



ELSEVIER

Physica D 108 (1997) 135–146

PHYSICA D

Non-linear acoustics in Galilean and relativistic barotropic fluids

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Received 31 March 1995; accepted 3 February 1997

Communicated by S. Fauve

Abstract

Non-linear waves described by the defocusing non-linear Schroedinger (NLS) equation admit a hydrodynamical representation in terms of Galilean potential flows and, using this correspondence, an autonomous equation for potential flow's non-linear acoustic has been recently derived by Nore et al. However, this equation does not contain simple solutions of the original one such as (dark) solitons. The purpose of the present article is to characterize the reasons behind this failure and to present an original method to build separate equations describing all different types of acoustic solutions (but one).

For reasons of generality, we work in a framework adapted to special relativistic hydrodynamics. All the results we derive have Galilean counterparts which are also discussed. In particular, we argue that there exist an infinity of different acoustic sectors for relativistic barotropic fluids, and we prove this result for fluids with a particularly simple equation of state. Solitons are naturally captured by our approach and a few explicit examples are worked out. Conserved quantities for the acoustic regime are also derived.

1. Introduction

Much attention has been devoted to non-linear waves, both from a mathematical and a physical point of view, with applications ranging from optics to superconductivity and hydrodynamics (see, for example, [1] and references therein). Recently, Nore et al. [2], using the fluid dynamical representation of the (defocusing) NLS equation given by Madelung's transform, established a second-order scalar wave equation to describe non-linear acoustics (up to a certain order). Equations of this type are very useful to understand non-linear phenomena such as renormalization of pulse velocity. However, their equation does not do justice to the wealth a different non-linear acoustic phenomena contained in the original NLS equation. In particular, the NLS dark-solitons, in the acoustic limit, cannot be obtained from the equation presented in [2]. The reason for this is that the authors of [2] actually considered just a single 'long wavelength' scaling, where the wave vector scales as the density inhomogeneities, whereas, there actually exist an infinity of non-equivalent scalings which correspond to what we call different acoustic sectors. The acoustic

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solitons of the NLS equation just happen to belong to another acoustic sector than the one presented in [2]. Independently, Debbasch and Brachet [3] presented a relativistic generalization of the usual Galilean Madelung's transform, with which it is possible to derive an 'honest to God' relativistic fluid dynamical representation of the non-linear Klein–Gordon (NLKG) equation. This relativistic generalization permits a de facto generalization of the different Galilean results pertaining to non-linear acoustics to the relativistic case. It turns out that the methods developed to obtain these results can actually be applied more generally to barotropic fluids, both in the Galilean and relativistic regime, provided that we restrict our attention to potential flows. Here a barotropic fluid will be defined as one in which the local thermodynamical state can be specified with a single thermodynamical variable. This type of idealized fluid naturally enters the description of isentropic flows for which the entropy remains constant in the space–time region occupied by the fluid [4].

This article is organized as follows: In Section 2, we present some basic results on potential flows of relativistic barotropes. We first generalize to arbitrary equations of state the results obtained in [3] for Bose-condensates at vanishing temperature and then, we introduce a general framework suitable for discussing properly wave phenomena. In Sections 3 and 4, we address our main topic: barotropes a priori possess an infinity of different acoustic sectors and there exist a simple variational method through which one can obtain, for all sectors but one, a scalar non-linear wave equation which describes the acoustic at any (given) order. In Section 5, two particular examples are worked out in relative detail to show how the method can be practically implemented on the special case of semi-classical relativistic superfluids at $T = 0$ K, for which any flow is automatically potential [3]. The Galilean limit is also investigated naturally, to provide a link with [2]. Section 6 is devoted to the direct obtention of correct expressions for the different conserved quantities in the acoustic regime. Some possible extensions of this work are finally discussed in the conclusion. Appendix A provides the reader with some exact solutions of the NLKG equation which belong to various sectors discussed in the article.

Notations. In this article, greek and latin tensorial indices will, respectively, run from 0 to 3 and from 1 to 3. The position of a point in space–time will be denoted by $x = (x^\mu) = (ct, \mathbf{x})$, where c is the light velocity and the signature of the Minkowski metric η is conventionally chosen to be negative, so that $\eta = \text{diag}(1, -1, -1, -1)$. Moreover, the partial derivative with respect to x^μ (resp. x_μ) will be usually indicated by using μ as subscript (resp. superscript) to the quantity which is derived and \square will stand for the usual d'Alembertian operator defined by $\square = \partial_\mu \partial^\mu$. Finally, ∇ and Δ represent, respectively, the 3d gradient and Laplacian operators.

2. Fundamentals on potential flows of special relativistic barotropes

In order to support the existence of interesting non-linear acoustic phenomena such as solitons, the barotropes we study have to include some dispersive terms in their dynamics. For simplicity reasons, we will restrict this discussion ab initio to the case where these dispersive terms are identical to those present in the (relativistic) flows of Bose-condensate at $T = 0$ K. The extension of the results obtained in this article to the case of more general dispersion terms is relatively straightforward and will be presented elsewhere. Potential flows of special relativistic barotropes with the same type of dispersion as a Bose-condensate at vanishing temperature can be adequately described by a single complex scalar field $\Psi(x^\mu)$, the dynamics of which derives from a generalized non-linear Klein–Gordon (GNLKG) Lagrangian density L :

$$L(\Psi_\mu^*, \Psi_\mu, \Psi^*, \Psi) = \Psi_\mu^* \Psi^\mu - m^2 c^2 |\Psi|^2 - f(|\Psi|^2). \quad (2.1)$$

In the preceding equation, m represents the mass of the (real) particles out of which the fluid is made and the scalar function f specifies the equation of state of the barotrope. For a Bose-condensate at $T = 0$ K, f may be supposed

to be essentially a quadratic polynomial in $|\Psi|^2$ such as the one given in (2.22). Introducing the squared modulus ρ and the phase θ of Ψ , L reads

$$L(\rho_\mu, \theta_\mu, \rho, \theta) = \rho \theta_\mu \theta^\mu - m^2 c^2 \rho - f(\rho) + \frac{1}{4} \frac{\rho_\mu \rho^\mu}{\rho}. \quad (2.2)$$

The last term in (2.2) is clearly responsible for the dispersive nature of the fluid. Moreover, the 4-velocity u of the barotrope and its scalar density n can be defined by:

$$u_\mu = -\theta_\mu (\theta_\alpha \theta^\alpha)^{-1/2} \quad (2.3)$$

and

$$n = \frac{\rho}{m} (\theta_\alpha \theta^\alpha)^{1/2} \quad (2.4)$$

so that the conserved current j associated to the $U(1)$ (or phase)-invariance of L takes the correct hydrodynamical form

$$j = nu. \quad (2.5)$$

Defining then the enthalpy density w by

$$w = \frac{\rho}{m} \theta_\alpha \theta^\alpha, \quad (2.6)$$

one finds the usual relation for relativistic potential flows:

$$\theta_\mu = -H u_\mu, \quad (2.7)$$

where H is the enthalpy per particle,

$$H = w/n. \quad (2.8)$$

From (2.4) and (2.6), one has

$$\rho = m \frac{n^2}{w}. \quad (2.9)$$

If one wants to eliminate the gradients of θ in the expression for n and w , one can use the equation of motion for ρ derived from L :

$$\frac{1}{2} \partial_\mu \left(\frac{\rho^\mu}{\rho} \right) = \theta_\mu \theta^\mu - m^2 c^2 - \frac{df}{d\rho} - \frac{1}{4} \frac{\rho_\mu \rho^\mu}{\rho^2} \quad (2.10)$$

to obtain

$$H^2 = m^2 c^2 + \frac{df}{d\rho} + \frac{(\sqrt{\rho})_\mu^\mu}{\sqrt{\rho}}. \quad (2.11)$$

This equation, complemented by the two immediate relations

$$n = \rho H / m \quad (2.12)$$

and

$$w = nH = \rho H^2 / m, \quad (2.13)$$

directly gives n and w as functionals of ρ and its derivatives:

$$n(\rho, \rho_\mu, \rho_\nu) = \frac{\rho}{m} \left(m^2 c^2 + \frac{df}{d\rho} + \frac{(\sqrt{\rho})_\mu^\mu}{\sqrt{\rho}} \right)^{1/2}, \quad (2.14)$$

$$w(\rho, \rho_\mu, \rho_\nu) = \frac{\rho}{m} \left(m^2 c^2 + \frac{df}{d\rho} + \frac{(\sqrt{\rho})_\mu^\mu}{\sqrt{\rho}} \right). \quad (2.15)$$

It is important to realize at this point that, in the relativistic regime, ρ cannot generally be eliminated from these expressions in order to obtain w (or any other thermodynamical quantity) as a functional of the density n and its derivatives, simply because (2.14) contains dispersive terms in the form of derivatives of ρ . This means that the natural thermodynamical variable is ρ and not n . In particular, one cannot a priori obtain the pressure p by simply integrating the usual differential relation

$$dp = n d(w/n) = n dH, \quad (2.16)$$

since, basically, (2.16) derives from the fact that H is the Legendre transform, with respect to n , of the internal energy per particle $U = \varepsilon/n$, considered as a function of n - not ρ - and, possibly, some other variables. This practically means that, from the point of view of thermodynamics, a definition of the pressure through w and n is somewhat arbitrary for dispersive barotropes. If, however, dispersive terms are omitted, (2.14) can be solved (at least locally) in ρ and a usual thermodynamical structure can then be recovered from (2.16). Implementing this idea for a non-dispersive barotrope, one gets

$$p(\rho) = \int n(\rho) d(H(\rho)) = \int \frac{\rho H}{m} \frac{dH}{d\rho} d\rho = \int \frac{\rho}{2m} \frac{d(H^2)}{d\rho} d\rho. \quad (2.17)$$

If there is no dispersion, (2.11) becomes

$$H^2 = m^2 c^2 + \frac{df}{d\rho} \quad (2.18)$$

and, integrating (2.17) by parts, we obtain

$$p(\rho) = \frac{1}{2m} \left(\rho \frac{df}{d\rho} - f(\rho) \right). \quad (2.19)$$

The pressure can then be written as a function of n by solving for ρ the non-dispersive form of Eq. (2.14):

$$n(\rho) = \frac{\rho}{m} \left(m^2 c^2 + \frac{df}{d\rho} \right)^{1/2} \quad (2.20)$$

and by inserting the result in (2.19).

For simplicity reasons, we will retain (2.19) as the definition of the pressure in the dispersive case. This is the choice that we already made in studying superflows of a Bose-condensate at $T = 0$ K [3]; we will also retain the usual definition for the internal energy density ε :

$$\varepsilon = w - p, \quad (2.21)$$

keeping in mind that, if there is dispersion, $U = \varepsilon/n$ cannot be interpreted as Legendre transform of H with respect to the pressure p defined by (2.19).

Let us conclude this section by observing that, for a semi-classical Bose-condensate at vanishing temperature [3], Ψ can be interpreted as the common wave function of the bosons in the condensate and obeys the usual NLKG equation; the function f , after convenient normalization, can therefore be supposed to read [3]

$$f(\rho) = \rho(\rho - 2) \quad (2.22)$$

so that the equation of state is then simply

$$p = \frac{m}{2} \left(\frac{n^2}{w} \right)^2. \quad (2.23)$$

A sometimes different discussion of the preceding identifications and some details about the way to obtain from them the correct Galilean limit is given in [3] for the case of the Bose-condensate. A complete presentation of the Galilean limit on the more general case corresponding to an arbitrary equation of state would follow the developments in [3] very closely and should not be repeated here. Let us just note that, in order to obtain directly the Galilean dynamics from the one derived from L , one has to introduce a new complex field Φ related to Ψ by

$$\Phi = e^{imc^2t} \Psi = \sqrt{\rho} e^{i\phi}, \quad (2.24)$$

and having written the equations of motion in terms of ρ and ϕ , let c tend to infinity. It then turns out that, in this limit, the particle density n^G , defined as the limit of n/c [3], is always identical to ρ , even if the dispersive terms in L are taken into account. This means that the enthalpy density w can then be written as a function of n^G and its derivatives. As a matter of fact, the limit expression of $H = w/(n/c) = Hc$ reads

$$H \approx mc^2 - \frac{1}{2m} \frac{\Delta n^G}{\sqrt{n^G}} + \frac{1}{2m} \frac{df}{dn^G}. \quad (2.25)$$

The first term in the preceding expression represents the rest mass energy per unit volume, the second one – the only one which involves derivatives of n^G – traces back to the dispersive effects in the fluid while the third one is a function of the particle density only and can be interpreted as the Galilean enthalpy per particle H^G . Moreover, p^G , the Galilean limit of the pressure p , can be obtained directly from $w^G = n^G H^G$ by using (2.16). As for the Galilean limit of $U = Uc$, it also involves three different contributions. The first two are identical with the first two terms appearing in the Galilean equivalent of H and the third one, U^G , is simply related to H^G and p^G by the usual formula

$$\varepsilon^G = n^G U^G = w^G - p^G. \quad (2.26)$$

We therefore recover, in the Galilean limit, a usual thermodynamical structure in terms of ε^G , w^G , n^G and p^G . In particular, H^G is then the Legendre transform of U^G with respect to the particle density.

3. Identification of the acoustic sectors

Let us suppose f admits an extremum for a certain non-vanishing value ρ_{\min} of ρ ; it is then easy to verify that the field Ψ_{eq} , defined by

$$\Psi_{\text{eq}} = \sqrt{\rho_{\min}} e^{-imc^2t}, \quad (3.1)$$

identically satisfies the equations of motion derived from L . The corresponding density n_{eq} and 4-velocity u_{eq} are then given by (2.3) and (2.4):

$$n_{\text{eq}} = c\rho_{\min}, \quad (3.2)$$

$$u_{\text{eq}} = (1, \mathbf{0}) \quad (3.3)$$

so that this state corresponds to a fluid at rest with uniform particle density. Let us now investigate small perturbations around this unperturbed equilibrium state. Following the traditional hydrodynamical language [4], we shall say that

such a perturbation (n, u) is an acoustic one if $n - n_{\text{eq}}$, $w - w_{\text{eq}}$ and $\mathbf{u} - \mathbf{u}_{\text{eq}}$ are infinitesimal quantities of the same order, conventionally chosen to be the first (order). Eqs. (2.3), (2.4), (2.9) and (2.23) then imply that, for an acoustic perturbation, $\delta = \rho - \rho_{\text{min}}$ and the four derivatives ϕ_μ are also to be considered of the first order. Terms of order higher than one in δ represent non-linear acoustic effects. There is another small parameter in the problem: the ratio ξ/λ , of the coherence-length, at whose scale dispersive effects become dominant, to the characteristic length associated to the (spatial) variations of the acoustic solution. We will consider particular asymptotics in which both δ and ξ/λ are small and are related by

$$\xi/\lambda = \delta^{1-\alpha}, \quad (3.4)$$

where α is a real positive parameter inferior to one. This scaling implies that the variations of ϕ are of order α and ∂_μ is of order $1 - \alpha$, which ensures the correct scaling for ϕ_μ . The GNKKG equation and its Galilean counterpart, the GNLS equation, should therefore admit an infinity of acoustic sectors, each one being characterized by a value of the parameter α . Some explicit solutions of the NLKG equation in the acoustic regime are given in Appendix A and prove the existence of such sectors, at least for values of α superior or equal to $\frac{1}{2}$. Up to now, the only scalings that have been studied for the NLKG equation seem to be the one corresponding to $\alpha = 0$, with the further limitation that the work has only been done in the Galilean limit [2], and the one corresponding to $\alpha = 1$, in the linear regime only [3]. However, these scalings are definitely not the only interesting ones; as a matter of fact, for f given by (2.22), $\rho_{\text{min}} = 1$ and the acoustic soliton described in Appendix A belongs to the sector associated to $\alpha = \frac{1}{2}$.

4. General presentation of the scheme

Let us choose a value for α and investigate the corresponding acoustic sector by a perturbation calculation, pushed up to order $\beta \geq \alpha$. Since the procedure that is to be described now can only be successfully implemented if $\alpha < 1$, we will suppose this condition to be realized in what follows and discuss the case $\alpha = 1$ only briefly at the end of this Section 5.1.

In terms of δ and ϕ , the Lagrangian density reads:

$$L(\delta_\mu, \phi_\mu, \delta, \phi) = \frac{1}{4(\delta + \rho_{\text{min}})} \delta_\mu \delta^\mu + (\delta + \rho_{\text{min}}) \phi_\mu \phi^\mu - 2m(\delta + \rho_{\text{min}}) \phi_t - f(\delta + \rho_{\text{min}}), \quad (4.1)$$

from which we obtain the following Lagrange equations:

$$\frac{1}{2(\delta + \rho_{\text{min}})} \square \delta - \frac{1}{4(\delta + \rho_{\text{min}})^2} \delta_\mu \delta^\mu - \phi_\mu \phi^\mu + 2m \phi_t + f(\delta + \rho_{\text{min}}) = 0 \quad (4.2)$$

and

$$\frac{1}{(\delta + \rho_{\text{min}})} [\delta_\mu \phi^\mu - m \delta_t] + \square \phi = 0. \quad (4.3)$$

The basic idea behind the proposed scheme is to solve perturbatively the equation of motion of δ , namely Eq. (4.2), in order to obtain, at order $(\beta + \alpha)$, an expression of this quantity as a function of the derivatives of ϕ . This can always be done, as long as $\alpha < 1$. The obtained expression can then be used in (4.1) to obtain, at the same order, an effective lagrangian density which will be a functional of the derivatives of ϕ only. This will automatically furnish a non-linear wave equation for ϕ (only) at order β and the associated conserved currents at order $(\beta + \alpha)$. There is naturally a price to pay for the elimination of δ : the final equation for ϕ will turn out to be of the fourth order in time while the original equations (4.2) and (4.3) are only of the second order.

We will now illustrate briefly how this method works if the dynamics of the fluid corresponds to the usual NLKG dynamics, with f given by (2.22); this will in particular apply if we are interested in acoustic waves propagating in a semi-classical Bose-condensate with equation of state (2.23). We will first address the important case $\alpha = \frac{1}{2}$, which is then the correct scaling for acoustic solitons; the case $\alpha = 0$ will also be rapidly reviewed, first because it is one of the only two scalings that seems to have been considered systematically until now (albeit in the Galilean limit only) and because it will also give us the opportunity to compare the implementation of the proposed method for two different scalings.

5. Two examples

5.1. Case $\alpha = \frac{1}{2}$

If we suppose (2.22), (4.2) particularizes to

$$\frac{1}{2(\delta+1)}\square\delta - \frac{1}{4(\delta+1)^2}\delta_\mu\delta^\mu - \phi_\mu\phi^\mu + 2m\phi_t + 2\delta = 0. \quad (5.1)$$

Let us now find the non-linear wave equation corresponding to the acoustic sector $\alpha = \frac{1}{2}$ up to terms of order $\beta = \frac{5}{2}$. Eq. (5.1) reads

$$\delta = -\frac{1}{4}(1-\delta)\square\delta + \frac{1}{8}\delta_\mu\delta^\mu + \frac{1}{2}\phi_\mu\phi^\mu - m\phi_t + o(\delta^3). \quad (5.2)$$

This in turn implies that

$$\square\delta = -\frac{1}{4}\square^2\delta + \frac{1}{2}\square(\phi_\mu\phi^\mu) - m\square\phi_t + o(\delta^3), \quad (5.3)$$

which gives

$$\square^2\delta = -m\square^2\phi_t + o(\delta^3). \quad (5.4)$$

Inserting this result in (5.3), we get

$$\square\delta = \frac{1}{4}m\square^2\phi_t + \frac{1}{2}\square(\phi_\mu\phi^\mu) - m\square\phi_t + o(\delta^3). \quad (5.5)$$

Similarly, (5.2) implies that

$$\delta_\mu\delta^\mu = m^2\phi_{t\mu}\phi_t^\mu + o(\delta^3) \quad (5.6)$$

so that

$$\delta(1 + \frac{1}{4}m\square\phi_t) = -\frac{1}{4}[\frac{1}{4}m\square^2\phi_t + \frac{1}{2}\square(\phi_\mu\phi^\mu) - m\square\phi_t] + \frac{1}{8}m^2\phi_{t\mu}\phi_t^\mu + \frac{1}{2}\phi_\mu\phi^\mu - m\phi_t + o(\delta^3), \quad (5.7)$$

which finally gives

$$\delta = -\frac{1}{4}[\frac{1}{4}m\square^2\phi_t + \frac{1}{2}\square(\phi_\mu\phi^\mu) - m\square\phi_t] + \frac{1}{8}m^2\phi_{t\mu}\phi_t^\mu + \frac{1}{2}\phi_\mu\phi^\mu - m\phi_t + \frac{1}{4}m^2\phi_t\square\phi_t + o(\delta^3), \quad (5.8)$$

or, reordering the terms by increasing order,

$$\delta = -m\phi_t + \frac{1}{4}m\square\phi_t + \frac{1}{2}\phi_\mu\phi^\mu - \frac{1}{8}[\frac{1}{2}m\square^2\phi_t + \square(\phi_\mu\phi^\mu)] + \frac{1}{8}m^2\phi_{t\mu}\phi_t^\mu + \frac{1}{4}m^2\phi_t\square\phi_t + o(\delta^3). \quad (5.9)$$

From the preceding expression, we easily deduce

$$\delta^2 = m^2\phi_t^2 - m\phi_t(\frac{1}{2}\square\phi_t + \phi_\mu\phi^\mu) + o(\delta^3) \quad (5.10)$$

and, using (4.1), (5.6), (5.9) and (5.10), we obtain the following expression for the effective lagrangian density L , which only involves the derivatives of ϕ :

$$L(\phi_{t\mu}, \phi_\mu, \phi) = -2m\phi_t + \phi_\mu\phi^\mu + m^2\phi_t^2 - m\phi_t\phi_\mu\phi^\mu + \frac{1}{4}m^2\phi_{t\mu}\phi_t^\mu + o(\delta^3). \quad (5.11)$$

The extremization of the action associated to L gives the equation of motion of ϕ in the form [5]

$$\partial_\mu \left(\frac{\partial L}{\partial \phi_\mu} \right) - \partial_{t\mu} \left(\frac{\partial L}{\partial \phi_{t\mu}} \right) = 0. \quad (5.12)$$

Expliciting the partial derivatives, we obtain

$$\left(m^2 + \frac{1}{c^2} \right) \phi_{tt} - \Delta\phi + \frac{m^2}{4} \Delta\phi_{tt} - \frac{m^2}{4c^2} \phi_{tttt} + 2m\nabla\phi \cdot \nabla\phi_t + m\phi_t \Delta\phi - \frac{3m}{c^2} \phi_t\phi_{tt} = 0. \quad (5.13)$$

This is the desired non-linear wave equation which governs the behaviour of the phase ϕ in the scaling $\alpha = \frac{1}{2}$, up to terms of order $\frac{5}{2}$. At the same order, ρ is related to ϕ through δ by (5.9). Moreover, because of the procedure used to derive it, (5.13) is automatically covariant (at this order), though not manifestly covariant since ϕ is not a Lorenz scalar.

Let us stress at this point that the corner stone of the method lies in solving (4.2) for δ at the desired order. To accomplish this, we have to first obtain an expression for the first and second derivatives of this quantity in terms of the derivatives of ϕ only, at the right order. This can only be done by successive differentiations of (4.2), under the condition that these differentiations indeed increase the order of the different terms in (4.2). This means that our method only works if the order of ∂_μ is strictly positive, i.e. $\alpha < 1$.

5.2. Case $\alpha = 0$

If one repeats the same procedure for this other scaling, pushing the perturbation calculations up to terms of the fourth order, one easily finds that

$$L(\phi_{t\mu}, \phi_\mu, \phi) = -2m\phi_t + \phi_\mu\phi^\mu + m^2\phi_t^2 - m\phi_t\phi_\mu\phi^\mu + \frac{1}{4}m^2\phi_{t\mu}\phi_t^\mu + \frac{1}{4}(\phi_\mu\phi^\mu)^2 + o(\delta^4), \quad (5.14)$$

and the equation of motion for ϕ is therefore

$$\begin{aligned} \left(m^2 + \frac{1}{c^2} \right) \phi_{tt} - \Delta\phi + \frac{m^2}{4} \Delta\phi_{tt} - \frac{m^2}{4c^2} \phi_{tttt} + 2m\nabla\phi \cdot \nabla\phi_t + m\phi_t \Delta\phi - \frac{3m}{c^2} \phi_t\phi_{tt} \\ + \frac{3}{2c^4} \phi_t^2\phi_{tt} + \frac{3}{2}(\nabla\phi)^2\Delta\phi - \frac{(\nabla\phi)^2\phi_{tt}}{2c^2} - \frac{\phi_t^2\Delta\phi}{2c^2} - \frac{2\phi_t}{c^2} \nabla\phi \cdot \nabla\phi_t = 0. \end{aligned} \quad (5.15)$$

Letting c tend to infinity and noticing that, in this limit, (5.15) implies that $\Delta^2\phi$ and $\Delta\phi_{tt}$ are identical up to terms of order 5, we recover the equation which was proposed by Nore et al. [2] to describe, at the fourth order, the acoustic sector of the NLS equation corresponding to $\alpha = 0$.

It is important to realize that, although the first terms in (5.13) and (5.15) are apparently identical, they do not have the same meaning and relative importance in both equations. For example $\nabla\phi \cdot \nabla\phi_t$ and $\Delta\phi_{tt}$ are both of order $\frac{5}{2}$ in (5.13) while, in (5.15), their order are, respectively, 3 and 4.

6. Conserved quantities

6.1. Particle number conservation

As we already pointed out, this conservation law is well known to be associated to the phase invariance of the Lagrangian density $L(\Psi^{*\mu}, \Psi_\mu, \Psi^*, \Psi)$ (Eq. (2.1)) [3,5,6] or, equivalently, to the invariance with respect to translation in ϕ space of $L(\delta_\mu, \phi_\mu, \delta, \phi)$ (Eq. (4.1)). Naturally, the effective Lagrangian density L will also display the same invariance and one can deduce from it the corresponding conserved current directly. Let us work this out on the two examples proposed in the preceding section.

In both cases, the Lagrangian density L depends on the first derivatives of ϕ and of ϕ_t only and the equation of motion takes the form (5.12); this makes it clear that the conserved current $j = (j^0, \mathbf{j})$ we are searching for is given by

$$j^\mu = -\frac{1}{2} \left[\frac{\partial L}{\partial \phi_\mu} - \partial_t \left(\frac{\partial L}{\partial \phi_{t\mu}} \right) \right], \quad (6.1)$$

the factor $-\frac{1}{2}$ being there to ensure the correct normalization of j . For $\alpha = \frac{1}{2}$, we easily obtain from (6.1):

$$\mathbf{j} = -\nabla\phi + m\phi_t\nabla\phi + \frac{1}{4}m^2\nabla\phi_{tt} + o(\delta^2), \quad (6.2)$$

$$j^0 = mc - \phi_t(1 + m^2c) + 3\frac{m}{c}\phi_t^2 - \frac{m^2}{2c}\phi_{ttt} + o(\delta^2). \quad (6.3)$$

and similarly, for $\alpha = 0$,

$$\mathbf{j} = -\nabla\phi + m\phi_t\nabla\phi + \frac{m^2}{4}\nabla\phi_{tt} + \left(\frac{\phi_t^2}{c^2} - (\nabla\phi)^2 \right) \nabla\phi + o(\delta^3), \quad (6.4)$$

$$j^0 = mc - \phi_t(1 + m^2c) + 3\frac{m}{c}\phi_t^2 - \frac{m^2}{2c}\phi_{ttt} + \left(\frac{\phi_t^2}{c^2} - (\nabla\phi)^2 \right) \frac{\phi_t}{c} + o(\delta^3) \quad (6.5)$$

6.2. Stress–energy–momentum tensor

Since the effective Lagrangian density L has the same space–time symmetries as the original Lagrangian L , one can also deduce directly from it the (canonical) stress–energy–momentum tensor. However, the calculation is not completely straightforward because L depends on some second derivatives of ϕ .

Suppose that we are given a Lagrangian density Λ which is a function of a field A and of its first and second derivatives. Let us define the two conjugate momenta p and π by

$$p_\mu = \frac{\partial \Lambda}{\partial A^\mu} \quad (6.6)$$

and

$$\pi_{\mu\nu} = \frac{\partial \Lambda}{\partial A^{\mu\nu}}. \quad (6.7)$$

A direct calculation shows that, up to a multiplicative constant, the canonical stress–energy–momentum tensor T is given by

$$T_{\mu\nu} = p_\mu A_\nu + \pi_{\mu\alpha} A_\nu^\alpha - (\partial^\alpha \pi_{\mu\alpha}) A_\nu - \Lambda \eta_{\mu\nu}. \quad (6.8)$$

Following [3], we choose $1/2m$ as normalizing factor and obtain, if $\alpha = \frac{1}{2}$,

$$T_{\mu\nu} = (1 - m\phi_t) \frac{\phi_\mu \phi_\nu}{m} - \frac{m}{4} \phi_{t\mu} \phi_\nu - c\phi_\nu \left(1 + \frac{\phi_\alpha \phi^\alpha}{2} \right) \eta_{\mu 0} + \frac{mc}{4} \phi_{t\alpha} \phi_\nu^\alpha \eta_{\mu 0} - \frac{L}{2m} \eta_{\mu\nu} + o(\delta^3), \quad (6.9)$$

with L given by (5.11).

Similarly, if $\alpha = 0$, we get from (6.8):

$$T_{\mu\nu} = \left(1 - m\phi_t + \frac{\phi_\alpha \phi^\alpha}{2} \right) \frac{\phi_\mu \phi_\nu}{m} - \frac{m}{4} \phi_{t\mu} \phi_\nu - c\phi_\nu \left(1 + \frac{\phi_\alpha \phi^\alpha}{2} \right) \eta_{\mu 0} + \frac{mc}{4} \phi_{t\alpha} \phi_\nu^\alpha \eta_{\mu 0} - \frac{L}{2m} \eta_{\mu\nu} + o(\delta^4) \quad (6.10)$$

the correct expression for L being now (5.14)

Let us remark that this canonical stress–energy–momentum tensor is evidently not symmetrical; it should therefore be conveniently symmetrized if one wishes to obtain from it the angular-momentum density; we will not delve into more technical details about this point here.

7. Conclusion

We have argued that the GNLKG equation and its Galilean equivalent, the GNLS equation, interpreted as dynamical equations for potential flows of barotropes, admit an infinity of acoustic sectors and we have developed a variational scheme to study all of them but one analytically. The way to obtain the usual conserved currents has also been discussed and the Galilean limit has been investigated to provide a link with previous works. Moreover, we have also constructed explicitly exact solutions of the NLKG equation which, in the acoustic limit, belong to various of these different acoustic sectors. The natural extension of this work to rotational flows of barotropic and non-barotropic fluids is currently underway. An interesting context in which the methods developed in the present article may prove to be useful is wave turbulence and especially the study of its asymptotic time–behaviour [7].

Appendix A

In this section, we will show that any acoustic sector of the NLKG equation characterized by an α superior or equal to $\frac{1}{2}$ is not empty; the proof will be obtained by exhibiting exact acoustic solitons which belong to any such sector.

In terms of the function Φ defined by (2.24), the equation of motion derived from (2.1) and (2.22) reads

$$\frac{\Phi_{tt}}{c^2} - 2im\Phi_t = \Delta\Phi - 2\Phi(|\Phi|^2 - 1). \quad (A.1)$$

We will search for travelling wave solutions of (A.1) of the form:

$$\Phi(x, t) = u(x - Vt) \exp(i\kappa t) = (\rho(X))^{1/2} \exp[i(\lambda(X) + \kappa t)], \quad (A.2)$$

where x is one of the three cartesian space-coordinates and $X = x - Vt$.

Inserting the ansatz (A.2) in (A.1), one gets two separate (real) equations:

$$V\rho_X \left(1 - \frac{\kappa}{mc^2} \right) = \frac{1}{m\Gamma^2} (\rho\lambda_X)_X, \quad (A.3)$$

$$\begin{aligned} & \frac{1}{c^2} [V^2(\sqrt{\rho})_{XX} - \sqrt{\rho}(\kappa - V\lambda_X)^2] + 2m\sqrt{\rho}(\kappa - V\lambda_X) \\ & = (\sqrt{\rho})_{XX} - \sqrt{\rho}\lambda_X^2 - 2\sqrt{\rho}(\rho - 1) \end{aligned} \tag{A.4}$$

with, as usual,

$$\Gamma = (1 - V^2/c^2)^{-1/2}. \tag{A.5}$$

Eq. (A.3) can be directly integrated into

$$\lambda_X = A + B/\rho, \tag{A.6}$$

where

$$A = mV\Gamma^2(1 - \kappa/mc^2) \tag{A.7}$$

and B is an arbitrary integration constant. For the equation of state (2.22), the equilibrium state of the fluid corresponds to $\rho = 1$ and, naturally, $\lambda_X = 0$. This fixes the value of B to $B = -A$.

Using (A.6) and (A.7), one can then integrate (A.4) and obtain

$$\frac{1}{4\Gamma^2}\rho_X^2 = F(\rho), \tag{A.8}$$

$$F(\rho) = \rho^3 - \rho^2 \left[2 + m^2V^2 \left(1 - \frac{\kappa}{mc^2} \right)^2 + 2m\kappa \left(1 - \frac{\kappa}{2mc^2} \right) \right] + 2C\rho - m^2V^2 \left(1 - \frac{\kappa}{mc^2} \right)^2, \tag{A.9}$$

where C is an arbitrary (real) integration constant.

It can be shown that [8,9], if F has three distinct zeros ρ_1, ρ_2, ρ_3 , ($\rho_1 > \rho_2 > \rho_3$), then (A.8) has periodic solutions in the form of conoidal waves:

$$\rho(X) = \rho_2 - (\rho_2 - \rho_3)cn^2[\Gamma(\rho_1 - \rho_3)^{1/2}X]. \tag{A.10}$$

In (A.10), the parameter s^2 of the cn function is related to the zeros of F by

$$s^2 = (\rho_2 - \rho_3)/(\rho_1 - \rho_3). \tag{A.11}$$

Because the ρ_i 's are the zeros of F one also has the following relations:

$$\rho_1\rho_2\rho_3 = m^2V^2(1 - \kappa/mc^2)^2, \tag{A.12}$$

$$\rho_1 + \rho_2 + \rho_3 = \rho_1\rho_2\rho_3 + 2 + 2m\kappa(1 - \kappa/2mc^2). \tag{A.13}$$

Let us now choose:

$$\rho_2 = 1, \tag{A.14}$$

$$\rho_3 = 1 - \varepsilon, \tag{A.15}$$

$$\rho_1 = 1 - \varepsilon + \eta, \tag{A.16}$$

$$s^2 = \varepsilon/\eta \tag{A.17}$$

with $\eta = \varepsilon^\beta > \varepsilon > 0$. This choice ensures that both $\delta = (\rho - 1)$ and λ_X are of order ε . For small enough ε and $\eta(m^2c^2 > \varepsilon\eta - \varepsilon^2)$, (A.13) then gives

$$\kappa = mc^2 \left(1 - \sqrt{1 + \frac{\varepsilon^2 - \varepsilon\eta}{m^2c^2}} \right), \tag{A.18}$$

which proves that κ is of order $\varepsilon\eta$. Eq. (A.12) then implies

$$mV = (1 + (\varepsilon^2 - \varepsilon\eta)/m^2c^2)^{-1/2}(1 + \eta - 2\varepsilon - \varepsilon\eta + \varepsilon^2)^{1/2}, \quad (\text{A.19})$$

which proves that V is of order zero in ε and η . Since the phase ϕ of the function Φ verifies:

$$\phi_x = \lambda_X, \quad (\text{A.20})$$

$$\phi_t = -V\lambda_X + \kappa, \quad (\text{A.21})$$

the preceding considerations prove that both ϕ_x and ϕ_t are, as δ , of order ε .

Moreover, a straightforward calculation shows that, for $\varepsilon < 2\beta^{-1}$, δ_X/δ scales as $\eta^{1/2}$. This proves that the solution under consideration belongs, for small enough ε , to the sector characterized by $\alpha = 1 - \frac{1}{2}\beta$. Since η is superior to ε , β is inferior to 1 and α lies therefore between $\frac{1}{2}$ and 1.

Let us finally remark [9] that, if ρ_2 approaches ρ_1 and ε correspondingly approaches η , the soliton given by (A.5), (A.6) and (A.10) becomes the acoustic soliton whose density ρ and phase derivatives are given by

$$\rho(x, t) = 1 - \varepsilon \operatorname{sech}^2(\Gamma\sqrt{\varepsilon}(x - V_t)), \quad (\text{A.22})$$

$$\phi_x = -\phi_t/V = \Gamma^2 mV(1 - 1/\rho) \quad (\text{A.23})$$

with $mV = (1 - \varepsilon)^{1/2}$. This shows that solitons, in the acoustic regime, belong to the sector associated with $\alpha = \frac{1}{2}$.

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