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Possible line of critical points for a random field Ising model in dimension 2

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Résumé. — Nous étudions un modèle d'Ising particulier en champ aléatoire. Sur chaque site, le champ aléatoire est soit $+\infty$ avec une probabilité $p/2$, $-\infty$ avec une probabilité $p/2$ ou 0 avec une probabilité $1-p$. En utilisant des arguments de lois d'échelle des systèmes finis, nous montrons que pour p petit, la fonction de corrélation moyenne de deux spins à une distance R décroît comme $R^{-\eta(p)}$ où l'exposant $\eta(p) = 2\pi p + \mathcal{O}(p^2)$. Les hypothèses faites pour obtenir ce résultat et les généralisations possibles à d'autres modèles d'Ising en champ aléatoire sont discutées.

Abstract. — We study a particular random field Ising model in dimension 2 at 0 temperature. On each site the random field is either $+\infty$ with probability $p/2$, $-\infty$ with probability $p/2$ or 0 with probability $1-p$. Using finite size scaling arguments, we show that for small p, the average correlation function between two spins at distance R decreases like $R^{-\eta(p)}$ where the exponent $\eta(p) = 2\pi p + \mathcal{O}(p^2)$. The assumptions made to obtain this result and the possible generalizations to other random field models are discussed.

The random field Ising model [1-5] (RFIM) has been for a long time a controversial subject. The question of the lower critical dimensionality d_ℓ (above which ferromagnetic order can exist) has been much debated between those [2] who claim that $d_\ell = 2$ according to the Imry-Ma [1] picture and those [3] who assert that $d_\ell = 3$ is a direct consequence of the dimensionality shift $d \to d-2$ (a ferromagnetic spin model in a random magnetic field in dimension d has the same exponents as the same model without a random field in dimension $d-2$). The controversy is now beginning to be resolved because recent studies [4] of the properties of an interface in presence of a random field give more confidence in the fact that $d_\ell = 2$ whereas some difficulties [5] have been found in the arguments which gave $d \to d-2$. Anyhow, even if one accepts that $d_\ell = 2$, the properties of the RFIM right at $d = 2$ are not at all clear.

The purpose of the present Letter is to give arguments in favour of a Kosterlitz-Thouless phase at zero temperature in dimension 2 for a random field Ising model. The model is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i \tag{1}$$

where J is the ferromagnetic interaction, $\sum_{\langle ij \rangle}$ denotes the sum over nearest neighbours on a square lattice, the spins σ_i are Ising spins and h_i is the random field on the ith site. The peculiarity of this model lies in the probability distribution $\rho(h_i)$ of the field h_i : the random field can only be 0, $+\infty$ or $-\infty$:

$$\rho(h_i) = (1-p)\,\delta(h_i) + \frac{p}{2}\left[\delta(h_i+\infty) + \delta(h_i-\infty)\right]. \tag{2}$$

This model is in several aspects simpler than other random field models : for example it can be solved exactly in one dimension [6] at any temperature and for any value of p whereas other random field models remain unsolved in 1D at finite temperature.

Our study of the 2D case at 0 temperature consists of two steps. First, we calculate the correlation length $\xi_n(p)$ for $p \ll 1$ on an infinite strip of finite width n. We obtain that $\xi_n(p)$ increases linearly with n for large n

$$\xi_n \sim An \tag{3}$$

and we calculate the constant A for small p.

Then we use finite size scaling [7-9] which tells us that if $\xi_n(p)$ increases linearly with n, the 2D system is at criticality.

We shall calculate the critical exponent η which characterizes the power law decrease of the average spin-spin correlation function at criticality

$$\overline{\langle \sigma_0 \sigma_R \rangle} \sim R^{-\eta} \quad \text{for large } R \tag{4}$$

by using a relation [8, 9] between the exponent η and the amplitude A of equation (3).

Let us start by calculating $\xi_n(p)$ for a strip of width n with periodic boundary conditions in the limit $p \ll 1$. $\xi_n(p)$ is defined here by the exponential decrease of the average [10] correlation function $\overline{\langle \sigma_0 \sigma_L \rangle}$ between two spins at distance L along the strip

$$\overline{\langle \sigma_0 \sigma_L \rangle} \sim \exp\left(-\frac{L}{\xi_n(p)}\right). \tag{5}$$

Consider a strip of width n with $L+1$ columns numbered from 0 to L. Let us fix once and for all, the spins of column 0 to be $+$. We define $F_+(L)$ the number of unsatisfied bonds of the strip in its ground state if we take all the spins of column L to be $+$. $F_+(L)$ is simply related to the ground state energy.

Similarly, let us denote by $F_-(L)$ the number of unsatisfied bonds in the ground state if all the spins of column L are $-$. Since on each site between columns 1 and $L-1$, the field is randomly distributed according to (2), the difference $\Delta(L) = F_+(L) - F_-(L)$ has a certain probability distribution $Q_L(\Delta)$. When $L \to \infty$, $Q_L(\Delta)$ converges exponentially towards a limit probability distribution $Q_\infty(\Delta)$ and this exponential convergence gives the correlation length $\xi_n(p)$

$$Q_L(\Delta) - Q_\infty(\Delta) \sim \exp -\frac{L}{\xi_n}. \tag{6}$$

It is easy to see that (5) and (6) indeed define the same length ξ_n. The reason is that the correlation function $\overline{\langle \sigma_0 \sigma_L \rangle}$ is just given by

$$\overline{\langle \sigma_0 \sigma_L \rangle} = \sum_{\Delta<0} Q_L(\Delta) - \sum_{\Delta>0} Q_L(\Delta). \tag{7}$$

We shall use (6) to calculate the correlation length $\xi_n(p)$ in the limit $p \ll 1$. To do so, we need to make some remarks about the structure of the ground state. Because $p \ll 1$, the distance between two spins in an infinite field is very large. The consequence is that the ground state is composed of a succession of positive and negative domains along the strip. The frontier between 2 successive domains is always a straight interface which cuts only n bonds a cross the strip. In addition to that, each spin with an infinite field which belongs to a domain with a wrong sign. costs only 4 unsatisfied bonds. So in the limit $p \ll 1$ the ground state of an infinite strip is composed of a succession of very long domains separated by straight interfaces perpendicular to the strip and inside the domains there are isolated spins with an infinite field opposite to the sign of the domain. Keeping this structure in mind, one can write the recursion relation for $\Delta(L)$. With probability $1 - np$, all sites at column L have a zero field, therefore $\Delta(L+1) = \Delta(L)$. With probability $np/2$, there is one site at column L with $h_i = +\infty$, and thus $\Delta(L+1) = \max(\Delta(L) - 4, -n)$. This comes from the fact that $F_+(L+1) = F_+(L)$ and $F_-(L+1) = \min(F_-(L) + 4, F_+(L) + n)$ because the system has to choose the lowest energy between an isolated spin + in a domain − which costs 4 unsatisfied bonds or a frontier between column L and column $L+1$ which costs n unsatisfied bonds. Similarly with probability $np/2$, there is one site at column L with $h_i = -\infty$ and then $\Delta(L+1) = \min(\Delta(L) + 4, n)$. We see that Δ makes a random walk constrained to remain between n and $-n$ and the only allowed values of Δ are $\pm(n - 4K)$ with K integer. One can easily write the recursion relation for $Q_L(\Delta)$:

$$\text{if } |\Delta| < n \quad Q_{L+1}(\Delta) = (1 - np)\, Q_L(\Delta) + \frac{np}{2} Q_L(\Delta - 4) + \frac{np}{2} Q_L(\Delta + 4) \tag{8}$$

$$\text{if } \Delta = \varepsilon n \quad Q_{L+1}(\varepsilon n) = \left(1 - \frac{np}{2}\right) Q_L(\varepsilon n) + \frac{np}{2} Q_L(\varepsilon(n-2)) + \frac{np}{2} Q_L(\varepsilon(n-4)) \tag{9}$$

where $\varepsilon = 1$ or $\varepsilon = -1$.

From (8) and (9) we calculate the correlation length ξ_n given by (6). For an odd width $n \geqslant 3$, one finds

$$\exp\left(-\frac{1}{\xi_n}\right) = 1 - np + np \cos\left(\frac{2\pi}{n+3}\right)$$

which means that

$$\xi_n^{-1} = np\left[1 - \cos\left(\frac{2\pi}{n+3}\right)\right] \tag{10}$$

since $p \ll 1$. For even $n \geqslant 4$, one finds similarly

$$\xi_n^{-1} = np\left[1 - \cos\left(\frac{2\pi}{n+2}\right)\right]. \tag{11}$$

The cases $n = 1$ or $n = 2$ are irrelevant here since we are only interested in the large n behaviour. We want to emphasize that the whole calculation presented until now is valid only if $p \ll 1$. We even need $np \ll 1$ for our picture of the ground state to be true. Formulas (10) and (11) give exactly the linear term in the small p expansion of ξ_n^{-1}. The remarks made on the structure of the ground state and the fact that we consider on each column only 2 configurations (the spins are either all + or all −) are only valid to first order in p. To calculate the order p^2 in the expansion of ξ_n^{-1}, one should take into account situations where pairs of spins in an infinite field are at distances of the order of the strip width.

To derive equations (10) and (11), we used the fact that the ground state has a simple structure when $np \ll 1$. It is not obvious that (10) and (11) remain valid in the whole region $p \ll 1$ including $np \gg 1$. One can argue that in several examples, such as low temperature expansions for ferromagnets, the simplest way of deriving the expansion is to consider configurations with a finite number of overturned spins whereas the expansions remain valid for finite densities of overturned spins. In these examples, this procedure is perfectly justified. For the problem studied here, we did not find any convincing argument to prove or to disprove the validity of (10) and (11) for $np \gg 1$. One way to attack the problem could be to calculate the order p^2 of $\xi_n^{-1}(p)$. If this term is proportional to n^2 for large n, this means that (10) and (11) are wrong for $np \gg 1$. On the contrary if the order p^2 is linear in n, this would strengthen the validity of (10) and (11). Unfortunately we do not know for the moment how to calculate this order p^2 for all widths n. In the following, we shall assume that (10) and (11) remain valid for $p \ll 1$ even if $np \gg 1$.

It is then easy to see in (10) and (11) that for $p \ll 1$, the ξ_n have the behaviour (3) with A given by

$$A^{-1} = 2\,\pi^2\,p\,. \tag{12}$$

Let us now use finite size scaling. The first thing that finite size scaling [7-9] tells us is that the 2D system is at criticality if the correlation length $\xi_n(p)$ increases linearly with n. Here we find that ξ_n increases linearly with n in the region $p \ll 1$. Therefore we conclude that for p small enough there is a line of critical points.

The second thing is a relation between the coefficient A and the exponent η. It has been observed [8-9] for a large class of 2 dimensional models including the Ising, the Potts, the XY and the Baxter model that the coefficient A is related to the exponent η (see Eqs. (3) and (4)) in the following way

$$A^{-1} = \pi\eta\,. \tag{13}$$

There does not yet exist a completely satisfactory proof of this relation (see Refs. [8] and [9]). If we assume that (13) is also valid for the RFIM studied here, we get

$$\eta = 2\,\pi p + \mathcal{O}(p^2)\,, \tag{14}$$

since A^{-1} is known only up to order p. This way of calculating the small p expansion of η from the expansion of A is very similar to what has been done for the 2 dimensional XY model at low temperature [9].

In this work, we have calculated the $\xi_n(p)$ for $p \ll 1$. Under the assumptions that finite size scaling holds and that relation (13) holds also for this RFIM, we have found a line of critical points for small p and we have the expression of η for small p.

It would be interesting to continue the small p expansion of the $\xi_n(p)$ in order, to confirm the fact that $\xi_n(p)$ increases linearly with n, to have more terms in the expansion of η and to see the value of the end point p^* of the line of fixed points. It would also be interesting to generalize the results of this letter to more usual distributions of random fields, like a Gaussian of width $\overline{h^2}$. It is reasonable [11] to think that for $\overline{h^2} \ll J^2$, one has $\xi_n^{-1} \sim \overline{h^2}/n$. This would mean that $\eta \sim \overline{h^2}$ but the calculation of the coefficient is more difficult because Δ can now take a continuous set of values. We hope to make some progress on these aspects in the future.

Lastly, it would be also interesting to consider the probability distribution of the correlation functions since the average correlation functions do not always contain the whole information and the most probable correlation functions may have different behaviours [10].

Apart the question of the lower critical dimension and the possibility of a line of fixed points at $d = 2$, there remain several aspects to be understood in the random field Ising model : the complicated low temperature phase even in high dimension [12] or the possibility of Griffiths [13] singularities even when the average field is not zero [14].

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