



Scaling transformation and probability distributions for financial time series

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Accepted 13 August 1999

Communicated by G. Nicolis

Abstract

The price of financial assets are, since [Bachelier L. *Annales de l'Ecole Normale Supérieure* 1900;3:XVII:21–86], considered to be described by a (discrete or continuous) time sequence of random variables, i.e., a stochastic process. Sharp scaling exponents or unifractal behavior of such processes has been reported in several works [Mandelbrot BB. *J Business* 1963;36:394–419; Peters EE. *Chaos and order in the capital markets*. New York: Wiley, 1991; Mantegna RN, Stanley HE. *Nature* 1995;376:46–49; Evertsz CJG. *Fractals*. 1995;3:609–616; Bouchaud JP, Potters M. *Théorie des risques financiers*. Aléa Saclay, 1997]. In this paper we investigate the question of scaling transformation of price processes by establishing a new connection between non-linear group theoretical methods and multifractal methods developed in mathematical physics. Using two sets of financial chronological time series, we show that the scaling transformation is a non-linear group action on the moments of the price increments. Its linear part has a spectral decomposition that puts in evidence a multifractal behavior of the price increments. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

One of the pillars of modern physics is the covariance of theories under certain group actions. What is particular to a given application, such as initial and boundary conditions usually breaks the symmetries of the theory. The symmetry group of observed data is therefore usually much smaller than the covariance group of the theory. An example is hydrodynamics where the equations are invariant under space–time translation and scaling, but where the solutions are not, in general, invariant. For theories that are covariant under scaling (to be specific, we can think of Navier–Stokes or Korteweg–De Vries equation) the situation is clear: the scaling properties, such as the spectral decomposition of the solution at each time are given by the scaling properties of the initial condition, the boundary conditions and external forces. The study of scaling transformation properties in the domain of economics and finance is more complicated because the evolution equations (or even the theory) governing the dynamics are largely unknown. It is thus not possible, in this case, to separate these scaling properties into a general property of an underlying theory and into what is particular to the situation under study. The observed financial chronological data result, at least to some extent, from the particularities of each market and not only from a general abstract dynamics. Therefore, there is no a priori reason to expect the data to exhibit simple properties under scaling transformation. Keeping this simple observation in mind, we will base our analysis of the scaling transformation on methods adapted to physical systems with complex behavior:

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- (i) Multifractal analysis of fully turbulent systems introduced in [2] and within that approach further developed inversion techniques developed in [3] (see also [4]).
- (ii) Non-linear group representation theory developed in [5] and applied to many non-linear evolution equations in mathematical physics (see [6] for recent contributions).

We apply these methods to two sets of financial chronological series:

1. *Foreign exchange rate DM/\$*: The data set provided by Olsen and Associates contains worldwide 1 472 241 bid–ask quotes for US dollar–German mark exchanges rates from 1 October 1992 until 30 September 1993. Tick by tick data are irregularly spaced in recorded time. To obtain price values at a regular time, we use linear interpolation between the two recorded time that immediately precede and follow the regular time. We obtain in this way, for a regular time of 15 s, 1 059 648 data. Our study focuses on the average price which is the mean of the bid and ask price.

2. *Stock index CAC 40*: The data set provided by the “Société de Bourse Française” contains 1 045 890 quotes of the CAC 40 index from 3 January 1993 until 31 December 1996. Tick by tick data are regularly recorded every 30 s, during opening hours (everyday from 10 a.m. until 5 p.m. except weekends and national holidays). Our data base consist of the daily registers to which a constant has been subtracted such that the value at 10 a.m. is equal to the value of the previous day at 5 p.m. The subtracted jump process (with jumps at fixed times) can be analyzed on its own. This separation allows for a finer analysis of the rest of the process.

Using these data sets, we obtained three new results. First, the scaling transformation of the moments of the observed probability distribution is a non-linear representation that is well approximated by a linear representation for small scaling parameters. This linear representation turns out to be diagonal. Secondly, the function of the order of the moment, defined by the spectrum of the generator is (non-trivially) concave. This shows, by definition [2], that the data are multifractal. Note that the concavity in the case of FX market (DM/\$) can partially be deduced from [7] and is confirmed, independently of our work, by Fisher et al. [8]. Our third new result is an explicit expression of the family of probability distributions of price increments corresponding to different time increments.

For larger values of the scaling parameter, the linear approximation breaks down and the non-linear terms of the representation has to be considered. The analysis of this paper can also be applied to the SP 500 index, where the results should be compared to the (unifractal) scaling behavior found in [1]. It should also be compared with, from the point of view of finance, more fundamental approach of stochastic time transformation (subordinate processes) that were applied to SP500 [9,10]. These points are left for future investigations.

2. The mathematical framework

We suppose that the financial variable is described by a stochastic process $(u(t))_{t \geq 0}$ such that the set of increments, $u(t + \tau) - u(t)$, $\tau \geq 0$, has a well-defined transformation property under scaling of the time increment τ , $\tau \mapsto a\tau$, $a > 0$. To avoid complications, irrelevant for the quite crude application reported in this paper, we suppose that $(u(t))_{t \geq 0}$ is stationary. Moreover, we will only consider the absolute value $|u(t + \tau) - u(t)|$ of increments. Let $w(\tau) = |u(\tau) - u(0)|$, $\tau \geq 0$. This means that for each (scaling) $a > 0$, there is a map T_a of the set $\mathcal{W} = \{w(\tau) \mid \tau \geq 0\}$ such that $T_a(w(\tau)) = w(a\tau)$. A group action T of the scaling (dilatation) group \mathbb{D} (the set of strictly positive real numbers) on the set \mathcal{W} is then defined, i.e., $T_{ab}(x) = T_a(T_b(x))$ and $T_e(x) = x$ for $a, b \in \mathbb{D}$, $x \in \mathcal{W}$, where $e = 1$ is the identity element in \mathbb{D} . In the cases under consideration in this paper, it follows from the observed time series that the estimated probability distribution $p_{w(\tau)}$ in \mathbb{R} of $w(\tau)$ is different for different $\tau > 0$. This is enough to ensure the existence of the action T , and moreover shows that T gives a group action \bar{T} , on the set $\mathcal{M} = \{p_{w(\tau)} \mid \tau \geq 0\}$ of probability distributions, defined by $\bar{T}_a(p_{w(\tau)}) = p_{w(a\tau)}$.

The group action \bar{T} is not linear, in spite of its appearance. To explicit properties of the scaling action \bar{T} , we change the coordinates of the elements in \mathcal{M} . As in the case of fully developed turbulence, we use the moments as coordinates. For $q \in \mathcal{M}$, let the moment vector be the sequence $S(q) = (S_r(q))_{r \geq 0}$, where $S_r(q) = \int_0^\infty x^r q(x) dx$ and $r \in \mathbb{R}^+$. Here we suppose that the set \mathcal{M} of probability measures is such that $S_r(q)$ exists for all orders r and moreover that q is determined by its moments of order $r \in \mathbb{R}^+$ (which is the case if

for example the Fourier transform of elements in \mathcal{M} are quasi-analytic). Let \mathcal{S} be the image (in the space $C(\mathbb{R}^+)$ of continuous real functions on \mathbb{R}^+) of \mathcal{M} under the coordinate transformation S . The image U of the group action \bar{T} is given by $U_a = S \circ \bar{T}_a \circ S^{-1}$, i.e., $U_a(S(p_{w(\tau)})) = S(p_{w(a\tau)})$. In the case U is a linear diagonal representation, it has the form $U_a = U_a^{(1)}$, where for given real numbers ζ_r with $r \in \mathbb{R}^+$,

$$U_a^{(1)}(m) = (a^{\zeta_r} m_r)_{r \geq 0} \tag{1}$$

for $a \in \mathbb{D}$ and $m \in C(\mathbb{R}^+)$, m_r corresponding to a moment of order r . We note that $\{\zeta_r \mid r \in \mathbb{R}^+\}$ is the spectrum of the generator of the representation $U^{(1)}$. When U is a non-linear perturbation of $U^{(1)}$, there are algorithms permitting its construction. However, they are outside the scope of this paper [5,6]. For commodity we denote $s_r(\tau) = S_r(p_{w(\tau)})$ which is the r th component of $U_\tau(S(p_{w(1)}))$. An accurate and explicit approximation of the inverse transformation S^{-1} , of the moment vectors $s_r(\tau)$ to probability distribution $p_{w(\tau)}$ has been developed in [3,4,11]. This permits us to obtain directly from experimental data, an explicit formula for the family $\mathcal{M} = \{p_{w(\tau)}\}_{\tau > 0}$ of probabilities. In fact, for each $\tau \in \mathbb{R}^+$, we can determine an element $p_{w(\tau)}$ by the formulas:

$$xp_{w(\tau)}(x) = \bar{p}(\ln x), \quad \alpha(r, \tau) = \frac{d \ln s_r(\tau)}{dr} \tag{2}$$

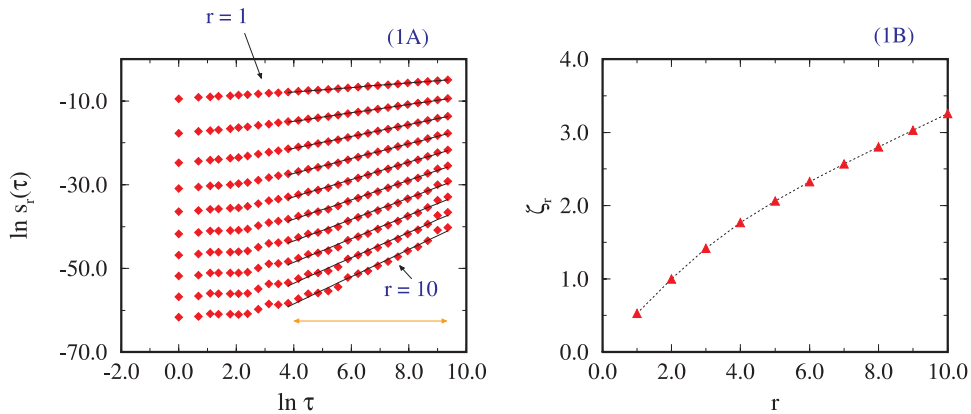


Fig. 1. (A) $\ln s_r(\tau)$ vs $\ln \tau$ for $r = 1, \dots, 10$ in the case of FX DM/\$ index. The scaling law $\ln s_r(\tau) = A_r + \zeta_r \ln \tau$ displayed as straight lines for $11 \leq \tau \leq 2896$ min (delimited by the double-ended arrow) gives ζ_r as the slope. (B) ζ_r determined by scaling law in the case of FX DM/\$ index.

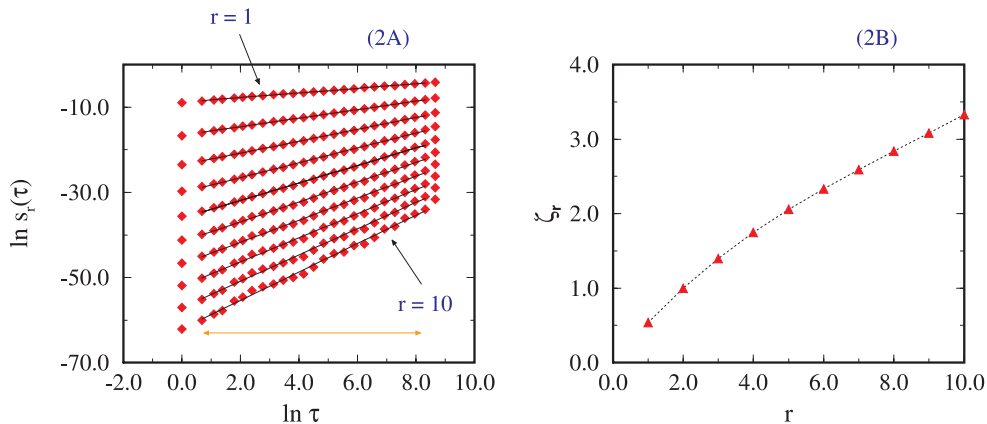


Fig. 2. (A) $\ln s_r(\tau)$ vs $\ln \tau$ for $r = 1, \dots, 10$ in the case of CAC40 index. The scaling law $\ln s_r(\tau) = A_r + \zeta_r \ln \tau$ displayed as straight lines for $1 \leq \tau \leq 2048$ min (delimited by the double-ended arrow) gives ζ_r as the slope. (B) ζ_r determined by scaling law in the case of CAC40 index.

$$\ln \bar{p}(\alpha(r, \tau)) = \ln s_r(\tau) - r \frac{d \ln s_r(\tau)}{dr} - \frac{1}{2} \ln(2\pi) - \frac{1}{2} \frac{d^2 \ln s_r(\tau)}{dr^2}, \tag{3}$$

where $r \in \mathbb{R}^+$.

3. Results

When the representation U is linear it follows from expression (1) that $\ln s_r(\tau) = A_r + \zeta_r \ln \tau$, where A_r and ζ_r are independent of τ . Fig. 1(A) shows that, in the case of FX DM/\$, this is satisfied, to a good approximation with time increments τ and moments of order r in the interval $11 \leq \tau \leq 2896$ min and $1 \leq r \leq 10$. In contrast, for CAC40 the domain of validity of the linear approximation also contains the

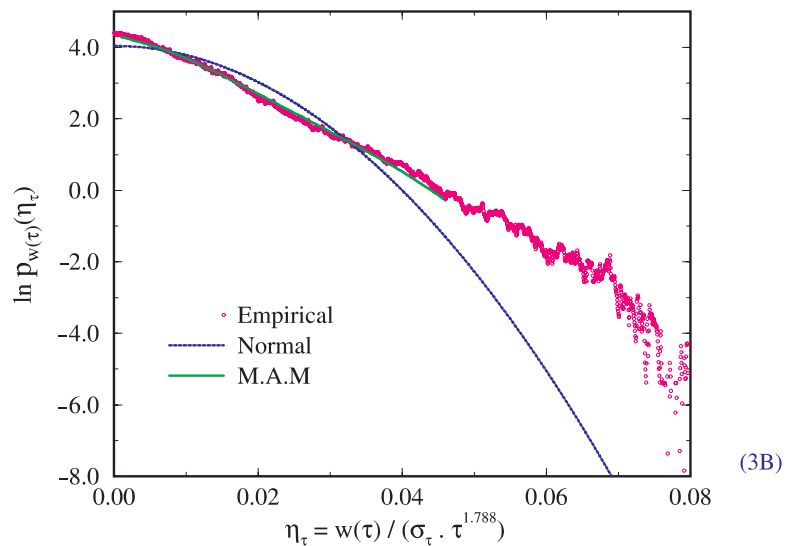
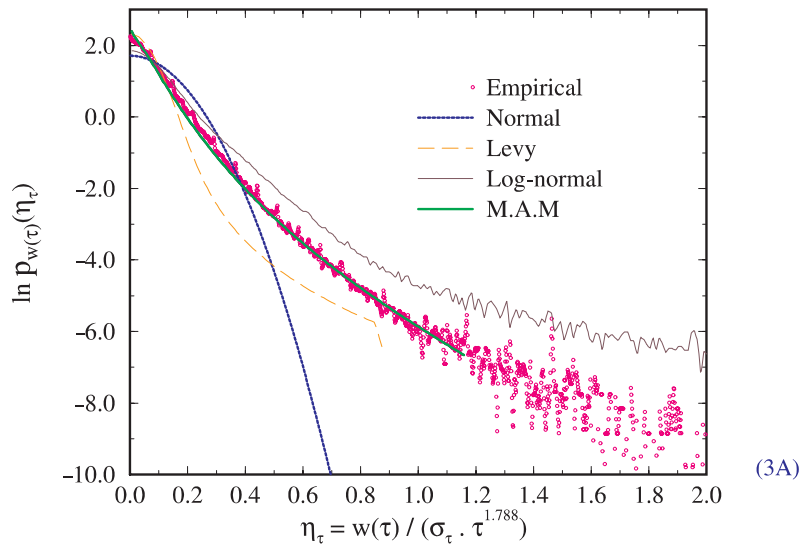


Fig. 3. (A) Presentation of the probability density function at $\tau = 8$ min for FX DM/\$ index and comparison with empirical data. (B) Same presentation at $\tau = 512$ min for FX DM/\$ index.

small values of τ : $1 \leq \tau \leq 2048$ min and $1 \leq r \leq 10$ (see Fig. 2(A)). Outside this domain in the (r, τ) plane, the linear representation approximation breaks down. Inside the domain of validity of the linear representation approximation, the spectrum of the generator is presented in Fig. 1(B) (resp., Fig. 2(B)) in the case of FX DM/\$ (resp., CAC40). The function $r \mapsto \zeta_r$ is in both cases (non-trivially) concave, which by definition (see [2]) shows that the system has a multifractal behavior.

Finally, we present in Fig. 3(A) and (B) (resp., Fig. 4(A) and (B)) probability densities (M.A.M) given by (2) and (3), for $\tau = 8$ min and $\tau = 512$ min in the case of FX DM/\$ index (resp., CAC40). In all the cases, the experimental probability distribution is well approximated, for a large range of price increments, by the corresponding probability distributions in the family $\{p_{w(\tau)}\}_{\tau>0}$ constructed by the inverse method developed in [4,11,12]. Other commonly used probability distributions are also presented in the figures for illustration.

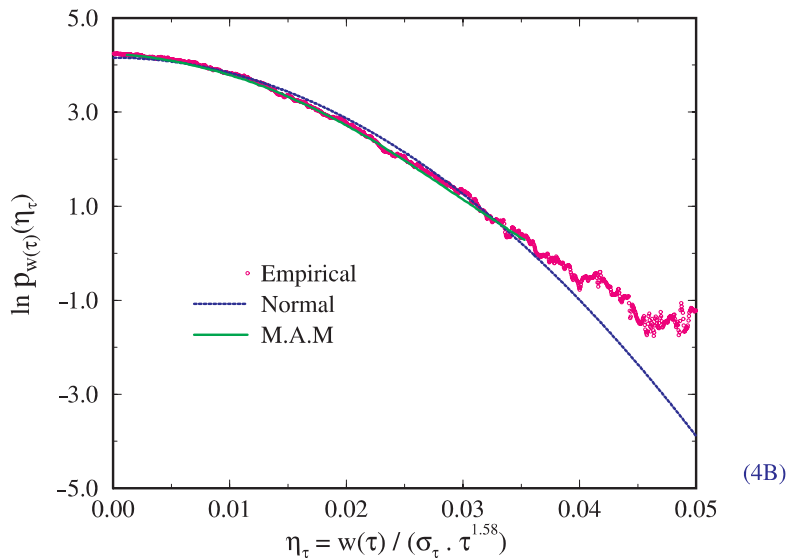
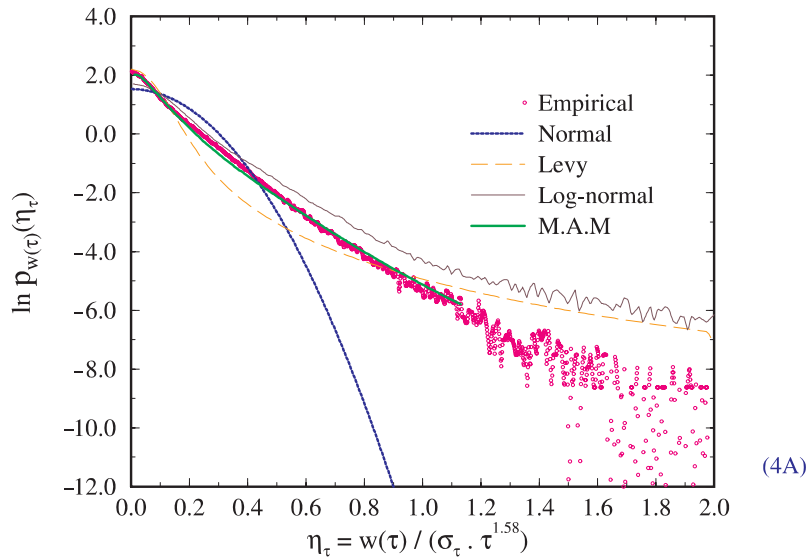


Fig. 4. (A) Presentation of the probability density function at $\tau = 8$ min for CAC40 index and comparison with empirical data. (B) Same presentation at $\tau = 512$ min for CAC40 index.

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