Scattering of First Sound by Superfluid Vortices

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The simplest interaction between sound and quantized superfluid vortices is investigated both analytically in the Born approximation and numerically, using the nonlinear Schrödinger equation. Compared with the scattering by a classical vortex, the quantum result reflects the structure of the vortex core and has an extra phase shift. The order of magnitude of the effects suggests possible experiments in rotating superfluid helium.

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The scattering of phonons by vortex filaments in superfluid helium has been calculated long ago by Pitaevskii, in order to compute the phonon part (dominant below $T=0.5$ K) of the mutual friction force between the normal fluid and the quantum vortices [1]. Pitaevskii's computation is based on a hydrodynamical description of the vortices. Such a classical description does not take into account the density variation in the vortex core. In this Letter, we commute, as suggested by Nozières and Pines [2], the scattering cross section using the nonlinear Schrödinger equation (NLSE) as a semiclassical model of neutral superflows [3]. Two different regimes are contained in NLSE: nonlinear dispersive acoustics and Eulerian dynamics of superfluid vortices with characteristic size, the so-called coherence length, ξ [4]. We derive an explicit formula for the first order acoustic scattering in the Born approximation which we integrate numerically to obtain the scattering cross section. In order to validate the Born approximation, we simulate numerically the response of NLSE vortices to an acoustic excitation using Fourier pseudospectral methods. Similarities to and differences from Pitaevskii's calculation are discussed in the conclusion. We also consider the order of magnitude for the total cross section for the scattering of sound in a rotating bulk of helium and discuss the possibilities of experimentally observing this effect.

The NLSE in 2D is $(\alpha,\beta,\Omega>0)$

$$\frac{\partial\psi}{\partial t}=i(\Omega\psi+\alpha\nabla^2\psi-\beta|\psi|^2\psi), \qquad (1)$$

which admits exact vortex solutions. A vortex at the origin is $\psi_0(\mathbf{x})=D(r)\exp(im\varphi)$, $m=\pm1$, where (r,φ) are polar coordinates. The function $D(r)$ is a function vanishing linearly at the origin, $D''(r)<0$, $D(r)=\sqrt{\Omega/\beta}+O(r^{-2})$ for $r\to\infty$ [5]. The equation for a small perturbation w around a vortex is obtained by putting $\psi=\psi_0+w$. Taking a second time derivative and keeping only linear terms in w, one has $\mathcal{L}w=S(w)$, $\mathcal{L}\equiv\partial_t^2-2\alpha\times\Omega\nabla^2+\alpha^2\nabla^4$, where the source $S(w)$ depends on $\psi_0(\mathbf{x})$. If $G(\mathbf{x},t)$ is the retarded Green function of $\mathcal{L}$, this last equation is equivalent to

$$w(\mathbf{x},t)=w_{\rm in}(\mathbf{x},t)+\int d\mathbf{x}'dt'G(\mathbf{x}-\mathbf{x}',t-t')S(w(\mathbf{x}',t')), \qquad (2)$$

where $w_{\rm in}$ is the value of w for $t\to-\infty$ and satisfies $\mathcal{L}w_{\rm in}=0$. We take

$$w_{\rm in}(\mathbf{x},t)=w_0\{\Omega\exp[i(\mathbf{k}\cdot\mathbf{x}-v_0t)]+(v_0-\Omega-\alpha\mathbf{k}^2)\times\exp[-i(\mathbf{k}\cdot\mathbf{x}-v_0t)]\}, \qquad (3)$$

where w_0 is a small amplitude, $\mathbf{k}=k\hat{\mathbf{x}}$ points in the direction of the x axis, and $v_0=\sqrt{2\alpha\Omega k^2+\alpha^2k^4}$. The form of $w_{\rm in}$ is dictated by the fact that it must also satisfy asymptotically the first order equation derived from (1) through $\psi=\psi_0+w$. The scattered wave $w_{\rm sc}$ in the Born approximation will be $w_{\rm sc}=G*S(w_{\rm in})$ [$*$ is the convolution product in (2)]. After a long calculation, we obtain in the radiation zone

$$w_{\rm sc}(\mathbf{x},t)=\frac{-\pi w_0\exp(i\pi/4)}{2\alpha[\pi k(\Omega^2+v_0^2)|\mathbf{x}|]^{1/2}}f(\theta,k)\{\Omega\exp[i(k|\mathbf{x}|-v_0t)]+i(v_0-\Omega-\alpha\mathbf{k}^2)\exp[-i(k|\mathbf{x}|-v_0t)]\}$$

with

$$f(\theta,k)=\int_0^{+\infty}dr\,rU(r)J_0\left(2kr\left|\sin\frac{\theta}{2}\right|\right), \qquad (4)$$

where θ is the polar angle between $\mathbf{x}$ and $\hat{\mathbf{x}}$ and

$$U(r)=-\alpha\beta(3\beta^2\rho^2+2\beta\Omega\rho+4\alpha\beta k^2\rho) \qquad (5)$$

with $\rho=|\psi_0|^2-\Omega/\beta$ and J_0 is the zero order Bessel function. One can easily estimate the corresponding differ-

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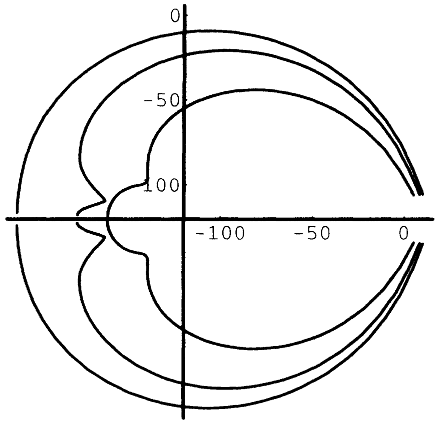

FIG. 1. Plot of the logarithm (in dB) of the differential cross section computed from (7) in polar coordinates for $k = 0.8$, 1.6, and 3.2, with $\xi\sqrt{2}$ as unit of length.

ential scattering cross section

$$\sigma(\theta,k) = \frac{\pi}{4\alpha^2 k(\Omega^2+v_0^2)} f(\theta,k)^2, \tag{6}$$

which gives for the total scattering section

$$S_{\rm tot}(k) = 2\int_0^\pi \sigma(\theta,k)\,d\theta. \tag{7}$$

By scaling appropriately time, space, and amplitude, we can set without loss of generality $\Omega = 1$, $\alpha = \frac{1}{2}$, and $\beta = 1$. This amounts to a choice of units such that the coherence length $\xi = 1/\sqrt{2}$, the sound velocity $c = 1$, and the superfluid condensate density $\rho = 1$.

The scattered wave (4) depends on the vortex density profile through the potential $U(r)$. This profile is obtained numerically by expanding on mapped Chebychev polynomials and minimizing the appropriate free energy. With ten Chebychev polynomials, this method reproduces the results of [6] with 5 digit accuracy. Then (4) is computed directly for r smaller than a cutoff and by replacing $U(r)$ and the Bessel function with their large r expansions above the cutoff. The total cross section (7) is finally computed using the asymptotic form of σ for small values of θ. The values for the cross sections obtained by this procedure are represented in Figs. 1 and 2 for different values of k and θ with $\xi\sqrt{2}$ as unit of length.

To validate the above results, we performed numerical simulations of NLSE using a standard Fourier pseudo-spectral method [7]. To study the scattering of a weak acoustic wave by superfluid vortices in a periodic domain, we have to use for initial data a periodic array of vortices. We have chosen to work with a square lattice of alternate sign vortices (similar to a 2D NaCl crystal) which has the advantages of containing + and − vortices and of a simple preparation by minimization of free energy. The minimum is found by integrating the real Ginzburg-Landau equation (RGLE):

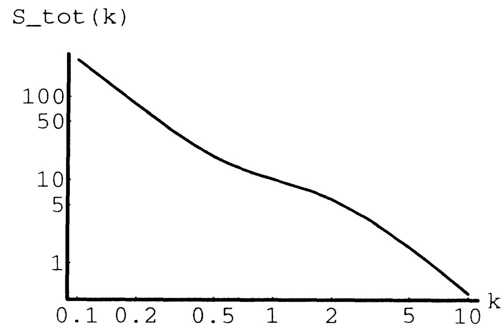

FIG. 2. Plot of the total cross section (8) vs k, with $\xi\sqrt{2}$ as unit of length.

$$\frac{\partial\psi}{\partial t} = \psi + \frac{1}{2}\nabla^2\psi - |\psi|^2\psi. \tag{8}$$

We first initialize a periodic complex field with a system of zeros at the locations of the vortices. We then let the field evolve through RGLE dynamics on a time scale sufficient for amplitude relaxation [4]. The initial data for the scattering computation is the superposition of a weak acoustic wave with the vortex lattice. Figure 3 displays the total amplitude of acoustic waves at $t = 80$ time units. The scattering is mainly forward, which is a characteristic of quadrupolar radiation. The numerical scattered amplitude is fitted with formula (4), thus determining a scattering coefficient $f(\theta,k)$ (see Fig. 4). This coefficient is compared to the Born approximation in Table I. The agreement is seen to be better when the box is larger, that is, when the vortices are more widely separated. The numerical simulations thus validate (4) in this limit.

Note that Pitaevskii's [1] classical cross sections are

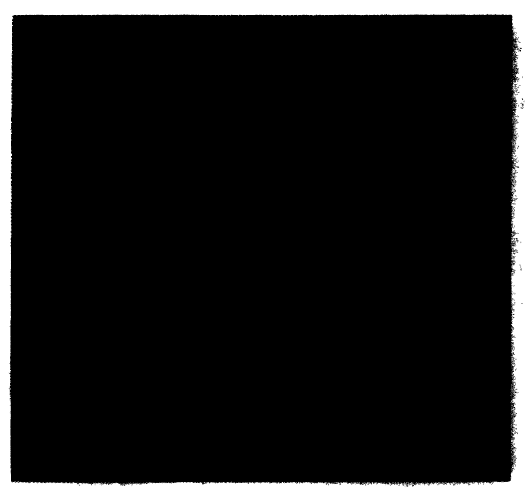

FIG. 3. Density plot of the scattering of an incident plane wave ($k = 1.6$, $w_0 = 0.01$) by direct simulation of NLSE (1). The density is proportional to $|\psi|$.

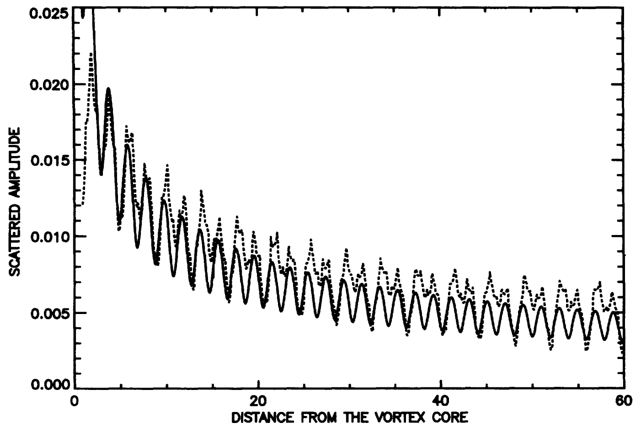

FIG. 4. Fit of the scattered amplitude obtained by numerical simulation (dotted lines) with formula (4) in the Born approximation (solid lines) for $\theta=5°$ and $k=1.6$.

valid in the limit of small k whereas the cross sections obtained from NLSE are valid for k of the order of the inverse core radius. In the latter case, the effects stemming from the density variation in the core region are dominant. The main qualitative difference for the scattering of sound between a classical filament and a topological vortex is the phase dependence of the scattering amplitude. Indeed, in the classical case [1,8], the amplitude is antisymmetric in θ, whereas (4) is symmetric.

The attenuation length λ of first sound corresponding to the scattering of acoustic phonons in a rotating bulk of helium turning at Ω rad s^{-1} is equal to $\lambda=1/2\times10^7\Omega\xi\times\sqrt{2}S_{\rm tot}(k)$. An acoustic wave at $k=0.1$ (in the units of Fig. 2) corresponds to a frequency of 50 GHz with a sound velocity of 300 m s^{-1}. The order of magnitude of λ is found to be 30 cm for $\Omega=10$ rad s^{-1}. This value suggests that it might be possible to detect experimentally the scattering of sound by quantum vortices in a geometry similar to that used in [9]. Such an experiment would give direct access to the density in the vortex core. The phase dependence of the scattering amplitude could be established by interference. However, note that Fetter [10,11] showed that the vortex core in an imperfect Bose gas is not empty, due to many body effects which are not included in the NLSE model of superflows. The differences between the present calculations and future experiments could provide new information needed to understand many body effects on the structure of quantum vortices in real helium II.

In summary, we have computed the cross sections for the scattering of sound by quantum vortices in a semiclassical model that takes into account the density variation in the vortex core.

TABLE I. Comparison of the scattering amplitude obtained from numerical simulations in different periodical boxes with the Born approximation.

θ	2.5°	5°	10°
$L=40\pi$	5.25	5.27	5.19
$L=80\pi$	5.75	5.45	4.8
$L=160\pi$	5.8	5.1	4.7
Formula (4)	6.5	5.35	4

Finally, note that, at a frequency of 50 GHz, the elementary excitations can be described as acoustic phonons. For frequencies an order of magnitude higher, the excitations are in the vicinity of the roton minimum in the dispersion curve. In this regime, a nonlocal generalization of NLSE [12] could be used. However, a classical model [13], ignoring the vortex core, is able to account for the roton part of the mutual friction, with the same approximations as Pitaevskii, in a semiempirical way.

The numerical simulations were performed on the CRAY-2 of the "Centre de Calcul Vectoriel pour la Recherche" de l'Ecole Polytechnique and on local workstations. Laboratoire de Physique Statistique associé au CNRS et aux Universités Paris VI et Paris VII.

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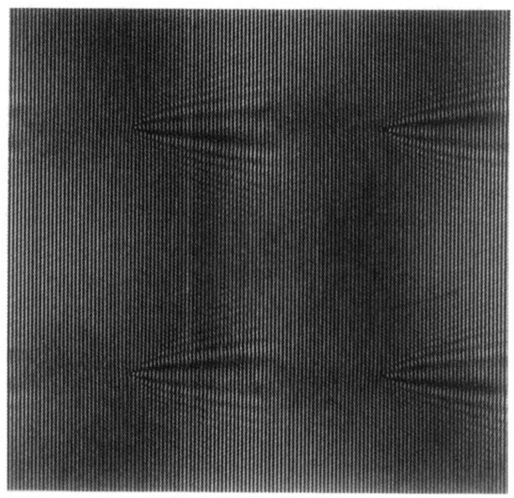

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