FEIGENBAUM'S RATIOS OF TWO-DIMENSIONAL AREA PRESERVING MAPS

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We generalize simple renormalization methods to calculate Feigenbaum's ratios for a two-dimensional area preserving map. Our results are in satisfactory agreement with known results and give approximate estimates for the ratio associated to more general cascades like 1, 3, 3^2, ..., 3^n, ....

Maps of the interval have attracted a lot of attention in the last few years (some recent references can be found in ref. [1]). The interest of physicists for these mathematical models was increased by the work of Feigenbaum [2,3] who discovered that the period-doubling cascades have universal critical properties: for a large class of one-dimensional maps $T_\mu$ which depend on a deformation parameter $\mu$, there is a cascade of bifurcations $1 \rightarrow 2 \rightarrow 4 \rightarrow ... \rightarrow 2^n \rightarrow ...$. The values $\mu_n$ of the parameter where the cycle $2^n$ becomes unstable and where a cycle $2^{n+1}$ appears accumulate at some value $\mu_\infty$. Feigenbaum pointed out that for a large class of maps $T_\mu$, one has

$$\lim_{n \rightarrow \infty} \frac{\mu_n - \mu_\infty}{\mu_{n+1} - \mu_\infty} = \delta,$$  \hspace{1cm} (1)

$\delta$ is a universal number which depends only on the shape of the map $T_\mu$ near its maximum. When the maximum of $T_\mu$ is quadratic, the value of $\delta$ is

$$\delta = 4.66920.$$

(2)

As mentioned by Eckmann [4], the same value of $\delta$ was obtained by several authors [5-7] when they studied other dissipative systems and it is believed [8] that this value will be soon measured in the cascades observed in Rayleigh Benard experiments.

A renormalization group theory of these period-doubling cascades was proposed by Feigenbaum [2,3] and rigorous results were derived by Collet et al. [9]. For maps like $(x \rightarrow 1 - a|x|^{1+\epsilon})$ it was possible to expand $\delta$ for small values of $\epsilon$ [7,9]. In our work [7] with A. Gervois, we developed several approximate renormalization methods similar to real space renormalizations used in statistical physics. They led to rather accurate values of $\delta$. Moreover, in ref. [7], the period doubling phenomenon was generalized to other cascades like 1, 3, 3^2, ..., 3^n, ... . Though for these new cascades, the stability zones of the cycles (values of $\mu$ where the cycle is stable) are not contiguous, well defined periods $3^n$ appear at values $\mu'_n$ which accumulate to $\mu'_\infty$ in the same way as in period-doubling cascades:

$$\lim_{n \rightarrow \infty} \frac{\mu'_n - \mu'_\infty}{\mu'_{n+1} - \mu'_\infty} = \delta'.$$

(3)

The number $\delta'$ is also universal and characteristic of the factor 3. For maps with a quadratic maximum, its value is

$$\delta' = 55.247 ... .$$

(4)

Again this universal ratio can be recovered approximatively by simple renormalization equations.

In this letter, we use the same kind of renormalization equations to calculate the ratios $\delta$ and $\delta'$ for two-dimensional area preserving maps. The importance of these maps is that they are good examples of conservative dynamical systems. We generalize to the two-dimensional case the method called "equality of slopes" in ref. [7]. The principle of the method is the following: consider a two-dimensional map $(x, y) \rightarrow T_\mu(x, y)$. Here we take for $T_\mu$ the Hénon map in the area preserving case [10].
\[ T_\mu(x, y) = (1 - \mu x^2 + y, -x). \]  
\text{(5)}

For the simplest cycles (cycles of length 1, 2, 3, 4) one can calculate analytically the coordinates of the points of a cycle as functions of \( \mu \). We can also linearize the map \( T^{(n)}_\mu \) in the neighbourhood of the points of cycle \( n \). This linearized map is a \( 2 \times 2 \) matrix \( M \) given by

\[ M = \prod_{i=1}^{n} \begin{pmatrix} -2\mu x_i & 1 \\ -1 & 0 \end{pmatrix}, \]
\text{(6)}

where the \( x_i \) are the coordinates of the points of cycle \( n \). The eigenvalues \( \lambda_n \) of this matrix can be expressed as a function of \( \mu \). For \( n = 1, 2, 3, 4 \), the \( \lambda_n \) are roots of

\[ \lambda^2 + 2f_n(\mu) \lambda_n + 1 = 0, \]
\text{(7)}

with

\[ f_1(\mu) = (1 + \mu)^{1/2} - 1, \quad f_2(\mu) = 2\mu - 7, \]
\[ f_3(\mu) = 4\mu [1 + (\mu - 1)^{1/2}] - 5 - 3(\mu - 1)^{1/2}, \]
\[ f_4(\mu) = 8\mu^2 - 16\mu^{3/2} - 1. \]

(Note that the cycle \( n \) is elliptic if \( -1 \leq f_n(\mu) \leq 1 \).)

For period doubling problems, the idea of renormalization is to try to associate at each value of \( \mu \), a value \( \mu' \) such that \( T_\mu \) looks like \( T_\mu^{(2)} \). An approximate way to do so is to say that the linearization of \( T_\mu^{(n)} \) around a point of cycle \( n \) is identical to the linearization of \( T_\mu^{(2n)} \) around a point of cycle \( 2n \). Therefore:

\[ f_n(\mu) = f_{2n}(\mu'). \]  
\text{(8)}

This relation provides an approximate value for \( \mu_\infty \) (the value of \( \mu \) where the cascade 1 \( \rightarrow \) 2 \( \rightarrow \ldots \rightarrow 2^n \ldots \) accumulates) and of \( \delta \), \( \mu_\infty \) is the fixed point of renormalization (8)

\[ f_n(\mu_\infty) = f_{2n}(\mu_\infty'), \]
\text{(9)}

whereas \( \delta \) is given by:

\[ \delta = \frac{d\mu}{d\mu'} \mu_\infty'. \]  
\text{(10)}

The choice of small value of \( n \) constitutes an approximation which should be improved by increasing \( n \). We have done the calculations for \( n = 1 \) and \( n = 2 \) and the results are summarized in table 1. In the same way, one can study the cascade 1, 3, 3, 3, \( \ldots \), 3\( n \) \( \ldots \) which accumulates at value \( \mu'_\infty \) with a Feigenbaum ratio \( \delta' \). The re-
It would be interesting to know more about the ratios $\delta$ one can define for cascades like $p$, $p^2$, $p^3$, ..., $p^n$, ... and to see if the $p$ dependence of these universal ratios is simple. Finally, it seems more difficult in the conservative case to define a map (like $x \to 1 - a|x|^{1+\epsilon}$ in the dissipative case) where an analytic expansion of $\delta$ and $\delta'$ can be done.

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References
