

# Multifractal Scaling of Probability Density Function : a Tool for Turbulent Data Analysis

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**Abstract.** — The probabilistic reformulation of the multifractal model [1] is obtained directly from the structure functions written as integrals over cumulative distribution functions (c.d.f.) by the steepest descent method. The saddle point being a function of scale, we perform a change of variable to obtain expressions that are asymptotically valid in the inertial range. Starting directly from the inertial range behavior of the c.d.f., our algorithm yields values for the scaling exponents and codimension that are identical to those obtained from structure functions. Furthermore, a simple interpretation of multifractality in terms of global c.d.f. scaling is shown to collapse the inertial range c.d.f. into a single curve, directly related to the codimension. Our method determines a new length scale, larger than the integral scale, that gives a quantitative measure of the degree of multifractality of the data. Finally, some possible future applications are mentioned.

Inertial range intermittency in turbulence is characterized by scaling laws obeyed by the structure functions. Experimentalists often calculate the longitudinal structure functions, i.e. the  $p$ -th order moments of the longitudinal velocity increments:  $\langle \Delta v^p(\ell) \rangle = \langle (v(x+\ell) - v(x))^p \rangle$ . In the inertial range, these structure functions scales as  $\ell^{\zeta_p}$ . The deviation of the exponent  $\zeta_p$  from the Kolmogorov law  $\zeta_p = p/3$  provides a quantitative measure of intermittency [1]. Frisch et Parisi's multifractal model [2] uses the Legendre transform  $\zeta_p = \inf_h (hp + \mu(h))$  to relate the exponent  $\zeta_p$  to the codimension  $\mu(h) = 3 - D(h)$  of a set on which the velocity increments  $\Delta v(\ell)$  scale as  $\ell^h$ . More recently, Frisch [1] proposed a probabilistic reformulation of the multifractal model, based on the asymptotic behavior of the probability distribution functions (pdf's) of the velocity increments. This reformulation has the advantage of being independent of the existence of singularities, and is related to an experimentally measurable quantity. However, it is difficult to use this reformulation to determine  $\mu(h)$ , which is defined in the double limit of vanishing viscosity and length  $\ell$ .

The purpose of this letter is to use the method of steepest descents to establish asymptotic formulas for the pdf's which are valid for  $\ell$  in the inertial range. These formulas can then be

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used both to test the validity of the scaling hypotheses and also to extract  $\mu$  from experimental pdf's.

Because the velocity increments  $\Delta v(\ell)$  are of either sign, the structure functions can be defined in several ways [1,3,4]. In this letter, we have chosen to use the absolute values of the increments, as in [3].

The pdf's of the absolute velocity increments  $p_{\text{inc}}(|\Delta v(\ell)|, \ell)$  can be used to express the  $p$ -th order structure function  $S_p(\ell) = \langle |v(x+\ell) - v(x)|^p \rangle$  as

$$S_p(\ell) = \int_0^\infty u^p p_{\text{inc}}(u, \ell) du, \quad \text{with } u = \Delta v. \quad (1)$$

We also define the cumulative distribution function (cdf)

$$\frac{dP_{\text{inc}}(u, \ell)}{du} = -p_{\text{inc}}(u, \ell) \quad (2)$$

$$P_{\text{inc}}(+\infty) = 0. \quad (3)$$

Integrating (1) by parts,  $S_p(\ell)$  can be written:

$$S_p(\ell) = p \int_0^{+\infty} u^{p-1} P_{\text{inc}}(u, \ell) du. \quad (4)$$

The basic idea is to approximate the integrals (4) by the method of steepest descents [5]. Let

$$I = \int_{-\infty}^{+\infty} e^{x\Phi(t)} dt \quad (5)$$

where  $\Phi$  has a dominant saddle point  $t_c$  ( $\Phi'(t_c) = 0$ ) and  $x \rightarrow \infty$ . The method of steepest descents yields the asymptotic expression (through this letter, the asymptotic expression  $f(x) \sim g(x)$  for  $x \rightarrow x_0$  means  $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$ ).

$$I \sim e^{x\Phi(t_c)} \sqrt{2\pi} (-x\Phi''(t_c))^{-1/2} \quad (6)$$

However, the method cannot be directly applied to (4), because the saddle point  $u_s$  is a function of  $\ell$ . In such cases, a change of variables must be found such that the saddle point in the new variable becomes independent of  $\ell$  [5]. Using the assumption of the multifractal model that  $u_s$  scales as  $\ell^h$ , then an appropriate choice of variable is  $h = \log(u/u_0)/\log(\ell/\ell_0)$ , with  $u_0, \ell_0$  some given velocity and length scales. Note that this is equivalent, up to a scale dependent affine transformation that leaves the value of expression (6) unchanged, to taking  $\log u$  as the integration variable. Introducing the pdf in  $\log u$ :

$$\tilde{p}_{\text{inc}}(\log u, \ell) = u p_{\text{inc}}(u, \ell) \quad (7)$$

and the associated cdf

$$\tilde{P}_{\text{inc}}(\log u, \ell) = -\frac{d\tilde{P}_{\text{inc}}(\log u, \ell)}{d \log u} \quad (8)$$

$$\tilde{P}_{\text{inc}}(+\infty) = 0. \quad (9)$$

Equation (4) then becomes

$$S_p(\ell) = p \int_{-\infty}^{+\infty} u^p \tilde{P}_{\text{inc}}(\log u, \ell) d \log u \quad (10)$$

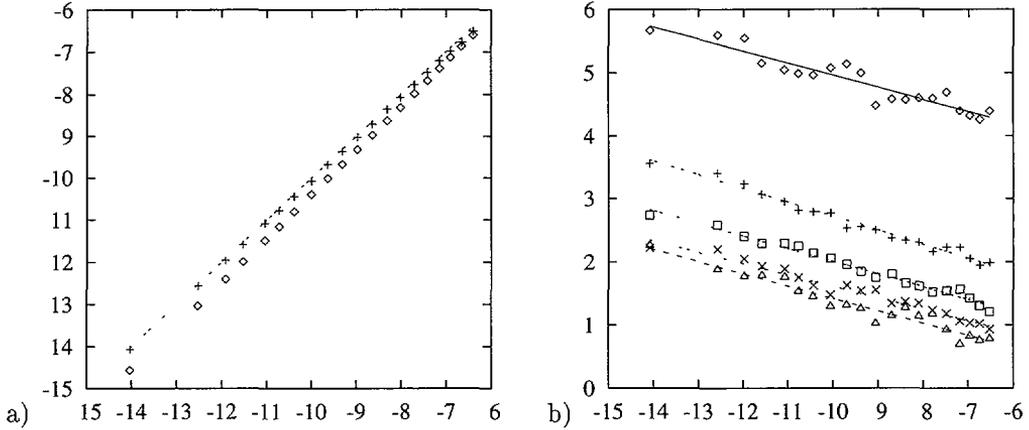


Fig. 1. — a)  $\log S_3(\ell)$  determined by (13) vs.  $\log\langle |u|^3 \rangle$  (true value). ( $\diamond$ ) without the  $l_{1/2}$  (14) contribution and ( $+$ ) with  $l_{1/2}$ . b)  $l_{1/2}^2$  vs.  $\log S_3(\ell)$  for  $p = 1 \dots 5$ . The scaling laws  $l_{1/2}^2 \sim -C(p) \log S_3(\ell)$  are displayed as straight lines.

or, in exponential form

$$S_p(\ell) = p \int_{-\infty}^{+\infty} \exp \left[ -\log \ell \left( -p \frac{\log u}{\log \ell} - \frac{\log \tilde{P}_{\text{inc}}(\log u, \ell)}{\log \ell} \right) \right] d \log u. \tag{11}$$

We are now in a position to apply the method of steepest descents (6) to (11) by identifying  $x$  with  $-\log \ell$  and  $\Phi(t)$  with  $\frac{-p \log u - \log \tilde{P}_{\text{inc}}(u, \ell)}{\log \ell}$ . Denoting by  $u_s$  the saddle point where  $\Phi(\log u)$  is maximal, i.e.

$$p = - \frac{\partial \log \tilde{P}_{\text{inc}}(\log u_s, \ell)}{\partial \log u}, \tag{12}$$

we obtain for  $\ell \rightarrow 0$  the asymptotic expression:

$$S_p(\ell) \sim p \exp(-\log \ell \Phi(\log u_s)) \sqrt{2\pi} \left[ \frac{-\partial^2 \log \tilde{P}_{\text{inc}}(\log u_s)}{\partial (\log u)^2} \right]^{-1/2} \tag{13}$$

The results that we shall present were obtained from pdf's computed on the experimental signal MODVITLON.D [6, 7]. The structure function and c.d.f. shown in this letter were computed from  $10^6$  successive integer data values, that were arbitrarily normalized by  $1/20000$ . Figure 1a shows that the saddle-point expression (13), gives a good approximation of the third order structure functions.

Expression (13) leads directly to Frisch's probabilistic reformulation [1]. Indeed, neglecting the subdominant second-order term (see below)

$$l_{1/2} = \sqrt{2\pi} \left[ \frac{-\partial^2 \log \tilde{P}_{\text{inc}}(\log u_s)}{\partial (\log u)^2} \right]^{-1/2} \tag{14}$$

the definition  $\zeta_p = \lim_{\ell \rightarrow 0} \log S_p(\ell) / \log \ell$  yields  $\zeta_p = ph + \mu$ , where  $h$  and  $\mu$  are defined by

$$h = \lim_{\ell \rightarrow 0} [\log u_s(p, \log \ell) / \log \ell] \tag{15}$$

$$\mu = \lim_{\ell \rightarrow 0} \left[ \log \tilde{P}_{\text{inc}}(\log u_s) / \log \ell \right]. \tag{16}$$

Starting from (11), (12), and using (15) and (16),  $\zeta_p$  can be expressed as

$$\zeta_p = \frac{\partial \log \tilde{P}_{\text{inc}}}{\partial \log \ell} \Big|_u = \frac{\partial \log \tilde{P}_{\text{inc}}}{\partial \log \ell} \Big|_p - h \frac{\partial \log \tilde{P}_{\text{inc}}}{\partial \log u} \Big|_\ell. \tag{17}$$

From (17), it can be verified that  $d\zeta_p/dp = h$ . The exponent  $\zeta_p$  is thus the Legendre transform of  $\mu(h)$ .

As shown by [3], the determination of the exponent of the scaling laws obeyed by  $S_p(\ell)$  is more accurate when the third-order structure function is used instead of  $\ell$  itself. Throughout this letter we shall use this method, called E.S.S. (Extended Self-Similarity) which is an expedient way to determine scaling laws within the inertial range where  $S_3(\ell)/\ell$  is constant to a good approximation.

The subdominant nature of the term  $l_{1/2}$  is a consequence of the scaling law  $l_{1/2}^2 \sim -C_0(p) \log \ell$ , (see Fig. 1b). This law can be derived from the definition of the saddle point (12) and from the multifractal behavior of  $\log(u_s/u_0) \sim h \log(\ell/\ell_0)$ , with  $h$  a function of  $p$ . The result is:

$$\frac{\partial^2 \log P_{\text{inc}}(\log u_s(\ell))}{\partial (\log u)^2} = -\frac{\partial p}{\partial \log u} \sim -\frac{C_p}{\log \ell}. \tag{18}$$

(15) and (16) yield the local scaling laws in the inertial range

$$\log u_s = h \log \ell + C_1(p) \tag{19}$$

$$\log \tilde{P}_{\text{inc}}(\log u_s) = \mu \log \ell - \log l_{1/2} + C_2(p) \tag{20}$$

$$\log S_p = \zeta \log \ell + C_3(p) \tag{21}$$

where the coefficients in (19-21) are determined by linear least-square fits (see Figs. 2a and 2b). Although  $l_{1/2}$  is subdominant in the limit  $\ell \rightarrow 0$ , it provides correction to the scaling law (20) with respect to the asymptotic expression (16). Thus, the approximation of  $S_p(\ell)$  by

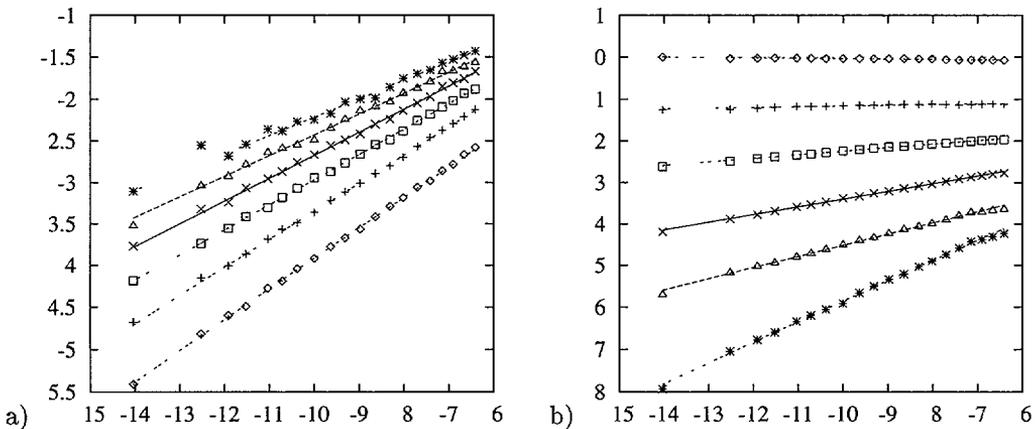


Fig. 2. — a)  $\log u_s(p, \log \ell)$  vs.  $\log S_3(\ell)$  for  $p = 1.. 6$ . The scaling law (19), displayed as straight lines, gives  $h(p)$  as the slope. b)  $\log \tilde{P}_{\text{inc}}(\log u_s, \log \ell) + \log l_{1/2}$  vs.  $\log S_3(\ell)$  for  $p = 1.. 6$ . The scaling laws (20), displayed as straight lines, directly gives the codimension  $\mu(h)$  as the slope.

Table I. — Values of exponent  $\mu_p$  and  $\zeta_p$  corresponding to straight lines in Figures 2a and 2b.  $\mu_p^*$  and  $\zeta_p^*$  are determined as  $\mu_p$  and  $\zeta_p$  but without the  $l_{1/2}$  term.

$p$	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5
$h_p$	0.36	0.34	0.32	0.31	0.29	0.27	0.26	0.25	0.24	0.23
$\mu_p$	-0.005	0.004	0.037	0.058	0.104	0.19	0.23	0.253	0.291	0.33
$\mu_p^*$	0.018	0.038	0.076	0.105	0.154	0.24	0.287	0.327	0.387	0.453
$\zeta_p$	0.357	0.525	0.692	0.847	1	1.147	1.275	1.393	1.498	1.591
$\zeta_p^*$	0.381	0.56	0.731	0.894	1.05	1.197	1.333	1.467	1.595	1.714

the method of steepest descents determines the scaling laws (19), (20) and (21). The slopes of these laws (for  $p \leq 6$ ) in the inertial range give the values of  $\mu(h)$  and  $\zeta_p$  displayed in Table I. Note that the correction to the exponents stemming from the  $l_{1/2}$  is in the range 0.05 – 0.1. The exponents  $\zeta_p$  obtained by the present method are identical to those of Benzi [3].

Because of multifractality  $h_p$  is non-constant and thus the straight lines in Figure 2a are non-parallel. A naive idea to quantify the “degree of multifractality” of the data is to extrapolate the straight lines in Figure 2a to scales much larger than the inertial scale and look for a common intersection. Such a procedure is not very accurate, but gives an order of magnitude for the intersection length scale  $\ell_\Lambda$  of order the integral scale  $\ell_I$ . Such a behavior is consistent with a global scaling law.

$$\tilde{P}_{\text{inc}}(\log u) \sim P_\Lambda \exp \left[ \log(\ell/\ell_\Lambda) \mu \left( \frac{\log(u/u_\Lambda)}{\log(\ell/\ell_\Lambda)} \right) - \frac{1}{2} \log(-\log(\ell/\ell_\Lambda)) \right] \tag{22}$$

This global scaling gives in the asymptotic ( $\ell \rightarrow 0$ ) regime the correct expression for  $\zeta_p$  (using expressions (12, 15, 16)). Furthermore, for values of  $\ell$  such that the subdominant term in (20) is not negligible, the  $1/2 \log(-\log(\ell/\ell_\Lambda))$  term in (22) will yield in (20) a value of  $\mu$  consistent to that in (22).

Equation (22) can be inverted into

$$\mu \left( \frac{\log(u/u_\Lambda)}{\log(\ell/\ell_\Lambda)} \right) = \frac{\log \tilde{P}_{\text{inc}}(\log u) - \log P_\Lambda + 1/2 \log(-\log(\ell/\ell_\Lambda))}{\log(\ell/\ell_\Lambda)} \tag{23}$$

This expression shows that if (22) holds, the c.d.f. 's corresponding to different values of  $\ell$  can be collapsed, by a simple scaling transformation, into a single function  $\mu_h$ . It can thus be used to determine values of  $\ell_\Lambda$  and  $P_\Lambda$  yielding the best collapse for  $\mu$  ( $u_\Lambda$  is not a free parameter, it can be expressed as  $(\varepsilon \ell_\Lambda)^{1/3}$  with  $\varepsilon$  determined as  $S_3(\ell)/\ell$  with  $\ell$  in the inertial range).

A convenient error function characterizing the collapse is the sum of the square of the differences between values for  $\mu$  obtained from (23) for all pairs of  $\ell$  contained in the inertial range. The sum is carried out for a set of values of  $h$ . To perform the minimization, we have used Powell's method [8]. The starting point for Powell's method is obtained by estimating the intersection points corresponding to the straight lines in Figures 2a and 2b. The result of this global procedure are presented in Figures 3a and 3b. The values obtained for  $\mu$  and  $\zeta_p$  are in good agreement with the one previously derived. The best value of  $\ell_\Lambda$  is found to be of order  $10 \ell_I$ . As a bonus, we have obtained a good representation of the c.d.f.'s corresponding to inertial scales, in terms of a single function  $\mu_h$  and the parameters  $\ell_\Lambda$ ,  $P_\Lambda$  only (see Fig. 4).

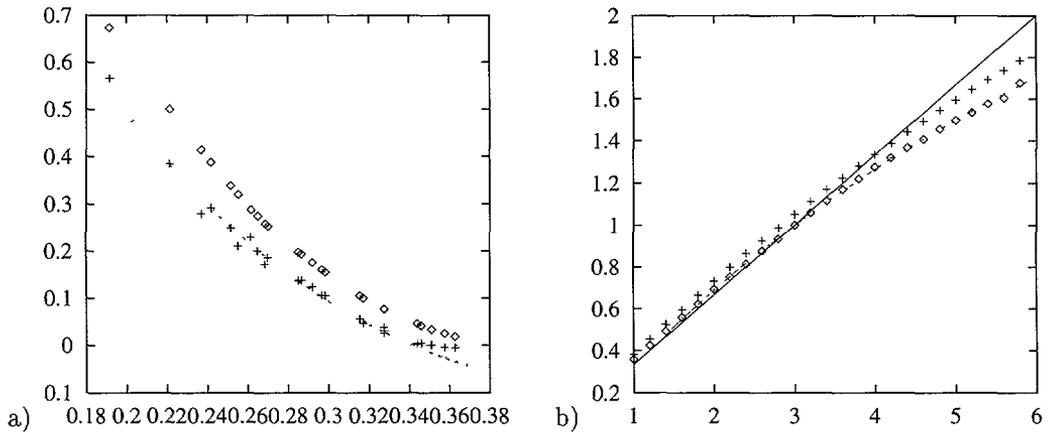


Fig. 3. — a)  $\mu(h)$  determined by global collapse (23), the dots (.) correspond to 8 c.d.f.'s with  $\ell$  in the inertial range,  $\mu_h$  determined by (20) without the  $l_{1/2}$  term ( $\diamond$ ), with  $l_{1/2}$  (+). b)  $\zeta_p$  determined by the Legendre transformation of the global collapse of  $\mu(h)$  (see Fig. 3a), the dots (.) correspond to 8 c.d.f.'s with  $\ell$  in the inertial range,  $\zeta_p$  determined by (21) without the  $l_{1/2}$  term (+), with  $l_{1/2}$  ( $\diamond$ ).

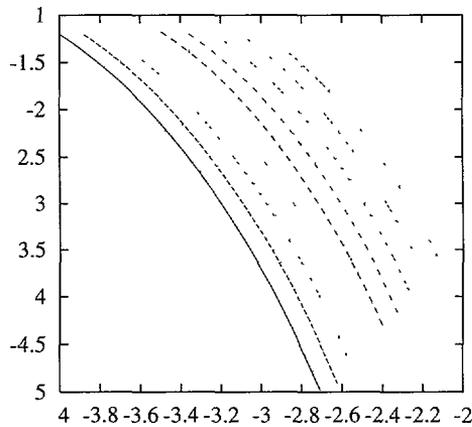


Fig. 4. — Global representation of the c.d.f. in inertial range. The dots correspond to c.d.f. determined by (22), the lines correspond to the experimental c.d.f.

Note that (22) is reminiscent of the asymptotic expression for the probability in large deviation theory [1]. Thus the c.d.f. for the velocity increment behaves like the c.d.f. of a random cascade model [1] after  $N = -\log(\ell/\ell_\Lambda)$  steps. Clearly there is no relevant dynamics going on between  $\ell_\Lambda$  and  $\ell_I$ , the former is only a geometric scale, indicating the origin of the global scaling law (22), valid only when  $\ell \ll \ell_I$ . In conclusion, let us remark that the scale  $\ell_\Lambda$  gives a quantitative geometrical measure of the degree of multifractality of data. Indeed for monofractal data, the slopes of the straight lines in Figure 2a would be independent of the order of the moment  $p$  and  $\ell_\Lambda$  would be infinite. A determination of  $\ell_\Lambda$  at increasingly large Reynolds numbers for a turbulent flow with fixed geometry, such as the one studied in [9,10], could be carried out with the methods developed in this letter. It would be interesting to know the behavior of  $\ell_\Lambda$  determined in this way at very large Reynolds number. Finally the present

method could be used to make a connection between the small scale structures seen in the numerical simulations and multifractality.

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