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LETTER TO THE EDITOR

On the Harris criterion for hierarchical lattices

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Abstract. We discuss the condition necessary for the Harris criterion to be valid on hierarchical lattices. We prove that disorder is always relevant when the specific heat exponent, α_p, is positive and show how to construct lattices for which disorder becomes relevant while α_p is still negative.

Hierarchical lattices (Berker and Ostlund 1979) have recently received a lot of attention (see Kaufman and Griffiths 1984 and references therein). This is because many statistical mechanical models defined on these lattices can be solved rather easily although they exhibit non-classical critical behaviour. Because of the success in treating pure systems several authors have attempted to see to what extent the properties of disordered systems on hierarchical lattices can be elucidated (Kinzel and Domany 1981, Gardner 1984, Collet *et al* 1984, Andelman and Berker 1984).

In the study of disordered systems, one of the first questions that can be asked is how can the presence of weak disorder modify the nature of a phase transition. The answer to this question is given by the Harris criterion (Harris 1974), which states that if the specific heat exponent of the pure system, α_p, is negative then weak disorder does not change the critical exponents (in renormalisation group language, disorder is irrelevant at the pure fixed point). If α_p is positive, however, the critical exponents should be changed by the presence of weak disorder (disorder is relevant).

The purpose of the present note is to show that it is easy to construct hierarchical lattices for which disorder is relevant even when α_p is strictly negative. We shall also prove that when α_p is positive, disorder is always relevant.

Let us start by looking at two simple hierarchical lattices constructed from the renormalisation transformations shown in figure 1. We shall consider a q-state Potts

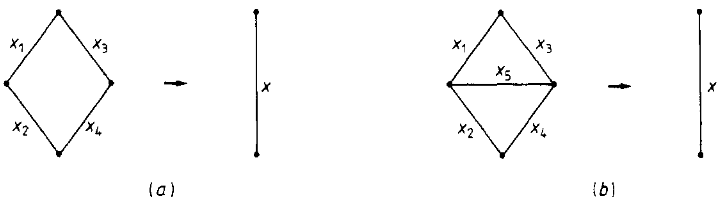

Figure 1. The renormalisation transformations for two hierarchical lattices on which (a) disorder becomes relevant at $\alpha_p = 0$, (b) disorder becomes relevant for $\alpha_p > -0.0525$.

model (Derrida and Gardner 1984) defined on these lattices and described by the Hamiltonian

$$\mathscr{H} = -\sum_{\langle ij \rangle} J_{ij}\delta_{\sigma_i\sigma_j}, \qquad \sigma_i = 1, 2, \ldots, q \tag{1}$$

where the sum extends over all pairs of nearest neighbours on the lattice. It will be convenient to use the variable

$$x = e^{\beta J}. \tag{2}$$

At each step of the renormalisation transformation, a set of n bonds $x_1, x_2, \ldots, x_n$ is replaced by a single bond x

$$x = F(x_1, x_2, \ldots, x_n). \tag{3}$$

The pure critical point, x_c, is given by

$$x_c = F(x_c, x_c, \ldots, x_c). \tag{4}$$

Defining

$$a_i = (\partial F/\partial x_i)(x_c, x_c, \ldots, x_c) \tag{5}$$

the specific heat exponent of the pure fixed point, α_p, obeys the equation (Derrida and Gardner 1984)

$$\left(\sum_{i=1}^{n} a_i\right)^{2-\alpha_p} = n. \tag{6}$$

The critical fixed point, x_c, is unstable and hence the thermal eigenvalue is greater than unity

$$\sum_{i=1}^{n} a_i > 1. \tag{7}$$

Therefore α_p is positive if and only if

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 > 1. \tag{8}$$

For the Potts model on the hierarchical lattices we consider (and also on Bravais lattices) α_p is negative for small q and increases with increasing q, passing through zero at a value which we shall denote q_c.

For the Potts model on the first hierarchical lattice shown in figure 1(a)

$$x = F_1(x_1, x_2, x_3, x_4) = \left(\frac{x_1x_2 + q - 1}{x_1 + x_2 + q - 2}\right)\left(\frac{x_3x_4 + q - 1}{x_3 + x_4 + q - 2}\right) \tag{9}$$

whereas for the second lattice (figure 1(b)) the renormalisation formula is

$$x = F_2(x_1, x_2, x_3, x_4, x_5)$$
$$= \frac{x_1x_2x_3x_4x_5 + (q-1)(x_1x_2 + x_3x_4 + x_5) + (q-1)(q-2)}{x_1x_3x_5 + x_2x_4x_5 + x_1x_4 + x_2x_3 + (q-2)(x_1 + x_2 + x_3 + x_4 + x_5) + (q-2)(q-3)}. \tag{10}$$

For our purposes the important difference between the two equations is that in (9) the four variables, x_i, $i = 1, \ldots, 4$, play symmetric roles, and hence the corresponding a_i are equal, whereas in (10) x_5 plays a special role (as is immediately apparent from the symmetry of figure 1(b)) and $a_1 = a_2 = a_3 = a_4 \neq a_5$. We shall now show that when all the a_i are equal, disorder becomes relevant when α_p changes sign (this result is derived in a different way by Da Cruz and Stinchcombe 1985). If, however, all the a_i are not equal, there is a range of values of q for which disorder is relevant even when α_p is negative.

To test whether disorder is relevant, it is sufficient to follow the width of a narrow distribution of bond strengths, x_i, concentrated around x_c, under the renormalisation group transformation (Derrida and Gardner 1984). We write

$$x_i = x_c + \varepsilon_i, \qquad x = x_c + \varepsilon, \tag{11}$$

$$\varepsilon = \sum_{i=1}^{n} a_i \varepsilon_i \tag{12}$$

for small ε_i and ε. Because the ε_i are independent random variables with zero mean,

$$\langle \varepsilon^2 \rangle = \left(\sum_{i=1}^{n} a_i^2 \right) \langle \varepsilon_i^2 \rangle. \tag{13}$$

Therefore disorder is relevant if and only if

$$\sum_{i=1}^{n} a_i^2 > 1. \tag{14}$$

Clearly one always has (Schwartz inequality)

$$\sum_{i=1}^{n} a_i^2 \geqslant \frac{1}{n} \left(\sum_{i=1}^{n} a_i \right)^2 \tag{15}$$

where the equality only holds when all the a_i are equal. Therefore, from (8), (14) and (15), when $\alpha_p > 0$ disorder is always relevant. When disorder is relevant, however, α_p can take either sign.

For the first lattice we consider (figure 1(*a*)) all the a_i are equal. Hence disorder becomes relevant at $q_c = 4 + 2\sqrt{2}$, where $\alpha_p = 0$. For the second lattice (figure 1(*b*)), however, the a_i are unequal. Using (10), a short calculation shows that $q_c = 5.9109$ but that disorder becomes relevant for $q_c > q > 5.302$ and $\alpha_p > -0.0525$.

In this letter, we have shown that for the Harris criterion to be valid on a hierarchical lattice, all the bonds in the elementary cluster must play symmetric roles in the renormalisation transformation. If this is not the case, disorder always becomes relevant when α_p is still strictly negative.

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