FEIGENBAUM'S RATIOS OF TWO-DIMENSIONAL AREA PRESERVING MAPS

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We generalize simple renormalization methods to calculate Feigenbaum's ratios for a two-dimensional area preserving map. Our results are in satisfactory agreement with known results and give approximate estimate for the ratio associated to more general cascades like 1, 3, 3^2, ..., 3^n, ...

Maps of the interval have attracted a lot of attention in the last few years (some recent references can be found in ref. [1]). The interest of physicists for these mathematical models was increased by the work of Feigenbaum [2,3] who discovered that the period-doubling cascades have universal critical properties: for a large class of one-dimensional maps T_μ which depend on a deformation parameter μ, there is a cascade of bifurcations $1 \to 2 \to 4 \to ... \to 2^n \to ...$. The values μ_n of the parameter where the cycle 2^n becomes unstable and where a cycle 2^{n+1} appears accumulate at some value μ_∞. Feigenbaum pointed out that for a large class of maps T_μ, one has

$$\lim_{n\to\infty} \frac{\mu_n - \mu_\infty}{\mu_{n+1} - \mu_\infty} = \delta, \qquad (1)$$

δ is a universal number which depends only on the shape of the map T_μ near its maximum. When the maximum of T_μ is quadratic, the value of δ is

$$\delta = 4.66\,920. \qquad (2)$$

As mentioned by Eckmann [4], the same value of δ was obtained by several authors [5–7] when they studied other dissipative systems and it is believed [8] that this value will be soon measured in the cascades observed in Rayleigh Benard experiments.

A renormalization group theory of these period-doubling cascades was proposed by Feigenbaum [2,3] and rigorous results were derived by Collet et al. [9]. For maps like ($x \to 1 - a|x|^{1+\epsilon}$) it was possible to expand δ for small values of ϵ [7,9]. In our work [7] with A. Gervois, we developed several approximate renormalization methods similar to real space renormalizations used in statistical physics. They led to rather accurate values of δ. Moreover, in ref. [7], the period doubling phenomenon was generalized to other cascades like 1, 3, 3^2, ..., 3^n, ... Though for these new cascades, the stability zones of the cycles (values of μ where the cycle is stable) are not contiguous, well defined periods 3^n appear at values μ'_n which accumulate to μ'_∞ in the same way as in period-doubling cascades:

$$\lim_{n\to\infty} \frac{\mu'_n - \mu'_\infty}{\mu'_{n+1} - \mu'_\infty} = \delta'. \qquad (3)$$

The number δ' is also universal and characteristic of the factor 3. For maps with a quadratic maximum, its value is

$$\delta' = 55.247\,.... \qquad (4)$$

Again this universal ratio can be recovered approximatively by simple renormalization equations.

In this letter, we use the same kind of renormalization equations to calculate the ratios δ and δ' for two-dimensional area preserving maps. The importance of these maps is that they are good examples of conservative dynamical systems. We generalize to the two-dimensional case the method called "equality of slopes" in ref. [7]. The principle of the method is the following: consider a two-dimensional map $(x,y) \to T_\mu(x,y)$. Here we take for T_μ the Hénon map in the area preserving case [10]

$$T_\mu(x,y) = (1-\mu x^2 + y, -x). \tag{5}$$

For the simplest cycles (cycles of length 1, 2, 3, 4) one can calculate analytically the coordinates of the points of a cycle as functions of μ. We can also linearize the map $T_\mu^{(n)}$ in the neighbourhood of the points of cycle n. This linearized map is a 2×2 matrix M given by

$$M = \prod_{i=1}^{n} \begin{pmatrix} -2\mu x_i & 1 \\ -1 & 0 \end{pmatrix}, \tag{6}$$

where the x_i are the coordinates of the points of cycle n. The eigenvalues λ_n of this matrix can be expressed as a function of μ. For $n = 1, 2, 3, 4$, the λ_n are roots of

$$\lambda_n^2 + 2f_n(\mu)\lambda_n + 1 = 0, \tag{7}$$

with

$$f_1(\mu) = (1+\mu)^{1/2} - 1, \quad f_2(\mu) = 2\mu - 7,$$

$$f_3(\mu) = 4\mu[1 + (\mu-1)^{1/2}] - 5 - 3(\mu-1)^{1/2},$$

$$f_4(\mu) = 8\mu^2 - 16\mu^{3/2} - 1.$$

(Note that the cycle n is elliptic if $-1 \leqslant f_n(\mu) \leqslant 1$.)

For period doubling problems, the idea of renormalization is to try to associate at each value of μ, a value μ' such that T_μ looks like $T_{\mu'}^{(2)}$. An approximate way to do so is to say that the linearization of $T_\mu^{(n)}$ around a point of cycle n is identical to the linearization of $T_{\mu'}^{(2n)}$ around a point of cycle $2n$. Therefore:

$$f_n(\mu) = f_{2n}(\mu'). \tag{8}$$

This relation provides an approximate value for μ_∞ (the value of μ where the cascade $1 \to 2 \to \dots \to 2^n \dots$ accumulates) and of δ. μ_∞ is the fixed point of renormalization (8)

$$f_n(\mu_\infty) = f_{2n}(\mu_\infty), \tag{9}$$

whereas δ is given by:

$$\delta = \mathrm{d}\mu/\mathrm{d}\mu'|_{\mu_\infty}. \tag{10}$$

The choice of small value of n constitutes an approximation which should be improved by increasing n. We have done the calculations for $n = 1$ and $n = 2$ and the results are summarized in table 1. In the same way, one can study the cascade $1, 3, 3^2, \dots, 3^n \dots$ which accumulates at value μ'_∞ with a Feigenbaum ratio δ'. The renormalization equation (8) becomes in this case:

$$f_n(\mu) = f_{3n}(\mu'), \tag{11}$$

and the procedure remains the same to find μ'_∞ and δ'. We have done in this case the calculations for $n = 1$ and the results are also given in table 1. They agree rather well with recent works [11–15] for the period doubling ratio δ and provide an estimate for the period tripling ratio δ'.

In order to show the accuracy of these calculations, we reproduce in table 2 the results obtained [7] with the same method for the well known one-dimensional map T_μ $(x \to 1 - \mu x^2)$. In this case the method is not very different from the one used by May and Oster [1] to calculate approximately δ.

The method we used in this letter gives rather accurate results with a very small amount of calculations. From tables 1 and 2, there is no doubt that the universal ratios δ and δ' are different for conservative systems (like the Hénon map) and dissipative systems (like x

Table 1
Results of the approximate renormalizations for the Hénon map $(x,y) \to (1-\mu x^2 + y, -x)$. The expected values are unknown except δ which has been recently obtained by several authors [11–15].

n	$2n$	μ_∞	δ
1	2	4.13278	9.0623
2	4	4.13621	8.6845
expected values		?	8.72 ...
n	$3n$	μ'_∞	δ'
1	3	1.18362	22.869
expected values		?	?

Table 2
Results of the approximate renormalization for the one-dimensional map $x \to 1 - \mu x^2$.

n	$2n$	μ_∞	δ
1	2	1.39039	5.123
2	4	1.40142	4.614
expected values		1.40115519	4.66920
n	$3n$	μ'_∞	δ'
1	3	1.78597	61.61
expected values		1.786440255	55.247

$\to 1 - \mu x^2$). It would be interesting to know more about the ratios δ one can define for cascades like p, $p^2, p^3, ..., p^n, ...$ and to see if the p dependence of these universal ratios is simple. Finally, it seems more difficult in the conservative case to define a map (like $x \to 1 - a|x|^{1+\epsilon}$ in the dissipative case) where an analytic expansion of δ and δ' can be done.

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