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Can Disorder Induce Several Phase Transitions?

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Abstract:

A few examples of disordered systems are described and the problem of averaging over the disorder is discussed. In particular, it is explained why one should be careful in averaging correlation functions. It is shown that the different possible averages of the correlation functions are related to different physical quantities and therefore all these averages have a physical meaning. This implies that different physical quantities may be singular at different critical points. A few remarks are done on numerical studies of disordered systems. Lastly three examples are discussed for which different physical quantities have different critical points: the Ising chain in a random field, the random walk on a disordered chain, the self-avoiding walk on a dilute lattice.

1. Introduction

Most of the classical problems in Statistical Mechanics have their disordered analogues. There are usually several ways to introduce randomness. For example, the Ising model on a regular lattice

$$\mathcal{H} = -\sum J\sigma_i\sigma_j \tag{1}$$

can be transformed into a disordered system by considering that the interactions J between nearest neighbor sites are random quenched variables. This means that the Hamiltonian $\mathcal{H}$ of the system is

$$\mathcal{H} = -\sum J_{ij}\sigma_i\sigma_j \tag{2}$$

and the J_{ij} are random independent variables distributed according to a given distribution $\rho(J_{ij})$. If one wants to study the physics of a diluted ferromagnet, then a possible choice for $\rho(J_{ij})$ can be just

$$\rho(J_{ij}) = p\,\delta(J_{ij} - J) + (1-p)\,\delta(J_{ij})\,. \tag{3}$$

If one wants to study spin glasses where one knows that there exist competing interactions, the distribution $\rho(J_{ij})$ must allow interactions J_{ij} of opposite signs. For example

$$\rho(J_{ij}) = (1/2\pi\sigma)^{1/2}\exp(-J_{ij}^2/2\sigma^2)\,. \tag{4}$$

Of course, these disordered Ising models can be easily generalized to become disordered Heisenberg models by replacing the Ising spins by n component vector spins.

Besides disordered magnetic systems, there are several classical problems of statistical mechanics for which the effect of quenched impurities is interesting. For example, one can study how a random walk is

affected by the presence of fixed impurities which can be either attractive or repulsive for the random walk. One can study the properties of a random walk on a dilute lattice: in that case, one expects that the diffusion constant vanishes when the concentration approaches the percolation threshold p_c and that at p_c the diffusion is anomalous. Another problem which is closely related is the problem of random resistor networks. One considers a bond percolation problem for which each occupied bond represents a resistor. For this problem, one would like to calculate the conductivity as a function of the concentration of bonds, in particular the critical behavior of this conductivity near the percolation threshold. Lastly a problem which will be discussed at the end of this talk is the self-avoiding walk (SAW) on a dilute lattice. This example is of particular interest because it is one of the most evident cases where the problem of averaging over disorder is posed. One can find recent reviews on disordered systems in references [1] and [2].

The two following features are common to all the examples described here and to most of the problems studied in the theory of disordered systems:

(1) The disorder is quenched: this means that the impurities (for example the interactions J_{ij}) have their positions fixed.

(2) The second important feature is that although the systems are inhomogeneous at a microscopic scale, they are homogeneous at a macroscopic scale.

2. The average over disorder

2.1. Free energy

For a pure Ising model, one knows that once the partition function $Z(\beta)$ has been calculated as a function of temperature $\beta = T^{-1}$ and also of the magnetic field, one can obtain all the physical quantities (the magnetization $m(\beta)$, the susceptibility $\chi(\beta)$, the specific heat $C(\beta)$, etc. . . .) by taking derivatives of $Z(\beta)$.

To study a disordered system like an Ising model with random interactions J_{ij} (eq. (2)), one must take into account the fact that all physical quantities depend on the configuration $\mathscr{C}$ of the disorder, namely here of the realization of the bonds J_{ij}. Therefore the partition function $Z(\beta, \mathscr{C})$ and all its derivatives $m(\beta, \mathscr{C})$, $\chi(\beta, \mathscr{C})$, $C(\beta, \mathscr{C})$ etc. . . . are functions of temperature but also of $\mathscr{C}$. Since each configuration $\mathscr{C}$ has a probability distribution $p(\mathscr{C})$, each physical quantity $Q(\beta, \mathscr{C})$ has a probability distribution $P(Q)$ given by:

$$P(Q) = \sum_{\mathscr{C}} p(\mathscr{C})\, \delta[Q - Q(\beta, \mathscr{C})]\,. \tag{5}$$

When the size of the system increases, one can prove in some cases or one hopes that the distribution $P(Q)$ of a physical quantity Q becomes narrower and narrower. Therefore the only quantity which can be observed in the thermodynamic limit is the most probable value $Q_{\text{m.p.}}$, the value around which most of the distribution is concentrated.

For some quantities Q, $Q_{\text{m.p.}}$ is very close to the average $\bar{Q}$ of Q and in the thermodynamic limit they become equal (fig. 1a)

$$Q_{\text{m.p.}} = \bar{Q}\,. \tag{6}$$

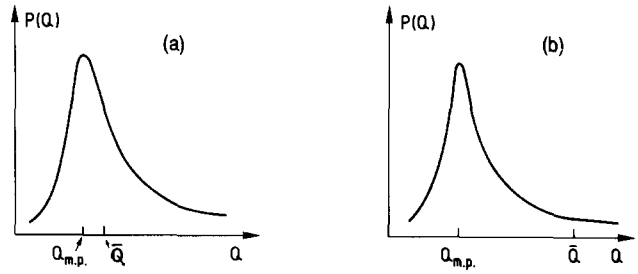

Fig. 1. (a) The average $\bar{Q}$ of a physical quantity is very close to its most probable value $Q_{m.p.}$. (b) $\bar{Q}$ and $Q_{m.p.}$ are very different.

For other quantities $\bar{Q}$ and $Q_{m.p.}$ are very different and they do not coincide at all in the thermodynamic limit (fig. 1b)

$$Q_{m.p.} \neq \bar{Q}. \tag{7}$$

To illustrate the fact that $Q_{m.p.}$ and $\bar{Q}$ may be equal or very different in the thermodynamic limit, let us take the simple example of an Ising chain with random nearest neighbor interactions. The Hamiltonian is

$$\mathcal{H} = -\sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1} \tag{8}$$

and the interactions J_i are assumed to be distributed according to a given distribution $\rho(J_i)$. The partition function for a system of $N+1$ spins can be easily calculated

$$Z = \prod_{i=1}^{N} c_i \tag{9}$$

and the c_i are given by

$$c_i = 2\cosh(\beta J_i). \tag{10}$$

It is clear that the partition function is a product of independent random variables. Therefore, for large N, the distribution of Z is a log-normal distribution peaked around $Z_{m.p.}$ which is very different from $\bar{Z}$:

$$Z_{m.p.} \simeq \exp[N\overline{\log c_i}] \tag{11}$$

$$\bar{Z} = \exp N[\log(\overline{c_i})] \tag{12}$$

$$Z_{m.p.} \ll \bar{Z}. \tag{13}$$

On the contrary the free energy F appears as a sum of independent random numbers and has a normal distribution for large N

$$\bar{F}_{m.p.} \sim \bar{F} = -NT\overline{\log c_i}. \tag{14}$$

So we see in this example that the partition function has the property of fig. 1b, therefore $\bar{Z}$ cannot be observed in the thermodynamic limit whereas the free energy is an example of the situation described in fig. 1a.

It is not very difficult to show that, these two facts remain valid for magnetic models with random short ranged interactions. The idea of the proof is explained in fig. 2. A very large system of N sites can be decomposed into p subsystems of N/p sites, each of them being in a new random configuration. Therefore the free energy f per site in the large system can be written as:

$$f = \frac{1}{p} \sum_{i=1}^{p} \left[-T\left(\frac{p}{N}\right) \log Z_i \right] + B \tag{15}$$

where Z_i is the partition function of the ith subsystem and B is a surface term which vanishes in the thermodynamic limit. Therefore in general for random short-ranged interactions, the average of the free energy has a physical meaning because the free energy has a normal distribution whereas the partition function has a log-normal distribution. It follows that one can average all the derivatives of the free energy: energy, entropy, magnetization, susceptibility etc....

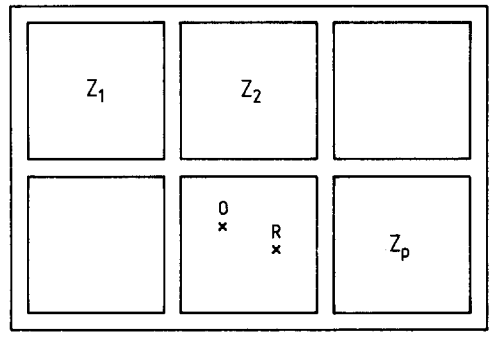

Fig. 2. A very large system of N sites can be decomposed into p subsystems. $\log Z \sim \Sigma_{i=1}^{p} \log Z_i$. In the thermodynamic limit one has to average $\log Z$. To calculate the correlation function $\langle \sigma_0 \sigma_R \rangle$ one can add two local fields h_0 and h_R. If one adds these two fields, the subsystem which contains 0 and R plays a special role and is no longer identical to the other subsystems. Therefore one cannot average the free energy anymore in presence of these two local fields.

2.2. *Correlation functions*

Let us now discuss the case of correlation functions [3]. For the Ising chain defined in (8), it is also easy to obtain the correlation function for any choice of the J_i:

$$\langle \sigma_0 \sigma_R \rangle = \prod_{i=0}^{R-1} t_i \tag{16}$$

with

$$t_i = \tanh(J_i/T) . \tag{17}$$

It is clear in this case that the correlation functions are products of random numbers. Therefore, for large R,

they have log-normal distributions like the partition function and then

$$\langle \sigma_0 \sigma_R \rangle_{\text{m.p.}} \neq \overline{\langle \sigma_0 \sigma_R \rangle} \,. \tag{18}$$

At first sight, this fact may look surprising since the correlation functions are derivatives of the free energy and we have seen that one can average the free energy as well as its derivatives. However to calculate $\langle \sigma_0 \sigma_R \rangle$, one can add to the Hamiltonian $\mathcal{H}$ magnetic fields on sites 0 and R. For this new Hamiltonian $\mathcal{H}'$

$$\mathcal{H}' = \mathcal{H} + h_0 \sigma_0 + h_R \sigma_R \tag{19}$$

one cannot average the free energy anymore as explained in fig. 2.

Both $\langle \sigma_0 \sigma_R \rangle_{\text{m.p.}}$ and $\langle \sigma_0 \sigma_R \rangle$ have a physical interest: $\langle \sigma_0 \sigma_R \rangle_{\text{m.p.}}$ is the value of the correlation function which can be observed with the highest probability whereas $\overline{\langle \sigma_0 \sigma_R \rangle}$ appears in the expression of the susceptibility

$$\chi = \sum_R \overline{\langle \sigma_0 \sigma_R \rangle} \,. \tag{20}$$

For the random Ising chain defined by (8), one can define several correlation lengths. First the two lengths $\xi_0(T)$ and $\xi_1(T)$ defined by:

$$\langle \sigma_0 \sigma_R \rangle_{\text{m.p.}} \sim \exp[R \, \overline{\log t_i}] = \exp - [R/\xi_0(T)] \qquad \text{for large } R \tag{21}$$

$$\overline{\langle \sigma_0 \sigma_R \rangle} = \exp[R \log(\overline{t_i})] = \exp - [R/\xi_1(T)] \,. \tag{22}$$

More generally, one could introduce a correlation length $\xi_p(T)$ for each moment of $\langle \sigma_0 \sigma_R \rangle$:

$$\overline{\langle \sigma_0 \sigma_R \rangle^p} = \exp[R \log(\overline{t_i^p})] = \exp - [R/\xi_p(T)] \,. \tag{23}$$

The fact that in general

$$1/\xi_2(T) > 2/\xi_1(T) \tag{24}$$

means that for large R

$$[\overline{\langle \sigma_0 \sigma_R \rangle^2} - (\overline{\langle \sigma_0 \sigma_R \rangle})^2]^{1/2} / \overline{\langle \sigma_0 \sigma_R \rangle} \gg 1 \,. \tag{25}$$

So it is very hard to measure numerically $\overline{\langle \sigma_0 \sigma_R \rangle}$ because of the large fluctuations of $\langle \sigma_0 \sigma_R \rangle$. This difficulty is also present in Monte Carlo calculations of lattice gauge models (see the talk of G. Parisi [4]).

2.3. Several phase transitions

For the random Ising chain, the t_i defined by (17) are always less than 1. Therefore all the correlation lengths $\xi_0(T), \xi_1(T), \ldots, \xi_p(T)$ never diverge at finite temperature. However, it is possible to exhibit

models for which these correlation lengths diverge at different temperatures. An example is the random cubic chain which was studied in collaboration with H.J. Hilhorst [3]. For this model, the correlation function $\langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_R \rangle$ is given exactly by

$$\langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_R \rangle = n \prod_{i=0}^{R-1} \tau_i \tag{26}$$

where n is the number of components of the spins and τ_i is given by:

$$\tau_i = \frac{\sinh(nJ_i/T)}{\cosh(nJ_i/T) + n - 1}. \tag{27}$$

For $n < 1$, one finds that $\xi_0(T)$ and $\xi_1(T)$ diverge at two different temperatures T_0 and T_1. It turns out that T_0 is the temperature where the zero field average free energy is singular whereas T_1 is the temperature where the average susceptibility diverges. In the limit where the distribution of J_i becomes a delta function, i.e. one suppresses the disorder, the two temperatures T_0 and T_1 coincide. Therefore the difference between T_0 and T_1 is a pure effect of disorder. The cubic model, for $n < 1$, is not very satisfactory because it is not realistic. However it has the advantage of showing by simple calculations that different quantities may be singular at different temperatures.

In the random Ising chain as well as in the random cubic chain, the correlation functions were always products of random numbers. For random magnets in higher dimension ($d > 1$), one can easily show that the first term in the high temperature expansion of $\langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_R \rangle$ remains a product of random numbers when the sites 0 and R are along the same axis of the cubic lattice. It would be very interesting to know the effect of the next terms of the high temperature expansion and to see whether $\xi_0(T)$ and $\xi_1(T)$ could diverge at different temperatures or, even if they diverge at the same T_c, they could have different critical behaviors.

For pure systems like the Ising model, one can use several ways to find the critical point: one can look for the point where the free energy is singular, or where the magnetization vanishes or where the susceptibility diverges etc.... All these definitions give the same critical point. On the contrary, for disordered systems, there is no reason that all these definitions give the same critical point. Griffiths [5] has shown that for dilute systems, the free energy starts to be singular at the critical temperature of the corresponding pure system. Therefore, for dilute systems, it is clear that the magnetization does not appear at the point where the free energy is singular.

At the end of this talk, I shall describe briefly a few examples of disordered systems where one can see several phase transitions. Before I do so, I would like to make a few remarks on the numerical calculations on disordered systems.

3. Numerical approaches

It has been shown in the talks on Monte Carlo calculations [6] for the $3d$ Ising model how one can use finite size scaling and Monte Carlo renormalization to obtain very accurate values of the critical point and of the exponents. In the case of disordered systems, any physical quantity Q has a statistical distribution $P_L(Q)$ for a finite system of linear size L. In principle, one should obtain these distributions for each L and then use a finite size scaling for these distributions $P_L(Q)$. In practice, it is hard to obtain the $P_L(Q)$ and usually one calculates numerically an average $\bar{Q}$ of Q over the disorder (see [7] and

references therein). Then one uses the finite size scaling for $\bar{Q}$. This procedure is very reasonable when $\bar{Q} \sim Q_{\text{m.p.}}$. However it is much less satisfactory in cases where $\bar{Q} \neq Q_{\text{m.p.}}$. Moreover, if in some problems, there are several critical points, one where the correlation length diverges, another where the susceptibility diverges etc. . . . , it is clear that the usual finite size scaling cannot work.

There exists a way to avoid the difficulty due to the fact that the physical quantities Q have a probability distribution $P_L(Q)$. One chooses as "the finite system of linear size L", a strip or a bar of width L but infinite in one direction. Because the strip is infinite in one direction, the distribution $P_L(Q)$ is a delta function. Therefore, there is no average to take over disorder. This method has now been used to study several disordered systems (see the talk of J. Vannimenus [8] and references therein).

4. Examples of disordered systems with several phase transitions

We discuss now briefly a few examples where the effect of disorder is to induce several critical points.

4.1. The Ising chain in a random field [9]

The Hamiltonian $\mathcal{H}$ of an Ising chain in a random field is

$$\mathcal{H} = -\sum_i J\sigma_i\sigma_{i+1} - \sum_i h_i\sigma_i \tag{28}$$

where the fields h_i are randomly distributed according to a given distribution $\tilde{\rho}(h_i)$. One can show easily that

$$\lim_{N\to\infty} \frac{1}{N} \log Z = J + \overline{h_i} + F(\varepsilon) \tag{29}$$

where

$$F(\varepsilon) = \lim_{N\to\infty} \frac{1}{N} \log\left[\operatorname{tr} \prod_{i=1}^{N} \begin{pmatrix} 1 & \varepsilon \\ z_i\varepsilon & z_i \end{pmatrix}\right] \tag{30}$$

with $z_i = \exp(-2h_i)$ and $\varepsilon = \exp(-2J)$.

We are now going to study the behavior of $F(\varepsilon)$ in the limit $\varepsilon \to 0$. First, it is easy to see that

$$F(0) = \max(0, \overline{\log z_i}) . \tag{31}$$

Let us now look at the expansion of $F(\varepsilon)$ around $\varepsilon = 0$. If we define $F_1(0)$ and $F_2(0)$ by

$$F_1(0) = \left.\frac{\mathrm{d}F}{\mathrm{d}(\varepsilon^2)}\right|_{\varepsilon^2=0} \tag{32}$$

$$F_2(0) = \left.\frac{\mathrm{d}^2F}{\mathrm{d}(\varepsilon^2)^2}\right|_{\varepsilon^2=0} \tag{33}$$

one finds that $F_1(0)$ is finite only if

$$\bar{z} < 1 \quad \text{or} \quad \overline{z^{-1}} < 1 \tag{34}$$

and $F_1(0)$ diverges if $\bar{z} \to 1^-$ or $\overline{z^{-1}} \to 1^-$. Similarly $F_2(0)$ is finite only if

$$\overline{z^2} < 1 \quad \text{or} \quad \overline{z^{-2}} < 1 \tag{35}$$

and $F_2(0)$ diverges if $\overline{z^2} \to 1^-$ or $\overline{z^{-2}} \to 1^-$.

In other words, let us consider that the distribution $\rho_\lambda(z_i)$ of z_i depends on a parameter λ: for example

$$\begin{cases} \rho_\lambda(z) = 2 & \text{if } \lambda < z < \lambda + \frac{1}{2} \\ \rho_\lambda(z) = 0 & \text{otherwise}. \end{cases} \tag{36}$$

Let us define λ_0 the value of λ for which

$$\overline{\log z} = 0 \tag{37}$$

and λ_p the values of λ for which

$$\overline{z^p} = 1. \tag{38}$$

Then we have the following critical points:

- $F(0)$ is singular at λ_0
- $F_1(0)$ is singular at λ_1 and λ_{-1} and exists only if $\lambda \notin [\lambda_1, \lambda_{-1}]$
- $F_2(0)$ is singular at λ_2 and λ_{-2} and exists only if $\lambda \notin [\lambda_2, \lambda_{-2}]$.

So we see that depending on the quantity we look at: $F(0)$, $F_1(0)$ or $F_2(0)$, we observe different critical points. Of course, if the distribution $\rho_\lambda(z)$ becomes a delta function, i.e. we suppress the disorder, then all these points λ_{-2}, λ_{-1}, λ_0, λ_1 and λ_2 become identical.

4.2. Random walk on a disordered chain

This example exhibits behaviors which are very similar to those of example 4.1. The problem has been studied in probability theory [10] and more recently by physicists [11], in particular in the context of $1/f$ noise [12]. The problem is defined by the following Master equation

$$\mathrm{d}P_n/\mathrm{d}t = W_{n,n+1}P_{n+1} + W_{n,n-1}P_{n-1} - W_{n+1,n}P_n - W_{n-1,n}P_n \tag{39}$$

where P_n is the probability for a particle to be on site n at time t and $W_{n,n+1}\,\mathrm{d}t$ is the probability of jumping from site $n+1$ to site n during the time $\mathrm{d}t$. The $W_{n,n+1}$ are randomly distributed according to a given distribution. Moreover, one does not assume any symmetry and in general, one has

$$W_{n,n+1} \neq W_{n+1,n}. \tag{40}$$

At time t, the different moments $\langle x(t)\rangle$, $\langle x^2(t)\rangle$ etc.... of the position are defined by

$$\langle x(t)\rangle = \sum_n nP_n\,; \qquad \langle x^2(t)\rangle = \sum_n n^2 P_n\,. \tag{41}$$

One can ask several questions:

a. Where is the particle when $t\to\infty$?

b. If the particle goes to infinity, what is its velocity V?

$$V = \lim_{t\to\infty} \frac{\langle x(t)\rangle}{t} \tag{42}$$

c. What is the diffusion constant D?

$$D = \lim_{t\to\infty} \frac{\langle x^2(t)\rangle - \langle x(t)\rangle^2}{2t}\,. \tag{43}$$

Each of these questions lead to define new critical points (or critical surfaces in the space of the distributions of the hopping rates $W_{n,n+1}$).

a. There is a critical point defined by:

$$\overline{\log(W_{n,n+1}/W_{n+1,n})} = 0\,. \tag{44}$$

If $\overline{\log(W_{n,n+1}/W_{n+1,n})}$ is negative (resp. positive) the particle will go to $+\infty$ (rep. $-\infty$) when $t\to\infty$.

b. For the velocity, there are two critical points:

$$\overline{(W_{n,n+1}/W_{n+1,n})} = 1 \qquad \text{or} \qquad \overline{(W_{n+1,n}/W_{n,n+1})} = 1\,. \tag{45}$$

If $\overline{W_{n,n+1}/W_{n+1,n}} < 1$, the particle goes to $+\infty$ with a finite velocity: $\langle x(t)\rangle \sim Vt$. If $\overline{W_{n+1,n}/W_{n,n+1}} < 1$, the particle goes to $-\infty$ with a finite velocity: $\langle x(t)\rangle \sim -Vt$. If $\overline{W_{n,n+1}/W_{n+1,n}} > 1$ and $\overline{W_{n+1,n}/W_{n,n+1}} > 1$, the velocity vanishes. Then one has

$$|\langle x(t)\rangle| \sim t^{|\alpha|} \qquad \text{for large } t \tag{46}$$

where the exponent α is less than 1 and is given by

$$\overline{(W_{n,n+1}/W_{n+1,n})^{\alpha}} = 1\,. \tag{47}$$

c. There are also two critical points for the diffusion constant D:

$$\overline{(W_{n,n+1}/W_{n+1,n})^2} = 1 \qquad \text{or} \qquad \overline{(W_{n+1,n}/W_{n,n+1})^2} = 1\,. \tag{48}$$

The diffusion constant D exists if $\overline{(W_{n,n+1}/W_{n+1,n})^2} < 1$ or $\overline{(W_{n+1,n}/W_{n,n+1})^2} < 1$ and it diverges when one approaches the critical points. All these critical points are very similar to those of the previous example. Again, if we consider a distribution $\rho_\lambda(W_{n,n+1})$ which depends on a parameter λ, when λ changes, one

crosses successive singularities: λ_2 and λ_{-2} given by equations (48) for D, λ_1 and λ_{-1} given by (45) for V, λ_0 given by (44) for the sign of $\langle x(t)\rangle$ in the limit $t\to\infty$.

In higher dimension [13], several studies indicate that there does not remain several critical points.

4.3. *The self-avoiding walk on a dilute lattice*

The problem of self-avoiding walk (SAW) on a random lattice has been studied a lot recently [14]. The conclusions of the different authors disagree and some [15] of them claim that the problem is trivial because some averages are easy to perform. My opinion is that these simple averages do not solve completely the problem which remains open.

For the SAW on a pure lattice, one of the first quantities which can be studied is the number Ω_N of different SAW of N steps starting at a given point. For large N, the behavior of Ω_N on a pure lattice is

$$\Omega_N \sim \mu^N N^{\gamma-1}. \tag{49}$$

If one defines $\chi(x)$ by

$$\chi(x) = \sum_{N=1}^{\infty} x^N \Omega_N \tag{50}$$

then $\chi(x)$ has a power law singularity at the point $x_c = \mu^{-1}$:

$$\chi(x) \sim (x_c - x)^{-\gamma}. \tag{51}$$

For the problem of a SAW on a dilute lattice, whose bonds are present with probability p, one should consider $\omega_N(0, \mathscr{C})$ the number of SAW of N steps starting from point 0 for a configuration $\mathscr{C}$ of the lattice. The number $\omega_N(0, \mathscr{C})$ depends on 0 and $\mathscr{C}$ because the lattice is disordered. Since by definition of a SAW, the walk visits a given bond of the lattice at most once, it is easy to average $\omega_N(0, \mathscr{C})$ over disorder

$$\overline{\omega_N(0, \mathscr{C})} = p^N \Omega_N. \tag{52}$$

One finds by looking at the numerical values of p_c and μ on different lattices that there is a range of concentrations: $\mu^{-1} < p < p_c$ where the lattice does not percolate but where $\overline{\omega_N}$ increases exponentially with N. This is due to the fact that $\omega_N(0, \mathscr{C})$ has very large fluctuations and has typical values very different from its average:

$$\omega_N(0, \mathscr{C})_{\text{m.p.}} \neq \overline{\omega_N(0, \mathscr{C})}. \tag{53}$$

To have an idea of the typical values of $\omega_N(0, \mathscr{C})$ one could try to calculate $\overline{\log(1+\omega_N)}$. One sees that $\chi(x, 0, \mathscr{C})$ defined by

$$\chi(x, 0, \mathscr{C}) = \sum_N x^N \omega_N(0, \mathscr{C}) \tag{54}$$

is very easy to average and has a very simple relation with $\chi(x)$ of the pure lattice

$$\bar{\chi}(x, 0, \mathscr{C}) = \chi(px) . \tag{55}$$

There is a singularity at $x_c = \mu^{-1} p^{-1}$ and one sees no trace of the percolation threshold p_c in this quantity. I think that although $\bar{\chi}$ has a simple expression where p_c does not play any role, other quantities like the average size of the SAW or $\overline{\log(1 + \omega_N)}$ should be singular at p_c. Unfortunately these averages are much more difficult to calculate.

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