A Primer in Classical Turbulence Theory

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Introduction

The aim of this short set of lectures is to make available in a self-contained form the basics of classical turbulence theory. A first section is concerned with orders of magnitude. How can one determine what the size of a parachute on a Martian probe should be? The second section is devoted to exact results. It culminates with the demonstration of the four-fifth law obeyed by the third order structure function. Finally, the last section is about intermittency. The multifractal scaling behavior of the pdf of velocity increments and the associated structure functions is investigated. A new double asymptotic relation between pdf and moments is derived.

1 Orders of magnitude and basic phenomenology

Imagine that you are faced with the following physical situation: a viscous fluid is stirred by a device that drives it with typical speed variations δu_I at length scale ℓ_I.

An important physical parameter of the flow is its Mach number $M_I = u_I/c_{\text{sound}}$, the ratio of the stirring speed to the sound velocity $c_{\text{sound}} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S}$. In these lectures, we will restrict our attention to low-Mach number flows, in other words we will be concerned with sub-sonic turbulence.

Low-Mach number flows are described [1] by the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \Delta \mathbf{u} \\ \text{div } \mathbf{u} = 0 \end{cases} \tag{1}$$

where ρ is the (constant) density of the fluid, p the pressure and ν the kinematic viscosity. The pressure field is determined at all times by div $\mathbf{u} = 0$. Indeed, this condition yields

$$\Delta p = -\rho \text{ div}((\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}) \tag{2}$$

a Poisson equation for p with right hand side computable from $\mathbf{u}$.

An important characteristic of a viscous flow is its Reynolds number

$$R_I = \frac{u_I \ell_I}{\nu}.$$

This number estimates the relative importance of the inertial terms $(\mathbf{u}.\nabla)\mathbf{u}$ with respect to viscous effects $\nu\Delta\mathbf{u}$. Indeed, the first term contains two velocities and one spatial derivative while the second term contains the viscosity, one velocity and two spatial derivatives. The order of magnitude of their ratio is thus (velocity $\times$ length)/viscosity.

We now suppose that the stirring is strong in the sense that $R_I >> 1$. This very large Reynolds number $R_I >> 1$ and small Mach number (in practice $M_I < 1/3$) regime is called “fully developed incompressible turbulence”. Experimentally, such a flow is extremely complex. The aim of the following considerations is to obtain a quantitative understanding of the order of magnitude of the turbulent velocity fluctuations.

1.1 The Richardson cascade

Let us suppose that our system is in a statistically stationary state. The stirring device is communicating energy to the flow. What happens to this energy? Once communicated to the fluid, in the form of kinetic energy $E_{\mathrm{cin}} = \frac{1}{2}\int \rho u^2 d^3\mathbf{x}$, it is conserved by the nonlinear terms of the Navier-Stokes equation. The viscous terms will dissipate it into heat, with power

$$W_d = \frac{\nu}{2}\int d^3\mathbf{x} \sum_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 .$$

However, this dissipation cannot take place at the injection scale as the Reynolds number associated to this scale is very large and thus the dissipation very small.

One is thus led to the physical image of the "Richardson cascade" [2]. The energy injected in the fluid at scale ℓ_I "cascades" down to smaller scales. This process stops when scales ℓ_d small enough for the energy to be dissipated into heat are reached. One can picture this cascade as a succession of eddies instabilities happening at scales $\ell_I > \ell > \ell_d$.

Using the concept of Richardson cascade, it is possible to understand why a modification of viscosity in a turbulent flow has no effect on its overall energy dissipation. This rather surprising fact is well supported experimentally. The point is that when ν is modified, it is the number of steps in the cascade that is changed. The small scales of the flow just adapt themselves in order to dissipate the energy injected at large scale by the stirring.

1.2 Kolmogorov scaling

In 1941, Kolmogorov [3] [4] [5] found quantitative expressions for the intensity of fluid motions at scale ℓ and for the dissipation scale ℓ_d.

Let us first remark that the Navier-Stokes equation is invariant under Galilean transformations: if $u(x,t)$ is a Navier-Stokes solution, then $u(x - u_0 t, t) + u_0$ is also a solution. A constant advection has thus no dynamical effect on the evolution of the flow. Accordingly, we define the intensity of the motions at scale ℓ to be the typical velocity variation over distance ℓ.

Furthermore, at scales much smaller than ℓ_I, the flow can be considered homogeneous. Quantities such as kinetic energy, or dissipated power are extensive. This just means that, if the statistical properties of the velocity $u(x,t)$ are homogeneous, the total kinetic energy E_c and the dissipated power W_d are proportional to the total mass of fluid. We will thus normalize these quantities by the total mass.

We will denote by ε the energy injection rate per unit mass (also equal to the cascade and dissipation rates). The dimension of ε is:

$$[\varepsilon] = W/\mathrm{kg} \ = L^2 T^{-3}.$$

Our turbulence is thus characterized by three parameters: the injection scale ℓ_I, the injection velocity δu_I and the viscosity of the fluid ν. These parameters have dimensions $[\ell_I] = L,\ [\delta u_I] = LT^{-1}$ and $[\nu] = L^2T^{-1}$.

The Richardson cascade leads us to a first hypothesis:

- $H_0 : \varepsilon$ is independent of ν.

The only combination of δu_I and ℓ_I with a correct dimension is $\delta u_I^3/\ell_I$. Thus (the symbol $\sim$ meaning proportional)

$$\varepsilon \sim \frac{\delta u_I^3}{\ell_I}. \tag{3}$$

The scaling law for $\delta u(\ell)$ will be obtained using the following two hypotheses:

- $H_I : \delta u(\ell)$ does not depend on ν (for $\ell_d < \ell < \ell_I$)
- $H_2 : \delta u(\ell)$ is a function of only ε and ℓ.

Dimensional analysis yields

$$\delta u(\ell) \sim (\varepsilon \ell)^{1/3}. \tag{4}$$

Using (3), (4) can be written

$$\delta u(\ell) \sim \delta u_I \left(\frac{\ell}{\ell_I}\right)^{1/3} \tag{5}$$

Let us remark that H_2 amounts to say that $\delta u(\ell)$ is a function of ℓ_I only through ε. Or, in other words, that there is no way by observing at scale ℓ to distinguish between two turbulences having the same ε but driven at two different injection scales.//

We can compute the Reynolds number associated with motions at scale ℓ

$$R_\ell \sim \frac{\delta u(\ell)\ell}{\nu} = \frac{\varepsilon^{1/3}\ell^{4/3}}{\nu}.$$

Viscous dissipation will take place at scales ℓ_d such that $R_{\ell_d} \sim 1$. It follows that

$$\ell_d \sim \nu^{3/4}\varepsilon^{-1/4}. \tag{6}$$

Using (3), (6) can also be written

$$\ell_d \sim \ell_I R_I^{-3/4}. \tag{7}$$

To end this section, let us remark that the scaling laws (3) together with (4) or (5) and (6) or (7) are in good agreement with experiments. They can be used to get order of magnitude estimates for real turbulence, as we now proceed to show.

1.3 Elementary examples

Turbulent dispersion Consider the time evolution of the separation of two particles initially separated by ℓ_0. Kolmogorov's law gives, for $\ell_d \ll \ell \ll \ell_I$,

$$\frac{d\ell}{dt} \sim (\varepsilon\ell)^{1/3}.$$

Integrating this equation, one gets

$$\ell^{2/3} - \ell_0^{2/3} \sim \varepsilon^{1/3} t.$$

Thus the diameter of a suspension of particles evolves as

$$\varepsilon^{1/2} t^{3/2}.$$

It is interesting to note that this law was established 15 years before Kolmogorov's law [6].

Terminal velocity in free fall Consider an object of size ℓ and mass M in free fall (the gravity is g) in a fluid of density ρ and kinematic viscosity ν. The terminal speed is obtained by equaling the weight Mg with the drag. In a laminar regime, the drag force is proportional to speed and viscosity. Dimensional analysis gives: $[\nu] = L^2T^{-1}$, $[F] = MLT^{-2}$, $[\rho] = ML^{-3}$ and thus $F_{\text{visc}} \sim \nu\rho\ell v$. In this laminar regime, one has

$$v_{\text{visc}} \sim \frac{Mg}{\nu\rho\ell}$$

(for a spherical object, there is a factor 6π in the denominator that is beyond dimensional analysis).

In the turbulent regime, the law $\varepsilon \sim \frac{v^3}{\ell}$ tells us that the power is proportional to the cube of velocity. As the power is the product of force by velocity, it follows that the force is proportional to the square of velocity. The dimensionally correct expression is:

$$F_{\text{turb}} \sim \rho\ell^2 v^2.$$

It follows that:

$$v_{\text{turb}} \sim \sqrt{\frac{Mg}{\rho\ell^2}}.$$

The ratio

$$\frac{F_{\text{turb}}}{F_{\text{visc}}} = \frac{\rho\ell^2 v^2}{\nu\rho\ell v} = \frac{\ell v}{\nu}$$

is simply the Reynolds number.

Let us compute the order of magnitude for the terminal speed of a paratrooper. The order of magnitudes are: $\rho_{\text{air}} \sim 1\text{Kg}/m^3$, $M \sim 100$ Kg, $g \sim 10$

m/s^2, $\ell \sim 1$m (without parachute). One finds $v_{\text{turb}} \sim \sqrt{1000} = 33$m/s, or about 120 Km/h. Taking $\ell = 10$ m (after opening the parachute) one finds a speed of 3,3 m/s. These orders of magnitude are correct (the real free fall speed is about twice as big).
The Reynolds number in free fall is about ($\nu_{\text{air}} \sim 10^{-5}$m^2/s)

$$R = \frac{1 \times 33}{10^5} \sim 3.10^6.$$

Note that in these conditions the laminar formula gives a ridiculous result for the terminal speed (of the order of the speed of light).

Parachutes on Mars The previous estimates for free fall terminal speed have practical engineering applications. This is illustrated by the following discussion on the working of the parachutes of NASA's Martian probe Pathfinder. This section was extracted from the JPL Pathfinder site on Internet at http://-mars.sgi.com/mpf/faqs_edl.html#parachute. The answers are by Rob Manning.

Q. If the Mars atmosphere is less than 1% than that of Earth, how can a parachute of the size you are using be sufficient? It would seem to me that you would need a parachute close to 1000' wide to achieve the same effect. Would you please explain the dynamics of placing a lander on Mars, and why a small parachute would work on Mars as it does on Earth?

A. You ask a very insightful question. The bottom line is you're right, parachutes this small **aren't** sufficient on Mars! On Mars Pathfinder, as on Viking, we use a "small" 40.5 ft (12.5 m) chute. It was scaled so that, with our lighter lander, it does about as much for the our descent speed as does Viking's. Our terminal velocity seconds before getting to the ground (where the atmosphere is "thickest") is still about 65 m/s (146 mph)!!

You are correct, it would indeed take a a larger chute to get slower "normal" Earth-like terminal velocities. Our chute on Mars is about the equivalent of a chute 38 times smaller in area on Earth (6.5 ft across!), and this includes the effect of Mars' lower gravity! A chute that could lower our lander to the Martian ground at a gentle 10 m/s (22 mph) would have to have an area about 42 times larger than our "little" chute (or a diameter of 263 ft)! That's 42 times the mass (and volume) of our 10 kg chute, or 420 kg, more than the mass of our entire lander! It wouldn't fit! We would need to have a "gossamer" (ultra-light weight material) parachute and then figure out how to get it open at high speeds!

This is why we turned to solid rockets to stop our lander just before we hit the ground. Viking, too, used liquid rockets to slow the terminal decent. Also Pathfinder's airbags protect the lander from the local terrain variations (bumps, craters, rocks, hills, etc.) after the rockets do their thing.

So why do we do we use a chute at all? Well, parachutes might not be all that good a laying a lander gently down on the Martian surface, but they do a spectacular job of braking something moving very fast. Remember, the drag FORCE a chute generates (therefore its deceleration), is proportional to the square of the velocity and only linearly proportional to the atmospheric density;

so even a thin atmosphere and a "small" chute will do much to slow our entry vehicle down once the heatshield's aerobraking has been mostly achieved.

This is also true of heatshield, our entry vehicle (like Viking's) enters the upper atmosphere at 7 km/s (or more than 15,000 mph!). Most of this is reduced by the friction with the heatshield. But even 2 minutes later, our vehicle is still screaming in at nearly 400 m/s (900 mph) when the parachute opens before slowing down to 65 m/s near the ground. I'd say that reducing our velocity by a factor of 6 (a factor of 36 in kinetic energy), isn't all that bad for only 10 kg of extra payload mass, wouldn't you?

So, the short answer is, you're right, parachutes **don't** work on Mars like they do on Earth (neither do airbags, but that is another story), but they do a great job when you need to slow down something that is whipping through the Martian atmosphere FAST!

2 Exact results

In this section, we will follow the methods and notations of references [1], [7] and [8].

2.1 General framework

A popular experimental setting is to take velocimetry data, using a hot-wire probe, in a turbulent flow with a large mean velocity. This is mostly done in wind tunnels, using grid generated turbulence.

Calling $\mathbf{u}$ the velocity on the probe and $\mathbf{u}_T = \mathbf{u} - \langle \mathbf{u} \rangle$ the turbulent velocity fluctuations. The turbulence rate of such a flow is defined as the ratio of the fluctuations (r.m.s.) to the mean velocity.

$$\tau = \frac{\sqrt{\langle \mathbf{u}_T^2 \rangle}}{|\langle \mathbf{u} \rangle|}.$$

In typical grid turbulence $\tau \sim 10^{-2}$. This small value of τ allows for two simplifications.

First, as the hot-wire probe is sensitive to the modulus of velocity, it is the longitudinal fluctuations that are measured. Indeed, writing $\mathbf{u}_T = \mathbf{u}_{T\parallel} + \mathbf{u}_{T\perp}$ with $\mathbf{u}_{T\perp} \cdot \langle \mathbf{u} \rangle = 0$ one gets

$$(\langle \mathbf{u} \rangle + \mathbf{u}_{T\parallel} + \mathbf{u}_{T\perp})^2 = \langle \mathbf{u} \rangle^2 + 2\langle \mathbf{u} \rangle \mathbf{u}_{T\parallel} + \mathbf{u}_{T\parallel}^2 + \mathbf{u}_{T\perp}^2$$

which shows that the transverse fluctuations have contributions that are second-order in τ and thus negligible.

Second, making use of the Galilean invariance, the temporal velocimetry series, taken at a fixed location, can be translated into spatial measurements. To do that, we make a Galilean transformation to the frame where $\langle \mathbf{u} \rangle = 0$. In this frame, one has

$$\mathbf{u}_{\mathrm{probe}}(t) = \langle \mathbf{u} \rangle + \mathbf{u}_T(-\langle \mathbf{u} \rangle t, t),$$

where the function $\mathbf{u}_T$ is defined in the mobile referential, whose spatial origin is on the probe at $t = 0$. The Taylor hypothesis amounts to consider that, provided that the time interval is not too large, one can consider that the turbulence is temporally frozen. In this way, one can measure the space variations of the turbulent velocity.

The signal is then analyzed in term of its increments. The measured increments $u(t + \delta t) - u(t)$ are considered as longitudinal spatial variations of the velocity

$$\delta u_{\|} = u_{\|}(x + \ell) - u_{\|}(x) \text{ with } \ell = -|\langle \mathbf{u} \rangle| \delta t$$

The results are presented as longitudinal structure functions of order p defined as

$$S_p(\ell) = \langle (\delta u_{\|})^p \rangle.$$

The goal of the rest of this section is to derive exact results related to the second and third order structure functions.

2.2 The Kármán-Howarth-Monin relation

We want to consider stationary turbulence. We thus need some device to inject energy. Mathematically, the simplest method is to add to the Navier-Stokes equations a volume force $\rho \mathbf{f}(x, t)$, that acts only at large scale. We thus consider the system defined by equations

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\rho} \nabla p + \mathbf{f} \\ div\ v = 0 \end{cases} \tag{8}$$

As $\rho \mathbf{f}$ is a volume force, $\mathbf{f}$ is a local acceleration, i.e. $[\mathbf{f}] = LT^{-2}$. We now make three hypotheses:

- H_1 - We suppose that $f(x, t)$ is stationary and homogeneous, which means that its statistical properties are invariant by time and space translation.
- H_2 - We admit that the Navier-Stokes equations have a statistically homogeneous solution (not in general stationary, in order to be able to discuss the decay case $\mathbf{f} = 0$).
- H_3 - We suppose that the quantities that we are about to define and manipulate exist and are finite. Let us remark that we do not, at this stage, impose isotropy.

Defining the velocity increments

$$\delta \mathbf{v}(\mathbf{r}, \boldsymbol{\ell}) = \mathbf{v}(\mathbf{r} + \boldsymbol{\ell}) - \mathbf{v}(\mathbf{r}),$$

we are interested in the quantity

$$\langle |\delta \mathbf{v}^2(\mathbf{r}, \boldsymbol{\ell})| \delta \mathbf{v}(\mathbf{r}, \boldsymbol{\ell}) \rangle$$

where the brackets denote an ensemble average (the average over the realizations of $\mathbf{f}$). Because of homogeneity, this quantity is a function of $\boldsymbol{\ell}$ only and not of $\mathbf{r}$.

Denoting by $v_i, f_i, r', v_i', \partial_i, \partial_i'$ et ∇_ℓ respectively $v_i(r), f_i(r),\ \mathbf{r}+\boldsymbol{\ell},\ v_i(\mathbf{r}+\boldsymbol{\ell}),\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial x_i'}$ and $\frac{\partial}{\partial \ell_i}$ the homogeneity implies for all ensemble average:

$$\partial_i\langle(\bullet)\rangle = -\partial_i'\langle(\bullet)\rangle = -\nabla_{\ell_i}\langle(\bullet)\rangle. \tag{9}$$

Starting from the Navier-Stokes equations (8) one obtains:

$$\begin{aligned}\partial_t \tfrac{1}{2}\langle v_i v_i'\rangle = &-\tfrac{1}{2}\partial_j\langle v_i v_j v_i'\rangle - \tfrac{1}{2}\partial_j'\langle v_i' v_j' v_i\rangle \\ &-\tfrac{1}{2\rho}\langle v_i'\partial_i p\rangle - \tfrac{1}{2\rho}\langle \partial_i' p' v_i\rangle \\ &+\tfrac{1}{2}\langle v_i' f_i\rangle + \tfrac{1}{2}\langle v_i f_i'\rangle \\ &+\tfrac{1}{2}\nu(\partial_{jj}+\partial_{jj}')\langle v_i v_i'\rangle\end{aligned} \tag{10}$$

where we have used the incompressibility to write the inertial terms in the first line. We have also commuted derivations and averages. The terms in the second line are zero, because of incompressibility. The terms involving the force can be grouped, using homogeneity, under the form:

$$\left\langle \mathbf{v}(r)\cdot\frac{\mathbf{f}(\mathbf{r}+\boldsymbol{\ell})+\mathbf{f}(\mathbf{r}-\boldsymbol{\ell})}{2}\right\rangle. \tag{11}$$

In the same way, the viscous term can be written

$$\nu\nabla_{\boldsymbol{\ell}}^2\langle \mathbf{v}(\mathbf{r})\mathbf{v}(\mathbf{r}+\boldsymbol{\ell})\rangle. \tag{12}$$

The inertial terms can be expressed using $\langle|\delta\mathbf{v}|^2\delta v_j\rangle$ as follows:

$$\begin{aligned}\langle|\delta\mathbf{v}|^2\delta v_j\rangle &= \langle(v_i'-v_i)(v_i'-v_i)(v_j'-v_j)\rangle \\ &= -\langle v_i' v_i' v_j\rangle + \langle v_i v_i v_j'\rangle \\ &\quad - 2\langle v_i v_i' v_j'\rangle + 2\langle v_i v_i' v_j\rangle \\ &\quad + \langle v_i' v_i' v_j'\rangle - \langle v_i v_i v_j\rangle\end{aligned} \tag{13}$$

The last two terms simplify, because of homogeneity.

Let us now evaluate $\nabla_{\boldsymbol{\ell}_j}\ \langle|\delta\mathbf{v}|^2\delta v_j\rangle$. Using incompressibility, the first two terms have zero contribution. We thus find that:

$$\nabla_{\boldsymbol{\ell}_j}\langle|\delta\mathbf{v}|^2\delta v_j\rangle = -2\partial_j'\langle v_i v_i' v_j'\rangle - 2\partial_j\langle v_i v_i' v_j\rangle \tag{14}$$

which is four times the inertial terms of (10).

Regrouping (10), (11), (12) and (14) we finally get the Kármán-Howarth-Monin relation:

$$\begin{aligned}\varepsilon(\boldsymbol{\ell}) \equiv &-\tfrac{1}{4}\nabla_{\boldsymbol{\ell}}\cdot\langle|\delta\mathbf{v}(\boldsymbol{\ell})|^2\delta\mathbf{v}(\boldsymbol{\ell})\rangle \\ = &-\partial_t\tfrac{1}{2}\langle \mathbf{v}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}+\boldsymbol{\ell})|\rangle \\ &+ \langle \mathbf{v}(\mathbf{r})\cdot\tfrac{\mathbf{f}(\mathbf{r}+\boldsymbol{\ell})+\mathbf{f}(\mathbf{r}-\boldsymbol{\ell})}{2}\rangle \\ &+ \nu\nabla_\ell^2\langle \mathbf{v}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}+\boldsymbol{\ell})\rangle.\end{aligned} \tag{15}$$

The quantity $\varepsilon(\boldsymbol{\ell})$ that was just defined can be interpreted in the following way.

Starting from the Navier-Stokes equation, one computes $-\partial_t\frac{1}{2}\langle\mathbf{v}(r)\cdot\mathbf{v}(\mathbf{r}+\boldsymbol{\ell})\rangle$. there is a contribution from the force and one from the nonlinear terms. $\varepsilon(\boldsymbol{\ell})$ is the contribution coming from the nonlinear terms.

Energy budget in spectral space Let us take the Fourier transform with respect to $\boldsymbol{\ell}$ of the Kármán-Howarth-Monin relation.

$$\begin{aligned}\frac{1}{(2\pi)^3}\int d^3\ell e^{i\boldsymbol{\ell}\mathbf{k}}\partial_t\frac{1}{2}\langle v(\mathbf{r})\cdot v(\mathbf{r}+\boldsymbol{\ell})\rangle = \frac{1}{(2\pi)^3}\int d^3\ell e^{i\boldsymbol{\ell}\mathbf{k}}\Big[\frac{1}{4}\nabla_{\boldsymbol{\ell}}\ \langle|\delta\mathbf{v}(\boldsymbol{\ell})|^2\delta\mathbf{v}(\ell)\rangle \\ +\nu\nabla^2_{\boldsymbol{\ell}}\langle\mathbf{v}(\mathbf{r})\mathbf{v}(\mathbf{r}+\boldsymbol{\ell})\rangle \\ +\left\langle\mathbf{v}(\mathbf{r})\cdot\frac{\mathbf{f}(\mathbf{r}+\boldsymbol{\ell})+\mathbf{f}(\mathbf{r}-\boldsymbol{\ell})}{2}\right\rangle\Big]\end{aligned} \tag{16}$$

Defining,

$$\begin{aligned}\mathcal{E}(\mathbf{k}) &= \frac{1}{(2\pi)^3}\int d^3\ell e^{i\boldsymbol{\ell}\mathbf{k}}\frac{1}{2}\langle\mathbf{v}(r)\cdot v(\mathbf{r}+\boldsymbol{\ell})\rangle \\ \mathcal{F}(\mathbf{k}) &= \frac{1}{(2\pi)^3}\int d^3\ell e^{i\boldsymbol{\ell}\mathbf{k}}\langle\mathbf{v}(r)\frac{f(\mathbf{r}+\boldsymbol{\ell})+f(\mathbf{r}-\boldsymbol{\ell})}{2}\rangle \\ \mathcal{T}(\mathbf{k}) &= \frac{1}{(2\pi)^3}\int d^3\ell e^{i\boldsymbol{\ell}\mathbf{k}}\frac{1}{4}\nabla_{\boldsymbol{\ell}}\ \langle|\delta\mathbf{v}(\boldsymbol{\ell})|^2\delta\mathbf{v}(\boldsymbol{\ell})\rangle\end{aligned} \tag{17}$$

(16) can be written

$$\frac{\partial}{\partial t}\mathcal{E}(\mathbf{k}) = \mathcal{T}(\mathbf{k}) - 2\nu\mathbf{k}^2\mathcal{E}(\mathbf{k}) + \mathcal{F}(\mathbf{k}) \tag{18}$$

This equation expresses the energy budget in spectral space. Indeed, the definition of $\mathcal{E}(\mathbf{k})$ (17) is the 3D generalization of the Winer-Kitchtine theorem that states that the spectral energy density of a signal is the Fourier transform of its correlation function. Indeed, using the relation

$$\int d^3k e^{ik\ell} = (2\pi)^3\delta^3(\boldsymbol{\ell}) \tag{19}$$

we find that

$$\int d^3k\mathcal{E}(\mathbf{k}) = \frac{1}{2}\langle\mathbf{v}^2\rangle. \tag{20}$$

The relations we have obtained up to now have been derived from the Navier-Stokes equations and homogeneity. They are thus valid in non-isotropic situations. In an isotropic case (for instance in grid turbulence) they can be greatly simplified. In particular, the quantities present in (18) are not in this case functions of the direction of the vector $\mathbf{k}$. In the isotropic case, it is customary to define the angle-averaged densities

$$\begin{aligned}E(k) &= 4\pi k^2\mathcal{E}(\mathbf{k}) \\ F(k) &= 4\pi k^2\mathcal{F}(\mathbf{k}) \\ T(k) &= 4\pi k^2\mathcal{T}(\mathbf{k})\end{aligned} \tag{21}$$

so that one can write, for instance, $\int_0^\infty E(k)dk = \frac{1}{2}\langle\mathbf{v}^2\rangle$. In the isotropic case, (18) reduces to

$$\frac{\partial}{\partial t}E(k) = T(k) - 2\nu k^2E(k) + F(k). \tag{22}$$

The terms in this equations are called the spectrum of respectively, energy transfer, energy dissipation and forcing.

In order to have a quantitative definition of the "Richardson cascade rate" one defines the energy flux by

$$\pi(k) = -\int_0^k dk\ T(k) \tag{23}$$

so that $T(k)$, which is the part stemming from the nonlinearities of the time variation of $E(k)$, can be written as

$$T(k) = -\frac{\partial}{\partial k}\pi(k) \tag{24}$$

Regrouping (17), (18), (21) and (23) one gets the following expression for the energy flux:

$$\pi(k) = \frac{1}{(2\pi)^3}\int_{|\mathbf{k}|<k} d^3k \int d^3\ell e^{i\mathbf{k}\boldsymbol{\ell}} \left[-\frac{1}{4}\nabla_{\boldsymbol{\ell}}\langle|\delta\mathbf{u}(\boldsymbol{\ell})|^2\delta\mathbf{u}(\boldsymbol{\ell})\right] \tag{25}$$

This relation justifies the notation $\varepsilon(\boldsymbol{\ell})$ that was adopted in the preceding section. Let us remark that (25) is also valid in the anisotropic case, if the energy budget is written in the cumulated form

$$\frac{\partial}{\partial t}\int_{|\mathbf{k}|<k} d^3k\mathcal{E}(\mathbf{k}) + \pi(k) = \int_{|\mathbf{k}|<k} d^3k[\mathcal{F}(\mathbf{k}) - 2\nu k^2\mathcal{E}(\mathbf{k})]. \tag{26}$$

The material of the following section will allow us to write (25) in a simpler way.

Isotropic case The general formulae defining the 3D Fourier transform

$$\begin{aligned} f(\boldsymbol{\ell}) &= \int d^3k e^{-i\mathbf{k}\boldsymbol{\ell}}\hat{f}(\mathbf{k}) \\ \hat{f}(\mathbf{k}) &= \tfrac{1}{(2\pi)^3}\int d^3\ell e^{ik\ell} f(\boldsymbol{\ell}) \end{aligned} \tag{27}$$

can be reduced to 1D integrals in the case where the function f depends only of ℓ. Indeed, in this case the definitions (27) give, in spherical coordinates k, θ, φ,

$$\begin{aligned} f(\ell) &= \int_0^\infty dk \int_0^\pi kd\theta \int_0^{2\pi} k\sin\theta d\varphi e^{-ikl\cos\theta}\hat{f}(k) \\ f(\ell) &= \int_0^\infty 4\pi k^2 dk \int_0^\pi \tfrac{d\theta}{2}\sin\theta e^{-ik\ell\cos\theta}\hat{f}(k) \end{aligned} \tag{28}$$

or, doing the θ integral

$$f(\ell) = \int_0^\infty 4\pi k^2 dk \left[\frac{e^{-ik\ell\cos\theta}}{2ik\ell}\right]_0^\pi \hat{f}(k) \tag{29}$$

and thus, finally

$$f(\ell) = \int_0^\infty 4\pi k^2 dk \frac{\sin(k\ell)}{k\ell}\hat{f}(k) \tag{30}$$

the inverse relation is

$$\hat{f}(k) = \frac{1}{(2\pi)^3}\int_0^\infty 4\pi\ell^2 d\ell \frac{\sin(k\ell)}{k\ell} f(\ell) \tag{31}$$

The relations (30) and (31) deduced from (27) can also be written, setting

$$\begin{aligned} F(k) &= 4\pi k^2\hat{f}(k) \\ f(\ell) &= \int_0^\infty dk \tfrac{\sin(k\ell)}{k\ell} F(k) \\ F(k) &= \tfrac{2}{\pi}\int_0^\infty d\ell\ k\ell\ \sin(k\ell) f(\ell) \end{aligned} \tag{32}$$

These relations allow the expression of $\pi(k)$ from (25) under the form

$$\pi(k) = \frac{2}{\pi}\int_0^k dk \int_0^\infty d\ell\ k\ell\ \sin(k\ell)\varepsilon(\ell) \tag{33}$$

or, doing the integral over k

$$\pi(k) = \frac{2}{\pi}\int_0^\infty d\ell\ \frac{\sin(k\ell) - k\ell\cos(k\ell)}{\ell}\ \varepsilon(\ell). \tag{34}$$

This integral can be put under the form

$$\pi(k) = \frac{2}{\pi}\int_0^\infty d\ell \left[\frac{\sin(k\ell)}{\ell} - \frac{d}{d\ell}\sin(k\ell)\right]\varepsilon(\ell), \tag{35}$$

integrating the last term by part, one finally gets the relation

$$\pi(k) = \frac{2}{\pi}\int_0^\infty d\ell \frac{\sin(k\ell)}{\ell}(1 + \ell\partial_\ell)\varepsilon(\ell) \tag{36}$$

2.3 Isotropic structure functions

The Kármán-Howarth-Monin relation involves correlation functions and a combination of derivatives of the third order structure function. In order to obtain relations that can be used in an experimental context, where only longitudinal components can be obtained, one must use isotropy in order to re-express everything in term of those measurable components.
The computation of the third order structure function being rather involved, we begin by computing the second order structure function. These second order results will allow us to define the Taylor scale in terms of measurable quantities.

Second order structure functions If homogeneity and isotropy are assumed a number of simplifications can be obtained. The second order structure function

$$B_{ij}(\boldsymbol{\ell}) = \langle (v'_i - v_i)(v'_j - v_j) \rangle$$

(with the same notations than in section (2.2)) cannot depend, because of isotropy on any special spatial direction. The only vector that can be present in its expression is $\boldsymbol{\ell}$. Calling $\ell^0_i = \frac{\ell_i}{\ell}$ the unitary vector in the direction of $\boldsymbol{\ell}$, the most general form for B_{ij} is

$$B_{ij} = A(\ell)\delta_{ij} + B(\ell)\ell^0_i\ell^0_j. \tag{37}$$

Setting

$$b_{ij} = \langle v_i v'_j \rangle \tag{38}$$

the definition of B_{ij} gives

$$B_{ij} = \langle v_i v_j \rangle + \langle v'_i v'_j \rangle - b_{ij} - b_{ji}. \tag{39}$$

Using the relations

$$\langle v_i v_j \rangle = \langle v'_i v'_j \rangle = \frac{1}{3} \langle v^2 \rangle \delta_{ij} \tag{40}$$

and

$$b_{ij}(\boldsymbol{\ell}) = b_{ji}(-\boldsymbol{\ell}) = b_{ji}(\boldsymbol{\ell}) \tag{41}$$

one gets

$$B_{ij} = \frac{2}{3} \langle \mathbf{v}^2 \rangle \delta_{ij} - 2b_{ij}. \tag{42}$$

On the other hand, incompressibility implies

$$\frac{\partial}{\partial \ell_i} B_{ij} = 0 \tag{43}$$

Using the relations

$$\frac{\partial \ell}{\partial \ell_k} = \ell_k^o \text{ et } \frac{\partial \ell_i^0}{\partial \ell_k} = \frac{1}{\ell}(\delta_{ik} - \ell_i^0 \ell_k^0) \tag{44}$$

or

$$\frac{\partial}{\partial \ell_i} \equiv \ell_i^0 \frac{\partial}{\partial \ell} + \frac{1}{\ell}(\delta_{ik} - \ell_i^0 \ell_k^0) \frac{\partial}{\partial \ell_k^0} \tag{45}$$

(37) gives

$$\ell_j^0 (A'(\ell) + B'(\ell)) + 2\ell_j^0 \frac{B(\ell)}{\ell} = 0 \tag{46}$$

or

$$A'(\ell) + B'(\ell) + \frac{2}{\ell} B(\ell) = 0. \tag{47}$$

hence,

$$B(\ell) + \frac{1}{2} \ell \partial_\ell (A + B) = 0. \tag{48}$$

So that, defining the longitudinal and transverse components as

$$\begin{array}{ll} A + B & = B_{\ell\ell} \\ A & = B_{tt} \end{array} \tag{49}$$

one gets

$$B_{\ell\ell} - B_{tt} + \frac{1}{2} \ell \partial_\ell B_{\ell\ell} = 0 \tag{50}$$

or

$$B_{tt} = (1 + \frac{1}{2} \ell \partial_\ell) B_{\ell\ell} \tag{51}$$

we thus find that, for $\ell \ll \ell d$
$B_{\ell\ell} = a\ell^2$, and

$$B_{tt} = 2a\ell^2 \tag{52}$$

Taylor scale these expressions can be used to relate a to the energy dissipation ε.
The relation $\varepsilon = \langle \frac{1}{2}\nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 \rangle$ gives

$$\varepsilon = \nu \left[\langle \frac{\partial v_i \partial v_i}{\partial x_j \partial x_j}\rangle + \langle \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\rangle\right] \tag{53}$$

and (37) yields

$$\langle v_i v_j' \rangle = \langle v_i v_j \rangle - \frac{1}{2} B_{ij} \tag{54}$$

thus, using (49), (51), and (52) gives

$$\langle v_i v_j' \rangle = \langle v_i v_j \rangle - \left[a\ell^2 \delta_{ij} - \frac{a}{2}\ell_i \ell_j\right] \tag{55}$$

Taking into account the relations

$$\frac{\partial}{\partial x_i'} = \frac{\partial}{\partial \ell} \text{ et } \frac{\partial}{\partial x_i} = -\frac{\partial}{\partial \ell_i} \tag{56}$$

one has

$$\langle \frac{\partial v_i}{\partial x_k}\frac{\partial v_j'}{\partial x_\ell'}\rangle = +\frac{\partial^2}{\partial \ell_k \partial \ell_\ell}[a\ell^2\delta_{ij} - \frac{a}{2}\ell_i\ell_j] \tag{57}$$

or

$$\langle \frac{\partial v_i}{\partial x_k}\frac{\partial v_j'}{\partial x_\ell'}\rangle = a\left[2\delta_{k\ell}\delta_{ij} - \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell}\right] \tag{58}$$

The first term of (53) is obtained in the limit $\ell \to 0$ as $j \to 1$, $\ell \to k$

$$\nu a \sum_{i,k}[2 - \delta_{ik}^2] = \nu a[18 - 3] = 15\nu a \tag{59}$$

The second term is obtained by setting $j \to k,\ \ell \to i$

$$\nu a \sum_{i,k}[2\delta_{ik}^2 - \frac{1}{2}(\delta_{ik}^2 + 1)] = \nu a \sum_{i,k}\left[\frac{3\delta_{ik} - 1}{2}\right] = 0 \tag{60}$$

One thus find

$$\varepsilon = 15\nu a. \tag{61}$$

(52) can thus be written

$$\begin{array}{l} B_{\ell\ell} = \frac{1}{15}\frac{\varepsilon}{\nu}\ell^2 \\ B_{tt} = \frac{2}{15}\frac{\varepsilon}{\nu}\ell^2 \end{array} \tag{62}$$

Defining the Taylor scale λ as

$$B_{\ell\ell} = v_{rms}^2 \frac{\ell^2}{\lambda^2} \tag{63}$$

with

$$v_{rms} = \sqrt{\langle \mathbf{v}^2 \rangle} \tag{64}$$

on gets

$$\varepsilon = 15\nu v_{rms}^2/\lambda^2 \tag{65}$$

or

$$\lambda = \sqrt{\frac{15\nu}{\varepsilon}} v_{rms} \tag{66}$$

In a Kolmogorov regime, one expects

$$\varepsilon \sim u_I^3/\ell_I \quad v_{rms} \sim u_I \tag{67}$$

and

$$\lambda \sim \sqrt{\frac{\nu u_I^2}{u_I^3/\ell_I}} = \ell_I \sqrt{\frac{\nu}{u_I \ell_I}} = \frac{\ell_I}{\sqrt{R_I}} \tag{68}$$

The Taylor scale λ and the velocity u_{rms} can be used to define a Reynolds number based on "internal" scales $R_\lambda = \frac{u_{rms}\lambda}{\nu}$, in a Kolmogorov regime, one has $R_\lambda \sim \sqrt{R_I}$. R_λ is extensively used in the experimental literature.

Third order structure functions We now turn our attention to third order structure functions. To wit, let us consider the quantities (with the same notations than in section (2.2))

$$b_{ij,m} = \langle v_i v_j v'_m \rangle \tag{69}$$

because of isotropy, $b_{ij,m}$ must be a function of δ_{ij}, $\ell_i^0 \equiv \frac{\ell_i}{\ell}$ and of ℓ. Taking into account the i, j symmetry, the most general form for $b_{ij,m}$ is

$$b_{ij,m} = C(\ell)\delta_{ij}\ell_m^0 + D(\ell)(\delta_{im}\ell_j^0 + \delta_{jm}\ell_i^0) + F(\ell)\ell_i^0\ell_j^0\ell_m^0 \tag{70}$$

the incompressibility implies

$$\partial'_m b_{ij,m} = 0 \tag{71}$$

to compute the divergence, let us recall the relations

$$\begin{aligned} \frac{\partial \ell}{\partial \ell_k} &= \ell_k^0 \\ \frac{\partial \ell_i^0}{\partial \ell_k} &= \frac{1}{\ell}(\delta_{ik} - \ell_i^0 \ell_k^0) \end{aligned} \tag{72}$$

which are obtained by computing the derivatives of $\ell = \sqrt{\ell_i^2}$, (72) implies in particular $\frac{\partial \ell_i^0}{\partial \ell_i} = \frac{2}{\ell}$ et $\ell_i^0 \frac{\partial \ell_j^0}{\partial \ell_i} = 0$.

Using these expressions, (71) yields

$$\begin{aligned} C'(\ell)\delta_{ij} &+ \tfrac{2}{\ell}C(\ell)\delta_{ij} \\ &+ 2D'(\ell)\ell_i^0\ell_j^0 + D(\ell)\tfrac{2}{\ell}(\delta_{ij} - \ell_i^0\ell_j^0) \\ &+ F'(\ell)\ell_i^0\ell_j^0 + F(\ell)\left[\tfrac{2}{\ell}\ell_i^0\ell_j^0\right] = 0 \end{aligned} \tag{73}$$

or

$$[C' + \frac{2}{\ell}(C + D)]\delta_{ij} + \left[(2D + F)' + \frac{2(F - D)}{\ell}\right]\ell_i^0\ell_j^0 = 0. \tag{74}$$

These equations can be written under the equivalent form

$$\begin{array}{ll} [\ell^2(3C+2D+F)]' = 0 \text{ (by taking the trace)} \\ C' + \frac{2}{\ell}(C+D) \quad = 0. \end{array} \tag{75}$$

The only solution of the first equation compatible with a finite $b_{ij,m}$ in $\ell = 0$ is:

$$3C + 2D + F = 0 \tag{76}$$

We can thus express D and F in function of C et C', under the form:

$$\begin{array}{l} D = -(C + \frac{\ell C'}{2}) \\ F = \ell C' - C. \end{array} \tag{77}$$

Using these expressions, one gets

$$\begin{array}{ll} b_{ij,m} = C\delta_{ij}\ell^0_m & -(C + \frac{\ell C'}{2})(\delta_{im}\ell^0_j + \delta_{jm}\ell^0_i) \\ & +(\ell C' - C)\ell^0_i\ell^0_j\ell^0_m. \end{array} \tag{78}$$

This expression yields the value of any component of the third order structure function

$$\begin{array}{l} B_{ijm} = \langle (v'_i - v_i)(v'_j - v_j)(v'_m - v_m)\rangle \\ B_{ijm} = 2(b_{ij,m} + b_{jm,i} + b_{mi,j}). \end{array} \tag{79}$$

One finds

$$\begin{array}{l} B_{ijm} = -2(\ell C' + C)(\delta_{ij}\ell^0_m + \delta_{jm}\ell^0_i + \delta_{mi}\ell^0_j) \\ \qquad +6(\ell C' - C)\ell^0_i\ell^0_j\ell^0_m \end{array} \tag{80}$$

Using this general result, we can express the longitudinal third order structure function as

$$S_3(\ell) = \langle (\delta v_\parallel(\boldsymbol{\ell}))\rangle = -6(\ell C' + C) + 6(\ell C' - C) \tag{81}$$

thus

$$S_3(\ell) = -12C. \tag{82}$$

The general result also yields the isotropic expression of

$$\begin{array}{ll} \langle |\delta \mathbf{v}|^2 \delta v_m\rangle = B_{iim} & \\ B_{iim} & = [-10(\ell C' + C) + 6(\ell C' - C)]\ell^0_m \\ B_{iim} & = [-4\ell C' - 16C]\ell^0_m \end{array} \tag{83}$$

thus

$$\begin{array}{l} \varepsilon(\ell) \equiv -\frac{1}{4}\nabla\boldsymbol{\ell}\langle |\delta\mathbf{v}|^2\delta\mathbf{v}\rangle \ = (\frac{2}{\ell} + \partial_\ell)(\ell C' + 4C) \\ \varepsilon(\ell) = \ell C'' + 7C' + \frac{8C}{\ell}. \end{array} \tag{84}$$

This expression is homogeneous to $\frac{C}{\ell}$, it can thus be cast under the form

$$(\ell\partial_\ell + \alpha)(\ell\partial_\ell + \beta)\frac{C(\ell)}{\ell} = \ell C''(\ell) + (\alpha + \beta - 1)C'(\ell) + (\alpha - 1)(\beta - 1)\frac{C(\ell)}{\ell} \tag{85}$$

one finds, by identification

$$\alpha + \beta = 8\ , \ (\alpha - 1)(\beta - 1) = 8 \tag{86}$$

thus

$$\alpha = 3, \beta = 5. \tag{87}$$

$$\varepsilon(\ell) = (3 + \ell\partial_\ell)(5 + \ell\partial_\ell)\frac{C(\ell)}{\ell} \tag{88}$$

Regrouping (82) and (88) one gets an expression that relates $\varepsilon(\ell)$ to the third order longitudinal structure function.

$$\varepsilon(\ell) = -\frac{1}{12}(3 + \ell\partial_\ell)(5 + \ell\partial_\ell)\frac{S_3(\ell)}{\ell} \tag{89}$$

The final expression for the energy flux in terms of the longitudinal third order structure function is obtained by putting (89) into (36), one gets

$$\pi(k) = \left(-\frac{1}{12}\right)\frac{2}{\pi}\int_0^\infty d\ell \frac{\sin(k\ell)}{\ell}(1 + \ell\partial_\ell)(3 + \ell\partial_\ell)(5 + \ell\partial_\ell)\frac{S_3(\ell)}{\ell} \tag{90}$$

This relation, together with (36) is what is needed to establish the four-fifth law, as we will see in details in the next section.

2.4 The four-fifth law

Relation (90) can be used to obtain the $\frac{4}{5}$ law. We need the following three hypotheses:

- H_1 the forcing term is acting only at small k
- H_2 we can take the limit $t \to \infty$, and in this limit (ν fixed) there is a finite dissipation rate per unit mass.
- H_3 we can then take the limit $\nu \to 0$, with a finite dissipation rate.

The energy budget relation (23) and (24)

$$\frac{\partial E(k)}{\partial t} = -\frac{\partial \pi(k)}{\partial k} - 2\nu k^2 E(k) + F(k) \tag{91}$$

give, using H_2

$$0 = -\frac{\partial \pi(k)}{\partial k} - 2\nu k^2 E(k) + F(k). \tag{92}$$

Integrating this relation over k one obtains

$$\varepsilon_{\text{injection}} = \int_0^\infty E(k)dk = \varepsilon_d = 2\nu\int_0^\infty k^2 E(k)dk \tag{93}$$

Using H_1 one gets

$$F(k) = 0 \;\text{ for } k \gg k_I \tag{94}$$

And H_3 gives (putting $\varepsilon = \lim \nu \to 0\ \varepsilon_d$)

$$\pi(k) = \varepsilon \text{ for } k \gg k_I. \tag{95}$$

The expression of $\pi(k)$ (90) is of the form

$$\pi(k) = \frac{2}{\pi}\int_0^\infty d\ell \frac{\sin(k\ell)}{\ell} G(\ell) \tag{96}$$

with

$$G(\ell) = -\frac{1}{12}(1+\ell\partial_\ell)(3+\ell\partial_\ell)(5+\ell\partial_\ell)\frac{S_3(\ell)}{\ell} \tag{97}$$

the large k behavior of $\pi(k)$ is dominated by the small ℓ behavior of $G(\ell)$. Indeed, putting $x = k\ell$ one finds

$$\pi(k) = \frac{2}{\pi}\int_0^\infty dx \frac{\sin x}{x} G\left(\frac{x}{k}\right) \tag{98}$$

and thus

$$\lim_{k\to\infty} \pi(k) = G(0)\frac{2}{\pi}\int_0^\infty dx \frac{\sin x}{x}. \tag{99}$$

the integral $\int_0^\infty dx \frac{\sin x}{x}$ can be computed as

$$\frac{1}{2}\int_{-\infty}^{+\infty} dx \frac{1}{2}\int_{-1}^{+1} dk e^{ikx} = \frac{1}{4}\int_{-1}^{+1} dk 2\pi\delta(k) = \frac{\pi}{2}. \tag{100}$$

One thus finds the relation, valid for $\ell \ll \ell_I$

$$-\frac{1}{12}(1+\ell\partial_\ell)(3+\ell\partial_\ell)(5+\ell\partial_\ell)\frac{S_3(\ell)}{\ell} = \varepsilon. \tag{101}$$

Setting $y = \frac{S_3(\ell)}{\ell}$, $x = \log(\ell)$ this reads

$$-\frac{1}{12}(1+\partial_x)(3+\partial_x)(5+\partial_x)y = \varepsilon \tag{102}$$

and, putting $y = -\frac{4}{5}\varepsilon + u$, we find

$$(1+\partial_x)(3+\partial_x)(5+\partial_x)u = 0 \tag{103}$$

thus

$$u = Ae^{-x} + Be^{-3x} + Ce^{-5x}, \tag{104}$$

The only solution, finite in $\ell \to 0$ $(x \to -\infty)$ is $A = B = C = 0$.
Thus

$$S_3(\ell) = -\frac{4}{5}\varepsilon\ell \tag{105}$$

Q.E.D.

3 Multifractal Asymptotic Models

3.1 Inertial intermittency

The $K41$ theory is the following expression for the r-th order structure function [8]

$$S_r(\ell) = C_r(\varepsilon\ell)^{r/3}. \tag{106}$$

We have demonstrated in the preceding section the exact result

$$C_3 = -\frac{4}{5} \tag{107}$$

The original success of the $K41$ theory was helped by early experimental verifications [9]. However the experiments later showed some imperfections in the theory, related to small-scale intermittency [8] [10] [11] [7].

Furthermore Landau [1] objected that, if the injection rate ε was fluctuating, the constant C_r could not be universal because $< \varepsilon^{r/3} > \neq < \varepsilon >^{r/3}$, for $r \neq 3$. Note that Landau's argument breaks down for $r = 3$, the only case where it was possible to derive an exact result!

More generally, one calls "intermittency" the variations in space and time of ε. When it is the cascade rate that fluctuates at inertial scales, one talks about "inertial intermittency".

In 1961, Kolmogorov and Obukhov introduced the log-normal model [12] [13], in order to take into account intermittency effects due to the spatial fluctuations of the energy dissipation. This model has never been directly related to the Navier-Stokes equations, but rather to experimental and numerical results. It paved the way to other intermittency models based, in a geometric context, on the concept of the Richardson cascade. These new approaches introduced the notion of fractal dimension [14] [15]. Examples are the β model, [16] [17], the random β model [18] [19], and the Parisi-Frisch [20] multifractal model.

In this section, we will be concerned with inertial intermittency models, where the structure functions follow scaling laws of the form

$$S_r(\ell) = C_r(\varepsilon\ell)^{r/3}(\ell/\ell_I)^{\zeta_r - r/3}. \tag{108}$$

The scaling exponent ζ_r is in general a nonlinear convex function of r. Such a scaling law is called "multifractal" (in contrast to simple "unifractal" scaling when ζ_r is linear in r). The following simple cascade model shows that such a behavior is simple to obtain.

3.2 Random cascade models

The random cascade model is a simple and explicit model where the moments follow, by construction, multifractal scaling laws. It was first introduced by Novikov and Stewart [16] in a special case, and then extended by Yaglom [21]. A number of authors have then studied it, including Mandelbrot [22], [23].

Consider the interval I_{ℓ_Λ} with initial length ℓ_Λ that is decomposed into 2^N sub-intervals of length

$$\ell = \ell_\Lambda \, 2^{-N}. \tag{109}$$

To each interval, we attribute a random variable u_ℓ, product of N identically distributed independent random variables $(v_i)_{i=1\ldots N}$, that obey the following hypotheses

$$v \geq 0, \langle v \rangle = 1, \langle v^r \rangle < \infty, \forall r > 0. \tag{110}$$

Where the symbol $\langle . \rangle$ denotes the statistical mean.

Consider the a real random variable v, following the hypotheses (110). By construction

$$u_\ell = \prod_{i=1}^{N} v_i. \tag{111}$$

The multiplicative process that constructs u_ℓ can be repeated indefinitely: $N \to \infty$.
We now compute the moments associated with u_ℓ

$$S(r) = \langle u_\ell^r \rangle = \langle \prod_{j=1}^{N} v_j^r \rangle = \prod_{j=1}^{N} \langle v_j^r \rangle = [\langle v^r \rangle]^N . \tag{112}$$

Or, in logarithmic form

$$\log S(r) = N \log \langle v^r \rangle. \tag{113}$$

According to the definition of ℓ (109), one finds

$$\log S(r) = \zeta_r \, \log \frac{\ell}{\ell_\Lambda} \tag{114}$$

with

$$\zeta_r = -\log_2 \langle v^r \rangle. \tag{115}$$

Thus, the order-r moments follow scaling laws with exponents ζ_r

$$S(r) = \left[\frac{\ell}{\ell_\Lambda} \right]^{\zeta_r} . \tag{116}$$

3.3 The Parisi-Frisch multifractal model

The Parisi-Frisch [20] model can be defined by considering that singularities corresponding to scaling exponents h are located over fractal sets S_h with Hausdorff dimension [14] $D(h) < 3$. Defining the fractal codimension

$$\mu_{PF}(h) = 3 - D(h). \tag{117}$$

Parisi-Frisch hypotheses We admit that the scaling exponents h belong to the interval $I = [h_{min}, h_{max}]$. To each exponent h one associates a fractal set $S_h \in \mathbb{R}^3$ with dimension $D(h)$ such that

$$\frac{\Delta u_\ell}{\Delta u_{\ell_I}} \sim \left[\frac{\ell}{\ell_I}\right]^h. \tag{118}$$

The exponents h_{min} et h_{max} and the dimension $D(h)$ are universal and do not depend on the turbulence production mechanism.

What is the probability to belong to set S_h ? We need to compute the probability P_ℓ to intersect a fractal set with dimension $D(h)$ with a ball of radius ℓ, [14] :

$$P_\ell = \frac{\text{Number of balls associated with } S_h}{\text{Number of balls associated with } \mathbb{R}^3}. \tag{119}$$

It is easy to figure out this probability when D in an integer. If $D = 1$, consider a segment of length ℓ. When divided in 2 ($\ell \to \ell/2$), one then gets 2 segments of length $\ell/2$. The number of segments is thus multiplied by 2 ($N \to 2N$).
The same argument gives:

$$D = 2,\, \ell \to \ell/2,\, N \to 4N \tag{120}$$

$$D = 3,\, \ell \to \ell/2,\, N \to 8N \tag{121}$$

So, using (121), we must write $N \sim \ell^{-D}$. Thus, from (119) :

$$P_\ell \sim \ell^{3-D(h)}. \tag{122}$$

We can now compute the order-r moment as:

$$\frac{S_r(\ell)}{(\Delta u_{\ell_I})^r} \sim \int_I \left[\frac{\ell}{\ell_I}\right]^{rh+3-D(h)} dM(h). \tag{123}$$

Where the explicit form of the measure $dM(h)$ is not needed. In the limit $\ell \to 0$, the power law with the smallest exponent is dominant. We thus obtain:

$$\frac{S_r(\ell)}{(\Delta u_{\ell_I})^r} \sim \left[\frac{\ell}{\ell_I}\right]^{\zeta_r} \tag{124}$$

with

$$\zeta_r = \inf_h (rh + 3 - D(h)) = \inf_h (rh + \mu_{PF}(h)). \tag{125}$$

Relation (125), defines ζ_r in terms of the Legendre transform of $\mu_{PF}(h)$. Note that this Legendre Transform formalism was first introduced by Polyakov [24].

The Frisch probabilistic reformulation The aim of the probabilistic reformulation is to avoid the (rather ill defined outside of measure theory) notions of singularities and fractal sets and to directly relate the codimension $\mu_{PF}(h)$ to the probability distribution function (pdf) of velocity increments. The actual definition is (see [8], in the following definition p actually denotes the cdf of velocity increments)

$$\mu(h) = \lim_{\ell \to 0} \lim_{\nu \to 0} \frac{\log p(\pm \ell^h, \ell)}{\log \ell}. \tag{126}$$

Where the double limit is non commutative.

3.4 Integral relations between pdf and moments

We now turn to the new asymptotic formalism developed in [25]. We restrict our attention to absolute value structure functions $S(r) =< |v(x+\ell) - v(x)| >$. We do this in order to deal with a formalism of minimum complexity. It would be simple, in principle, to introduce positive and negative fluctuations by separating the pdf into plus and minus parts as done in [8].

Let us denote by $p_{\mathrm{u}}(u)$ the pdf of the increment $u_\ell = |v(x+\ell) - v(x)|$. The order-$r$ moments read

$$S(r) = \langle u^r \rangle = \int_0^\infty u^r p_{\mathrm{u}}(u) du. \tag{127}$$

In what follows, we will make change of variable of the type $L_u = \log u$. Such change of variables introduce new pdf. We will denote, by convention, the densities by p_{u} for the variable u and $p_{\mathrm{L_u}}$ for the variable $L_u = \log u$. One has

$$p_{\mathrm{L_u}}(L_u)\, dL_u = p_{\mathrm{u}}(u)\, du$$

, where $u dL_u = du$ and thus $p_{\mathrm{L_u}}(L_u) = p_{\mathrm{u}}(e^{L_u}) e^{L_u}$. To avoid any confusion, let us remark that the symbols u or $\mathrm{L_u}$ indexing "p" are part of the name of the pdf. These symbols are not variables, in contrast to those between parentheses. The structure function $S(r)$ can then be expressed using $p_{\mathrm{L_u}}(.)$ under the form

$$S(r) = \int_0^\infty u^{r-1} p_{L_u}(\log u)\, du$$

which is a Mellin transformation [26].
Using the L_u variable, the moments can be expressed as

$$S(r) = \int_{-\infty}^{+\infty} e^{rL_u} p_{\mathrm{L_u}}(L_u) dL_u. \tag{128}$$

In what follows, me will write $p_{\mathrm{L_u}}(.)$ under the form $p(.)$.

Introducing the characteristic function associated with the pdf $p(L_u)$

$$Z_{L_u}(k) = \int_{-\infty}^{+\infty} e^{ikL_u}\, p(L_u) dL_u \tag{129}$$

one gets

$$S(r) = Z_{L_u}(-ir) \tag{130}$$
$$Z_{L_u}(k) = S(ik). \tag{131}$$

(here the index L_u of "Z" is part of the name of the characteristic function and does not stand for a variable). Inverting the Fourier transform, one gets

$$p(L_u) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ikL_u} Z_{L_u}(k)dk. \tag{132}$$

The relations (128) and (132) thus relate pdf and moments.

Steepest descent approximation Direct method

The idea is to evaluate the integrals (128) et (132) using the steepest descent approximation. The asymptotic validity of the approximation will latter be studied in detail.

Using the Laplace method, we can evaluate the integral (128),

$$S(r) = \int_{-\infty}^{+\infty} e^{rL_u} p(L_u) dL_u$$

. Setting

$$L_p(.) = \log p_{L_u}(.) = \log p(.)$$

. The order-r moments are defined by

$$S(r) = \int_{-\infty}^{\infty} e^{rL_u + L_p} dL_u \tag{133}$$

where L_p is a notation for $L_p(L_u)$. This integral admits a critical point in L_u if r verifies the relation

$$r = -\frac{dL_p}{dL_u}. \tag{134}$$

Doing a second order Taylor expansion and computing the Gaussian integral one obtains

$$S(r) \sim \left[-\frac{2\pi}{\frac{d^2 L_p}{dL_u^2}} \right]^{1/2} e^{rL_u + L_p}. \tag{135}$$

Setting $L_S = \log S(r)$, we thus obtain a parametric representation (the parameter is L_u) of the function $S(r)$

$$r = -\frac{dL_p}{dL_u} \tag{136}$$

$$L_S \sim -\frac{dL_p}{dL_u} L_u + L_p + \frac{1}{2} \log \left[-\frac{2\pi}{\frac{d^2 L_p}{dL_u^2}} \right]. \tag{137}$$

Inverse method

Relation (132) can be written, using (131):

$$p(L_u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{f(k)}\, dk \tag{138}$$

where

$$f(k) = -ikL_u + \log S(ik). \tag{139}$$

The integral (138) can be evaluated using a steepest descent method [27]. The function f defined in (139) being analytic, it is possible to deform the contour of integration from the real axis ($-\infty < k < +\infty$) to a new contour (C) in the complex plane such that the imaginary part of f is zero on (C). The saddle point of f is defined by $f^{'}(k_s) = 0$. Supposing the existence of such a saddle point in $k_s = -ir$ with $r \in \mathbb{R}$. The condition $f^{'}(k_s) = 0$ is equivalent to the relation $L_u = dL_S/dr$. Taylor-expanding f yields: $f(k) = f(k_s) + \frac{(k-k_s)^2}{2} f^{''}(k_s) + \mathcal{O}[(k-k_s)^3]$. We now look for k such that $\Im(f(k)) = 0$ with f defined by (139), close to the saddle point k_s. One has $\Im\left[f(k_s) + \frac{(k-k_s)^2}{2} f^{''}(k_s)\right] = \Im\left[(k-k_s)^2\right] = 0$. There is thus a double family of solutions $k = k_s + i\alpha$ et $k = k_s + \alpha$ with $\alpha \in \mathbb{R}$. As k_s is generically a saddle point of order two, taking into account the other terms in the Taylor expansion two lines crossing on the saddle point exist such that the imaginary part of f is zero on them. One is the imaginary axis itself, and the other is the curve (C) that will be taken as the integration contour. This curve is the steepest descent line for the real part of f. We thus have:

$$p(L_u) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{f(k)}\, dk = (2\pi)^{-1} \int_{(C)} e^{f(k)}\, d\gamma(k), \tag{140}$$

where $\gamma(k)$ is the total arc length taken along (C) at point k. Applying Laplace method to the integral (140) is the essence of the steepest descent method [27]. Using (130) and (139), we find

$$f(k_s) = -rL_u + L_S, f^{'}(k_s) = \left[-L_u + \frac{dL_S}{dr}\right] i, f^{''}(k_s) = -\frac{d^2 L_S}{dr^2}.$$

We find that using the relations $\frac{d}{dk} = \frac{d}{d(-ir)} = i\frac{d}{dr}$.

(140) evaluates as

$$p(L_u) = (2\pi)^{-1} \int_{(C)} e^{f(k)}\, d\gamma(k) \sim (2\pi)^{-1} \left[\frac{2\pi}{-[f(k_s)]''}\right]^{1/2} e^{f(k_s)} \tag{141}$$

or

$$p(L_u) \sim (2\pi)^{-1} \left[\frac{2\pi}{\frac{d^2 L_S}{dr^2}}\right]^{1/2} e^{-rL_u + L_S}. \tag{142}$$

The parametric representation (with parameter r) of the function $L_p(L_u)$ finally reads

$$L_u = \frac{dL_S}{dr} \tag{143}$$

$$L_p \sim -r\frac{dL_S}{dr} + L_S - \frac{1}{2}\log\left[2\pi\,\frac{d^2L_S}{dr^2}\right]. \tag{144}$$

The $\boldsymbol{\ell \longrightarrow 0}$ asymptotic In this section we show that, in the case of the random cascade model of section 3.2, the expressions (137) et (144) are the first terms of an asymptotic in $-\log(\ell/\ell_\Lambda)$.

The scaling law (116) strongly suggests to make in integrals (128) et (132) the following change of variables

$$\zeta_r = \frac{L_S}{\log(\ell/\ell_\Lambda)} \tag{145}$$

$$h = \frac{L_u}{\log(\ell/\ell_\Lambda)} \tag{146}$$

$$\mu = \frac{L_p}{\log(\ell/\ell_\Lambda).} \tag{147}$$

We thus get from (128)

$$S_r(\ell) = \int_{-\infty}^{\infty} e^{\{\log(\ell/\ell_\Lambda)(rh+\mu)\}} dL_u. \tag{148}$$

On the other hand, using (131) and (132), gives for the density

$$p(L_u) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{\{\log(\ell/\ell_\Lambda)(-ikh+\zeta_{(ik)})\}} dk. \tag{149}$$

The integrals thus have (as they should) the asymptotic parameter $-\log(\ell/\ell_\Lambda)$ factorized in the exponential.
In this way, (144) gives the asymptotically correct expression of the probability $L_p(L_u)$

$$L_u = \log(\frac{\ell}{\ell_\Lambda})\frac{d\zeta_r}{dr} \tag{150}$$

$$L_p = \log(\frac{\ell}{\ell_\Lambda})(-r\frac{d\zeta_r}{dr} + \zeta_r) - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left[\log(\frac{\ell}{\ell_\Lambda})\frac{d^2\zeta_r}{dr^2}\right] + \mathcal{O}\left[\frac{1}{\log(\ell/\ell_\Lambda)}\right]. \tag{151}$$

In the same way, using (146), (147), (145), and (137) gives the asymptotically correct relations

$$r = -\frac{d\mu}{dh} \tag{152}$$

$$L_S = \log(\frac{\ell}{\ell_\Lambda})(rh+\mu) + \frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left[-\frac{1}{\log(\ell/\ell_\Lambda)}\frac{d^2\mu}{dh^2}\right] + \mathcal{O}\left[\frac{1}{\log(\ell/\ell_\Lambda)}\right]. \quad (153)$$

Setting

$$\overline{\mu} = -r\frac{d\zeta_r}{dr} + \zeta_r, \quad (154)$$

relation (151) can be written

$$L_p = \overline{\mu}\log(\frac{\ell}{\ell_\Lambda}) - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left[\log(\frac{\ell}{\ell_\Lambda})\frac{d^2\zeta_r}{dr^2}\right] + \mathcal{O}\left[\frac{1}{\log(\ell/\ell_\Lambda)}\right].$$

We can then define the relation between μ (147) and $\overline{\mu}$ (154) using (147) and

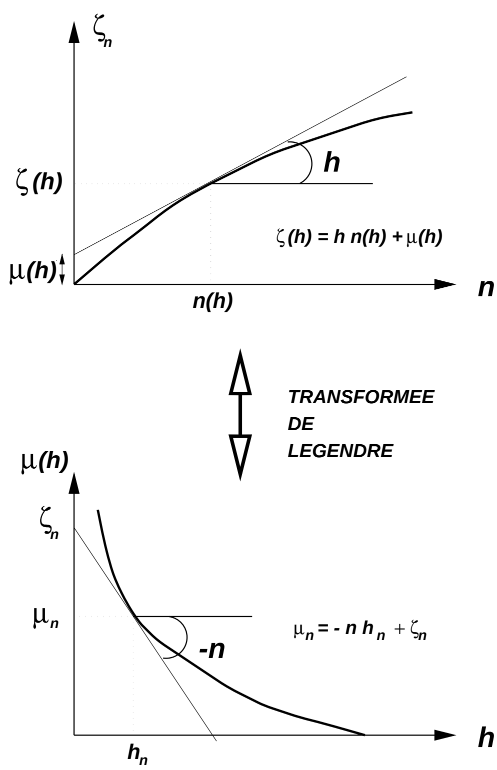

Fig. 1. The Parisi-Frisch Legendre transform.

(151) :

$$\mu = \overline{\mu} - \frac{1}{\log(\frac{\ell}{\ell_\Lambda})}\left\{-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left[\log(\frac{\ell}{\ell_\Lambda})\frac{d^2\zeta_r}{dr^2}\right]\right\}. \quad (155)$$

Expressing μ in function of $\overline{\mu}$, (153) can be written

$$L_S = (rh + \overline{\mu})\ \log(\frac{\ell}{\ell_\Lambda}) - \frac{1}{2}\log\left[\frac{d^2\zeta_r}{dr^2}\right] - \frac{1}{2}\log\left[-\frac{d^2\overline{\mu}}{dh^2}\right] + \mathcal{O}\left[\frac{1}{\log(\ell/\ell_\Lambda)}\right] \quad (156)$$

Where we have used the fact that according to (155) the correction between μ and $\overline{\mu}$ is in $\dfrac{1}{\log(\frac{\ell}{\ell_\Lambda})}$. Thus, at the required order, we can replace $\dfrac{d^2\mu}{dh^2}$ by $\dfrac{d^2\overline{\mu}}{dh^2}$ in (153).

The expressions (151) and (156) naturally yield the formalism of the Parisi-Frisch [20] model, which is the Legendre transformation that relates ζ_r to $\overline{\mu}(h)$. This Legendre transform is defined by $\zeta_r = \inf_h\,[rh + \overline{\mu}(h)]$, thus $d\overline{\mu}/dh = -r$ and admits a simple geometric representation (see Fig.(1)). In the same way the inverse transformation is defined by $\overline{\mu}(h) = \sup_r(\zeta_r - rh)$ (see Fig.(1)), or $d\zeta_r/dr = h$.

3.5 Explicit multifractal expressions for densities

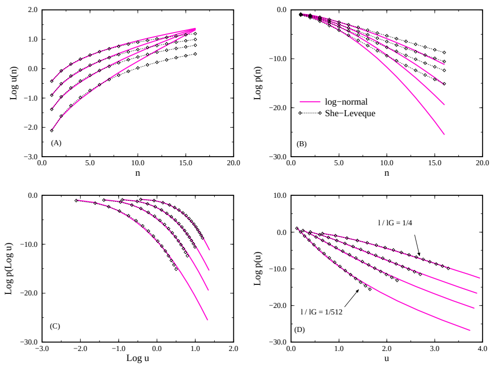

Fig. 2. Reconstruction of the pdf (see text) for different models (see table (1)) at scales $\ell/\ell_\Lambda = 1/4, 1/16, 1/64, 1/512$. Figure (A) : $\log u(n,\ell)$ as a function of the parameter r. Figure (B) : $\log p(n,\ell)$ as a function of the parameter r. Figure (C) : Classical representation $\log p(\log u,\ell)$. Figure (D) : Classical representation $\log p(u,\ell)$.

Experimental studies [28] [29] [30] show that the order-r structure functions follow multifractal scaling laws whose exponent ζ_r is a nonlinear function of r.

We now show that if the exponents ζ_r and the prefactors A_r of the scaling laws are known it is possible to use the the asymptotic relations of the previous section to reconstruct the pdf in the inertial range.

Let us first write the structure function as

$$\log S_r(\ell) = A_r + \zeta_r\ \log\ell \quad (157)$$

Table 1. Multifractal models of exponents ζ_n

models	exponents	parameters
Log-normal	$\zeta_n = n/3 + \eta\,(3n - n^2)/18$	$\eta = 0.2$
She-Lévêque	$\zeta_n = n/9 + 2\,(1 - (2/3)^{n/3})$	no parameters

$$= \log S_r(\ell_\Lambda) + \zeta_r \; \log \frac{\ell}{\ell_\Lambda} \tag{158}$$

with $A_r = \log S_r(\ell_\Lambda) + \zeta_r \; \log \ell_\Lambda$.

Let us now suppose that at the reference scale ℓ_Λ (that is of the order of the integral scale ℓ_I,see ref. [25]) the structure functions are Gaussian. Using the relation

$$\int_0^\infty e^{-x^2/(2\sigma^2)}\, x^r \, dx = \frac{\Gamma\left[\frac{r+1}{2}\right]}{\left[\frac{1}{2}\right]^{r/2} \sqrt{\pi}\,[\sigma^{-2}]^{r/2}} \int_0^\infty e^{-x^2/(2\sigma^2)}\, dx.$$

we find

$$\log S_r(\ell_\Lambda) = \log \Gamma \left[\frac{r+1}{2}\right] - \frac{1}{2}\, \log \pi + r \log u_\Lambda - \frac{r}{2}\, \log \left[\frac{1}{2}\right]. \tag{159}$$

where u_Λ is a velocity of the order of the integral velocity [25].

Reconstruction of inertial range pdf Using the asymptotic relations

$$L_u = \frac{d \log S_r(\ell)}{dr} \tag{160}$$

$$L_p \sim -r \frac{d \log S_r(\ell)}{dr} + \log S_r(\ell) - \frac{1}{2} \log \left[2\pi \, \frac{d^2 \log S_r(\ell)}{dr^2} \right] \tag{161}$$

and the expression of $\log S_r(\ell)$, (158) et (159) together with the exponents given in table (1) generates the pdf displayed on figure (2).

This method to reconstruct the pdf is new and was developed in [25]. It is apparent on figure (2) that the pdf change form when ℓ is varied.

References

1. L. Landau and E. Lifchiftz. *Mécanique des Fluides*, volume **6**. Edition MIR, 1989.
2. L. F. Richardson. *Weather prediction by numerical process.* Cambridge University Press England, 1922.
3. A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluide for very large reynolds number. *C. R. Acad. Sci. U.S.S.R*, **30**:9–13, 1941.

4. A. N. Kolmogorov. On degeneration decay of isotropic turbulence in an incompressible viscous liquid. *C. R. Acad. Sci. U.S.S.R*, **31**:538–540, 1941.
5. A. N. Kolmogorov. Dissipation of energy in locally isotropic turbulence. *C. R. Acad. Sci. U.S.S.R*, **32**:16–18, 1941.
6. L. F. Richardson. Atmospheric diffusion shown on a distance neighbor graph. *Proc. Roy. Soc.*, **110**:709–737, 1926.
7. A. S. Monin and A. M. Yaglom. *Statistical Fluid Mechanics*. MIT press, Cambridge, 1971.
8. U. Frisch. *Turbulence: the Legacy of A.N Kolmogorov*. Cambridge University Press, 1995.
9. G. K. Batchelor. *The theory of homogeneous turbulence*. Cambridge university press, Cambridge, 1960.
10. U. Frisch and S. A. Orszag. Turbulence : challenges for theory and experiment. *Physics Today*, page 24, April 1990.
11. G. K. Batchelor and A. A. Townsend. The nature of turbulent motion at large wave-numbers. *Proc. Roy. Soc.*, **A 199**:238–255, 1949.
12. A. N. Kolmogorov. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high reynolds number. *J. Fluid Mech.*, **13**:82–85, 1962.
13. A. M. Obukhov. Some specific features of atmospheric turbulence. *J. Fluid Mech.*, **13**:77–81, 1962.
14. K. Falconer. *Fractal Geometry : Mathematical Foundations and applications*. J. Wiley and Sons, 1993.
15. B. B. Mandelbrot. *Fractals : Form, Chance and Dimension*. Freeman, San Francisco, 1977.
16. E. A. Novikov and R. W. Stewart. The intermittency of turbulence and the spectrum of energy dissipation. *Izv. Akad. Nauk. SSSR, Ser. Geoffiz.*, pages 408–413, 1964.
17. U. Frisch, P. L. Sulem, and M. Nelkin. A simple dynamical model of intermittent fully developed turbulence. *J. Fluid Mech.*, **87**:719–736, 1978.
18. R. Benzi, G. Paladin, A. Vulpiani, and M. Vergasolla. On the multifractal nature of fully developed turbulence and chaotic systems. *J. Phys. A:Math*, **17**:3521–3531, 1984.
19. G. Paladin and A. Vulpiani. Anomalous scaling laws in multifractal objects. *Phys. Report*, **156**:147–225, 1987.
20. G. Parisi and U. Frisch. On the singularity structure of fully developed turbulence. In M. Ghil, R. Benzi, and G. Parisi, editors, *Turbulence and Predictability in Geophysical Fluid Dynamics, Proceedings Intern. School of Physics 'E. Fermi', 1983, Varenna, Italy*, pages 84–87. North Holland, Amsterdam, 1985.
21. A. M. Yaglom. Effect of fluctuations in energy dissipation rate on the form of turbulence characteristics in the inertial subrange. *Sov. Phys. Dokl*, **166**:49–52, 1966.
22. B. B. Mandelbrot. On intermittent free turbulence. In *Turbulence of fluids and plasmas*. Brooklyn Polytechnic Inst, 1968.
23. B. B. Mandelbrot. Intermittent turbulence in self-similar cascades : divergence of high moments and dimension of the carrier. *J. Fluid Mech.*, **62**:331–358, 1974.
24. A. M. Polyakov. Scale invariance of strong interactions and its application to lepton-hadron reactions. In *International School of High Energy Physics in Erevan 23 November - 4 December 1971 (Chernogolovka 1972)*. Acad. Sci. USSR, 1972.

25. J. M. Tchéou. *Analyse statistique multifractale en turbulence pleinement développée et application à la finance*. PhD thesis, Ecole Normale Supérieure de Cachan, France, 1997.
26. A. H. Zemanian. *Generalized integral transformations*. Pure Applied Mathematics, Interscience publishers, 1968.
27. C. M. Bender and S. A. Orszag. *Advanced mathematical methods for scientists and engineers*. International series in pure and applied mathematics, McGraw-Hill, 1978.
28. C. W. Van Atta and W. Y. Chen. Structure functions of the turbulence in the atmospheric boundary layer over the ocean. *J. Fluid Mech.*, **44**:145–159, 1970.
29. F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia. High order velocity structure functions in turbulent shear flow. *J. Fluid Mech.*, **140**:63–89, 1984.
30. Y. Gagne. *Etude Expérimentale de l'intermittence et des singularités dans le plan complexe en turbulence développée*. PhD thesis, Université de Grenoble, France, 1987.

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