{\mu^{\mathrm{G}}}{c}\,. \tag{2.3.19}$$

2.4. Acoustic modes

As explained in the preceding paragraph, the special relativistic Bose-condensate at $T = 0$ K can be considered, as its Galilean analogue, as some particular type of perfect fluid. The quantum nature of this fluid exhibits itself through the presence of the so-called quantum-pressure terms which can give birth to topological defects also known as vortices. But acoustic waves can also propagate in the Bose-condensate, as in any other fluid. Let us find the special relativistic dispersion relation for these waves. To do this, we will search for solutions of (2.2.4) representing small perturbations around the equilibrium state $\Phi(x) = 1$ i.e. for solutions of the form:

$$\Phi(x) = (1 + \varepsilon\rho(x))\,\mathrm{e}^{i\varepsilon\alpha(x)}\,, \tag{2.4.1}$$

where ρ and α are both of zeroth order. Inserting this ansatz in (2.1.3) and retaining only the first-order terms in ε, one obtains:

$$-2\rho_\mu K^\mu + \alpha^\mu{}_\mu = 0\,,$$
$$2\alpha_\mu K^\mu + \rho^\mu{}_\mu + 4\rho = 0\,. \tag{2.4.2}$$

If one seeks plane wave solutions to this system, one can check easily that the only solutions of this type which do not vanish identically must have a wave 4-vector k which verifies the dispersion relation:

$$4(k\cdot K)^2 + k^2(4-k^2) = 0\,. \tag{2.4.3}$$

Introducing the contravariant components of k, ω/c and $\boldsymbol{k}$ and setting those of K equal to $(mc, \mathbf{0})$, (2.4.3) can be written:

$$4m^2\omega^2 + \left(\frac{\omega^2}{c^2} - \boldsymbol{k}^2\right)\left(4 - \frac{\omega^2}{c^2} + \boldsymbol{k}^2\right) = 0\,. \tag{2.4.4}$$

Letting c tend to infinity, one obtains the usual Galilean dispersion relation:

$$\omega^2 = \frac{1}{m^2}(\boldsymbol{k}^2 + \tfrac{1}{4}(\boldsymbol{k}^2)^2)\,. \tag{2.4.5}$$

2.5. *Ginzburg–Pitaevski vortex*

Let us now present rapidly, in the hydrodynamical language, some fundamentals about the special relativistic equivalent of the Ginzburg–Pitaevski vortex (GPV) solution.

In this paragraph, when referring to equations of 2.2, we will implicitly assume that we have set *ab initio* the contravariant components of K equal to $(mc, \mathbf{0})$. With this in mind, we will say that a solution Ψ of (2.1.3) is a special relativistic Ginzburg–Pitaevski vortex (SRGPV) if there exists an inertial reference frame, the proper frame of the vortex, in which the function Φ associated to Ψ by (2.2.1) is a time independent (spatially-)cylindrical solution of (2.2.4). This definition directly implies that, in the proper frame of the vortex, the function Φ describes the SRGPV if (and only if) it also describes the usual Galilean GPV. In particular, if ρ and α are polar coordinates in this frame around the axis of the vortex, the phase of $\Phi(\rho, \alpha)$, $\phi(\rho, \alpha)$, will be an integral multiple of α:

$$\phi(\rho, \alpha) = q\alpha\,, \quad q \in \mathbb{Z}\,. \tag{2.5.1}$$

Therefore, in this frame, the phase $\theta(\rho, \alpha, t)$ of the original wave-function $\Psi(\rho, \alpha, t)$ will be given by:

$$\theta(\rho, \alpha, t) = q\alpha - mc^2 t\,, \quad q \in \mathbb{Z}\,. \tag{2.5.2}$$

Let C be any space-like closed contour around the vortex. The circulation:

$$I = \int_C \mathrm{d}x^\mu\, \theta_\mu \tag{2.5.3}$$

is a Lorenz scalar. Its evaluation in the proper frame of the vortex naturally gives:

$$I = 2\pi q\,. \tag{2.5.4}$$

To recast the preceding results in hydrodynamical language, we have first to evaluate the scalar $\theta_\mu \theta^\mu$. We obtain from (2.5.2):

$$\theta_\mu \theta^\mu = m^2 c^2 - \frac{q^2}{\rho^2}\,. \tag{2.5.5}$$

Since the possibility of presenting a solution of the NLKGE in hydrodynamical terms presupposes the particle current density associated to it, j, to be timelike, (2.5.5) entails that an hydrodynamical presentation of the SRGPV is only possible for $\rho > \rho_{\min} = |q|/mc = |q|\lambda_C$, where λ_C is the Compton wavelength of the bosons (cf. the short discussion on this point in 2.3). Using (2.3.2), (2.3.5), (2.3.11) and (2.5.5), we get that, for $\rho > \rho_{\min}$:

$$\frac{w}{n} = mc\left(1 - \frac{\rho_{\min}^2}{\rho^2}\right)^{1/2} \tag{2.5.6}$$

and

$$\boldsymbol{v}^{\mathrm{G}} = c\,\frac{\rho_{\min}}{\rho}\,\boldsymbol{\alpha}\,, \tag{2.5.7}$$

where $\boldsymbol{\alpha}$ is the unit orthoradial vector associated to α. If one uses the standard Galilean hydrodynamical presentation of the GPV, the velocity depends on ρ as $1/\rho$ and approaches infinity as ρ tends towards zero. In the special relativistic interpretation, this behaviour is clearly impossible. What happens is that the (3-)velocity still depends on ρ as $1/\rho$ but ceases to be defined for $\rho < \rho_{\min}$ and reaches the value c precisely at $\rho = \rho_{\min}$. Moreover, $\rho_{\min}$ tends evidently towards 0 as c tends to infinity. If we now take the contour C to lie entirely in the region $\rho > \rho_{\min}$, we obtain directly from (2.3.5):

$$I = \int_C \mathrm{d}x^\mu \left(\frac{w}{n} u_\mu\right) = 2\pi q\,. \tag{2.5.8}$$

However, the special relativistic vortex *cannot* be interpreted as a simple line distribution of vorticity, like its Galilean counterpart, as no hydrodynamical variable exist for $\rho < \rho_{\min}$.

3. Hydrostatics of bosons stars

The static, spherically symmetric solutions of the minimal coupling of the NLKG equation with the Einsteinian gravitational field has already been studied by various authors, in order to investigate the structure of what may be called a boson star. However, their point of view was essentially field theoretical; we will now review rapidly the structure of the basic equations to be used and, in the light of the preceding section, show that these stars satisfy a natural generalization of the Tolman–Oppenheimer–Volkoff equation (TOVE) [14].

Throughout this section we will conveniently set $c = \hbar = G = 1$ (see the discussion on the system of units at the beginning of Section 2). The appropriate action S for the coupled gravitational and complex scalar fields takes the form [14]:

$$S = \int \sqrt{-g}\, \mathrm{d}^4x \left(16\pi \frac{L_s}{2m} + R\right), \tag{3.1}$$

where L_s is the minimal curved space–time generalization of expression (2.1.2):

$$L_s = \nabla_\mu \Psi^* \nabla^\mu \Psi - m^2|\Psi|^2 - |\Psi|^4 + 2|\Psi|^2 \tag{3.2}$$

and R stands for the scalar curvature of the metric-compatible connection ∇_μ. The equation of motion for the scalar field is:

$$\nabla_\mu \nabla^\mu \Psi + m^2\Psi + 2\Psi(|\Psi|^2 - 1) = 0 \tag{3.3}$$

and a variation of (3.1) with respect to the metric leads to the Einstein equations. Introducing the same hydrodynamical variables as in Section 2, the stress–energy tensor (associated with Ψ) reads:

$$T_{\mu\nu} = wu_\mu u_\nu - pg_{\mu\nu} + \nabla_\mu\left(\frac{n}{\sqrt{w}}\right)\nabla_\nu\left(\frac{n}{\sqrt{w}}\right) + qg_{\mu\nu}\,. \tag{3.4}$$

Seeking the metric of a static spherically symmetric spacetime in the usual form [14]:

$$\mathrm{d}s^2 = f(r)\,\mathrm{d}t^2 - h(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{3.5}$$

the use of the preceding stress–energy tensor in the Einstein equations gives the following set of differential relations:

$$8\pi(\varepsilon(r) + q(r)) = (rh^2)^{-1}h' + r^{-2}(1 - h^{-1})\,, \tag{3.6}$$

$$8\pi(p(r) - q(r) + h^{-1}(\nu'(r))^2) = 8\pi P(r) = (rfh)^{-1}f' - r^{-2}(1 - h^{-1})\,, \tag{3.7}$$

$$8\pi(p(r) - q(r)) = \tfrac{1}{2}(fh)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}r}((fh)^{-1/2}f') + \tfrac{1}{2}(rfh)^{-1}f' - \tfrac{1}{2}(rh^2)^{-1}h'\,, \tag{3.8}$$

where $\nu(r)$ stands for $n(r)/\sqrt{w(r)}$. These equations are clearly identical with the ones which describe the interior of a static, spherically symmetric usual star, except for 'quantum pressure terms' which involve derivatives of ν. The solution to the problem consequently proceeds in a parallel way. (3.6) implies that:

$$h(r) = \left(1 - \frac{2M(r)}{r}\right)^{-1}, \tag{3.9}$$

where $M(r)$ is related to the internal energy density $\varepsilon(r)$ and quantum pressure $q(r)$ by:

$$M(r) = 4\pi \int_0^r (\varepsilon(r') + q(r'))r'^2\,\mathrm{d}r'\,. \tag{3.10}$$

Replacing $f(r)$ by $\exp(2\sigma(r))$ in (3.7) gives:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}r} = \frac{M(r) + 4\pi r^3(p(r) - q(r))}{r(r - 2M(r))} + 4\pi r\left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2 \tag{3.11}$$

and, after a rather tedious calculation, (3.8) takes the form:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(\varepsilon(r) + P(r) + q(r))(M(r) + 4\pi r^3 P(r))}{r(r - 2M(r))} - \frac{2}{r}(\nu'(r))^2\left(1 - \frac{2M(r)}{r}\right) \tag{3.12}$$

which is the equivalent, for the Bose-condensate, of the usual Tolman–Oppenheimer–Volkoff equation for ideal fluids. The differences are that the pressure p is replaced by the pressure-like quantities $P(r)$ or $(P(r) + q(r))$ and that an extra 'quantum pressure' term must be added to the standard relation. For a perfect fluid, the TOVE must be supplemented by an equation of state relating p to ε, in order to completely specify the (spherically symmetric) equilibrium configurations of the star under consideration. The case of the Bose-condensate is a little more complicated because (3.12) involves four different 'thermodynamical' quantities: P, ε, q and ν. Three other relations between these quantities are therefore required in order for the problem to be well-posed. The first one is, as in the ideal fluid case, the equation of state of the relativistic condensate, $p = mn^4/(2w^2)$, which, transcribed in terms of P, q and ν, reads:

$$P(r) = \tfrac{1}{2}m\nu^4(r) + \left(1 - \frac{2M(r)}{r}\right)(\nu'(r))^2 - q(r)\,. \tag{3.13}$$

The very definition of q (Eq. (3.3.8)) gives, after a straightforward calculation) the second relation:

$$q(r) = \left(\frac{2M(r)}{r} - 1\right)\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\nu(r)\frac{\mathrm{d}\nu}{\mathrm{d}r}\right) + \nu(r)\frac{\mathrm{d}\nu}{\mathrm{d}r}\left(\frac{1}{r} + 2\pi r\left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2\right)\right] - \frac{\nu(r)}{r^2}\frac{\mathrm{d}\nu}{\mathrm{d}r}\left[M(r) + 2\pi r^3(p(r) - \varepsilon(r) - 2q(r))\right]. \tag{3.14}$$

In (3.13) and (3.14), the scalar $M(r)$ has naturally to be understood as the functional of $\varepsilon(r)$ and $q(r)$ defined by (3.10). The last condition to be imposed comes from the fact that we are studying the structure of a static star, which, be definition, implies that, in the reference frame where the line element takes the form (3.5), all spatial components of the fluid 4-velocity u^μ vanish. In this frame, θ is then, according to (2.3.1), a function of the time-coordinate t only and, moreover:

$$\dot{\theta} = -\frac{w}{n}u_0\,, \tag{3.15}$$

w and n depend only on r, as well as u_0, since the solution we explore is supposed to be static. On the other hand, $\dot{\theta}$, like θ, is a function of t only. (3.13) therefore indicates that this function is constant and that θ varies linearly with t:

$$\theta = -\Omega t\,, \tag{3.16}$$

where Ω is some fixed real number, chosen to be positive if we are interested in stars made of matter (and not anti-matter). Since, by (2.3.1) and (3.5), u^0 is then equal to $f^{1/2}$, one gets:

$$\frac{w(r)}{n(r)}\,\mathrm{e}^{\sigma(r)} = \Omega \tag{3.17}$$

which, with the help of (3.11), provides the third required relation.

As has already been mentioned, the physics of these stars has been thoroughly studied and characterized by many authors; for a short review and extensive references, the reader is referred to [7]. Let us just remark here that, for example, the hydrodynamical derivation of (3.15) seems to us to be much simpler and intuitive that the purely field-theoretical one, to be found, e.g., in Ref. [6].

To conclude this section, let us now elaborate on the Newtonian–Galilean limit of the preceding equations. Following closely the usual procedure [14], we suppose that, in this case, $M(r) \ll r$, $4\pi r^3 P(r) \ll M(r)$ and $q(r) \ll \varepsilon(r)$. Since, in this regime, $\varepsilon(r)$ can be approximated very well by the mass density, we have:

$$M(r) \approx M(r) = m\int_0^r 4\pi r'^2 n^{\mathrm{G}}(r')\,\mathrm{d}r'\,. \tag{3.18}$$

On the other hand, using the fact that, in the Galilean case, $n/\sqrt{w}$ is equivalent to $\sqrt{n^{\mathrm{G}}/m}$, we get the following equivalent expression for $P(r)$:

$$P(r) \approx p(r) - \frac{1}{4m}\left[\frac{\mathrm{d}^2 n^{\mathrm{G}}}{\mathrm{d}r^2} - \frac{1}{n^{\mathrm{G}}}\left(\frac{\mathrm{d}n^{\mathrm{G}}}{\mathrm{d}r}\right)^2\right] - \frac{1}{2mr}\frac{\mathrm{d}n^{\mathrm{G}}}{\mathrm{d}r}\,. \tag{3.19}$$

The TOVE can then be recasted in the simpler asymptotic form:

$$\frac{1}{mn^{\mathrm{G}}}\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{M(r)}{r^2} + \frac{1}{m^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Delta\sqrt{n^{\mathrm{G}}}}{2\sqrt{n^{\mathrm{G}}}}\right). \tag{3.20}$$

It is easy to check that this is indeed the right equivalent to the TOVE for Newtonian starts of spin 0 bosons; as a matter of fact, one has just to couple the NLSE to Newtonian gravity by adding to it a (gravitational) potential term and by imposing, via the Poisson equation, that this potential is self-consistently created by the star itself. This was done by Rica in [15], where a detailed discussion of the Newtonian–Galilean solution can be found, together with possible astrophysical applications.

4. Homogeneous isotropic cosmological models

As has already been noticed, the minimal coupling of the gravitational field with an uncharged scalar field has already been extensively studied, notably for cosmological reasons, in relation to the construction of so-called inflationary models of the universe [16]. In this vein, let us now discuss some fundamentals about homogeneous isotropic universes obtained by minimally coupling, as in the preceding paragraph, the gravitational field and the *charged* scalar field Ψ. Expression (3.1) for the action remains valid and the metric to be sought for takes the form [14]:

$$ds^2 = dt^2 - a^2(t)\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right), \quad k=-1,0,1 \tag{4.1}$$

all variables now being functions only of the time coordinate t. The direct input of this *ansatz* in (3.1) gives the following equivalent expression for the action S:

$$S = \int d^4x \frac{\rho^2|\sin\theta|}{\sqrt{1-k\rho^2}} \times \left(\frac{16\pi a^3}{2m}[r_t^2 + r^2\theta_t^2 - m^2r^2 - r^4 + 2r^2] + 6a(k - a_t^2)\right). \tag{4.2}$$

The equation of motion for θ gives immediately that $a^3(t)\, r^2\theta_t$ is independent of t and its constant value will be hereafter conveniently denoted by C. The existence of C traces back to the particle number conservation, Eq. (2.1.6). The equations of motion for r and $a(t)$ then read respectively:

$$3a_tr_t + ar_{tt} = a\left(\frac{C^2}{a^6r^3} - m^2r - 2r^3 + 2r\right), \tag{4.3}$$

$$-\left(2\frac{a_{tt}}{a} + \frac{a_t^2}{a^2}\right) = \frac{4\pi}{m}\left(r_t^2 + \frac{C^2}{a^6r^2} - m^2r^2 - r^4 + 2r^2\right) + \frac{k}{a^2}. \tag{4.4}$$

These two equations admit an integral of motion I which is distinct from C and can be obtained by calculating the Hamiltonian density H associated to S and by replacing θ_t by C/a^3r^2. In this way one obtains:

$$I = \frac{4\pi}{m}a^3\left(r_t^2 + \frac{C^2}{a^6r^2} + m^2r^2 + r^4 - 2r^2\right) - 3a(k + a_t^2). \tag{4.5}$$

Eqs. (4.3) and (4.4) admit two simple limits. First, if one neglects the quantum pressure effects by removing the time-derivative of r in the action (4.2), (4.3) and (4.4) reduce to the standard Friedman–Robertson–Walker cosmology, for a fluid with $p = r^4/2m$ as equation of state (i.e., (2.3.9) with the r-dependance of n and w explicited). Second, when the constant of motion C is zero, the matter field dynamics reduces to that of a real scalar field and the model degenerates to Linde's chaotic inflation [17]. A complete study of the general solutions of Eqs. (4.3) and (4.4) will be presented elsewhere.

5. Conclusion

Introducing a generalization of the classical Madelung transformation, we have shown that the physics contained in the special relativistic NLKG equation, considered as a wave equation for a semi-classical superfluid at $T=0$ K, reduces, up to higher order 'quantum pressure terms', to standard irrotational dissipationless special relativistic hydrodynamics, naturally combined with the presence of topological defects. The dispersion relation for the acoustic modes propagating in this fluid has been derived. The special relativistic generalization of Ginzburg–Pitaevski quantum vortices has been presented in hydrodynamical language. We have also studied how this hydrodynamical description of a special relativistic Bose-condensate reduces, for small velocities, to the Galilean hydrodynamics derived from the NLS equation. These results prompted the search for the generalization of the usual Tolmann–Oppenheimer–Volkoff equation to bosons stars which is presented in Section 3 of this paper, together with its Newtonian limit. Finally, the equations governing the evolution of an isotropic 'toy-universe' filled with superfluid are presented and shown to give back, in suitable regimes, a standard Friedman–Robertson–Walker cosmology as well as Linde's chaotic inflation model. Let us remark that, in light of our present work, the general relativistic dynamics of a non-linear spin-0 field seems to be the simplest system in which a fluid is coupled to Einsteinian gravitation.

As a direct continuation of this work, we are currently studying, both analytically and numerically, the general relativistic phenomena corresponding to the Galilean vortex–vortex and vortex–sound interactions. Our forthcoming results will hopefully shed new light on many different issues, including cosmological ones.

Shifting to solid state physics, another application of the present study could be a more precise evaluation of possible relativistic effects and/or corrections arising in phenomena related to superconductivity.

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