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A combination of Monte Carlo and transfer matrix methods to study 2D and 3D percolation

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Résumé. — Nous présentons une méthode qui combine les idées de matrice de transfert et de Monte Carlo pour étudier le problème de la percolation de site en dimension 2 et 3. Nous utilisons cette méthode pour calculer les propriétés de rubans (2D) et de barreaux (3D). En utilisant les lois d'échelle des systèmes finis, nous obtenons des estimations du seuil et des exposants qui confirment des valeurs déjà connues. Nous discutons les avantages et les limitations de notre méthode en la comparant avec les calculs Monte Carlo habituels.

Abstract. — In this paper we develop a method which combines the transfer matrix and the Monte Carlo methods to study the problem of site percolation in 2 and 3 dimensions. We use this method to calculate the properties of strips (2D) and bars (3D). Using a finite size scaling analysis, we obtain estimates of the threshold and of the exponents which confirm values already known. We discuss the advantages and the limitations of our method by comparing it with usual Monte Carlo calculations.

1. Introduction.

Finite size scaling has become a widely used method to study numerically critical phenomena [1]. It consists in calculating the properties of finite systems (like squares or cubes) or of systems finite in all directions but one (strips or bars) and in extracting the critical behavior of infinite systems by looking at the size dependence of the physical properties.

There exist presently two main approaches to calculate the properties of finite systems. One is the Monte Carlo method [2] which can be used in a lot of situations (in particular in any dimension) but has the disadvantage that all the results are obtained with statistical errors : these error bars are expected to decrease only like the inverse of the square root of the computer time. The other method is the transfer matrix method [3] which allows to calculate with a very high accuracy the properties of finite systems or of strips but has the disadvantage that the size of the transfer matrix increases usually exponentially with the physical size of the system. For that reason, most of the works using transfer matrices were restricted to two dimensions.

For two dimensional systems, although transfer matrices limit strongly the size of the systems which can be studied, the fact that the properties of these finite systems can be calculated with a high accuracy gives estimates of the critical properties of infinite systems which are as good as, and sometimes better than, the results obtained from Monte Carlo simulations on much bigger systems. It is also possible that the finite size scaling is valid for smaller sizes when one chooses strip or bar geometries (transfer matrices) than when one chooses square or cube geometries (Monte Carlo).

The idea of trying the transfer matrix method to study 3 dimensional systems has already been proposed. For the 3d Ising model, it seems impossible to calculate the properties of bars of width larger than 5. The width 5 is already a very hard calculation [4]. Other 3 dimensional problems like localization [5] and random resistor networks [6] were also studied by using transfer matrices. In these cases, there is presently no way of calculating exactly the properties of narrow strips or bars. Thus the transfer matrix method is used to calculate the properties of a very long sample on which the values of the potentials (in the localisation problem) or of the resistors (in the conductivity problem) are chosen at random. If the sample were infinitely long, the results would be exact. However the typical length of bars which can be achieved is 10^5-10^8 and all the results are known with statistical errors. For the localisation problem and for the conductivity problem, it is possible to treat bars of width up to 20. Therefore it is possible to study bars of larger widths than for the Ising model but with the drawback of statistical errors.

In this paper, we present a way of extending the ideas which were used for the random resistor network problem to the study of 2d and 3d percolation. The main motivation is the following : since there is no hope to study exactly bars of width larger than 5 for the Ising or the percolation problem, we want to see whether a combination of transfer matrix ideas and of statistical methods can be used to study bars of larger widths. In this paper, we shall concentrate our effort on the site percolation problem because it is the easiest problem which can be studied by Monte Carlo methods : the configurations are generated at once with their good weight and there is no relaxation time to reach equilibrium. We shall describe a method which allows us to calculate the properties of very long strips or bars and then use the finite size scaling to estimate the percolation threshold and the critical exponents.

The paper is organised as follows. In section 2 we explain why we choose to calculate the moments of the sizes of clusters instead of the correlation length and we discuss how finite size scaling can be used to calculate the critical properties of the percolation problem. In section 3 we test the method in the case of the 2D and 3D Ising models for which we calculate, by the usual transfer matrix method, the second and the fourth derivatives of the free energy with respect to the magnetic field. Using the finite size scaling method described in section 2, we obtain estimates of the critical properties of the Ising model. In section 4, we make a standard Monte Carlo calculation of site percolation on squares and then we use the ideas of section 2 to calculate p_c and the exponents. In sections 5 and 6 we describe a combination of Monte Carlo and transfer matrix methods to calculate the properties of strips and bars. We then use the scaling of section 2 to estimate again p_c and the exponents. In the conclusion, we discuss the advantages and the limitations of the different approaches.

2. Finite size scaling.

For a lot of models, the easiest way of using finite size scaling is to calculate the correlation length ξ on strips of finite width and then write a phenomenological renormalization equation [3]. This was already done for the 2d percolation [7, 8] and led to rather accurate estimates of p_c and ν.

The correlation length ξ on strips is defined by the exponential decrease of the probability P_L that 2 sites at distance L along the strip are connected

$$P_L \sim \exp - L/\xi . \tag{1}$$

If one wants to calculate this length ξ by a Monte Carlo method, one needs to generate a lot of samples. For most of the samples the two points will not be connected and only very few events will give a non-zero contribution to P_L. Since P_L decreases exponentially with L, the number of samples should increase exponentially with L in order to have P_L with statistical errors independent of L. One can probably overcome this difficulty by introducing a bias but this makes the calculations more complicated.

In order to avoid the difficulty of calculating the correlation length, we decided to calculate other quantities for which the finite size scaling is simple. We have calculated the moments of the distribution of clusters. For a square of size $N \times N$, let us define $\langle M_2 \rangle$ and $\langle M_4 \rangle$ by

$$\langle M_2 \rangle = \left\langle \sum_i i^2 n_i \right\rangle \tag{2}$$

$$\langle M_4 \rangle = \left\langle \sum_i i^4 n_i \right\rangle \tag{3}$$

where n_i is the number of clusters of i sites of a square sample of linear size N and the averages are taken over a large number of samples. For each size N we shall calculate $\langle M_2 \rangle$ and $\langle M_4 \rangle$ as a function of the concentration p of occupied sites.

Similarly for a strip or a bar of width N and infinite in one direction one can define the moments m_2 and m_4 by

$$m_2 = \sum_i i^2 \tilde{n}_i \tag{4}$$

$$m_4 = \sum_i i^4 \tilde{n}_i \tag{5}$$

where the $\tilde{n}_i$ are the numbers of clusters of size i per unit length on a strip or on a bar. In principle one does not need to average in (4) and (5) since strips and bars are infinite in one direction and therefore the $\tilde{n}_i$ do not fluctuate.

The calculation of similar moments was already used in Monte Carlo studies of the 2D and 3D Ising models [9, 10] : in these works the moments $\langle M_2 \rangle$ and $\langle M_4 \rangle$ were calculated, M being the total magnetization of a square or a cube of linear size N. Exactly as in the Ising case one expects that for N large and $p - p_c \ll 1$, the moments M_2 and M_4 have a finite size scaling form

$$\langle M_2 \rangle \simeq N^{d+\gamma/\nu} F_2(N^{1/\nu}(p - p_c)) \tag{6}$$

$$\langle M_4 \rangle \simeq N^{2d+2\gamma/\nu} F_4(N^{1/\nu}(p - p_c)) . \tag{7}$$

For strips ($d = 2$) and bars ($d = 3$) since m_2 and m_4 are defined per unit length, the finite size scaling takes a slightly modified form :

$$m_2 \simeq N^{\gamma/\nu+d-1} G_2(N^{1/\nu}(p - p_c)) \tag{8}$$

$$m_4 \simeq N^{2\gamma/\nu+2d-1} G_4(N^{1/\nu}(p - p_c)) \tag{9}$$

From formulae (6) and (7), one can find several ways of estimating p_c and ν.

One method [9, 10] consists in drawing the ratio $\langle M_4 \rangle / \langle M_2 \rangle^2$ as a function of p for several N. If (6) and (7) were valid for every N, all the curves should intersect at the same point p_c. However, since finite size scaling is expected to work only for N large, the intersections will depend on N and it is only in the limit $N \to \infty$ that one shall get p_c. From the derivative of the ratio, at p_c one can also estimate ν :

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\langle M_4 \rangle}{\langle M_2 \rangle^2}\right) \sim N^{1/\nu}. \tag{10}$$

With this first method, one has an estimate of p_c and ν by comparing two sizes N and N'. A disadvantage of this method is that it requires the calculation of $\langle M_4 \rangle$. Usually the higher moments are dominated by the contribution of big, but rare clusters : therefore one needs the calculation to be rather long in order to produce enough of these big clusters. However, since $\langle M_2 \rangle$ and $\langle M_4 \rangle$ are calculated on the same samples, they are correlated and the error bar of the ratio $\langle M_4 \rangle / \langle M_2 \rangle^2$ is reduced as it is the case in the Monte Carlo calculations of the Ising model [10].

A second method consists in comparing the $\langle M_2 \rangle$ for 3 different sizes. At p_c, one expects from (6) that $\langle M_2 \rangle$ increases like a power of N. A possible way of finding p_c is to draw the ratios $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$ *versus* p which should be equal to $(2)^{\frac{\gamma}{\nu}+d}$ at p_c and to look for the points where all these curves intersect. Again since (6) is only valid for $N \gg 1$, the estimate of p_c will become better and better as one increases N. Once p_c is estimated, the slopes of the curves will give ν and the value of the ratio at the intersection will give γ/ν. With this method, one needs only to calculate the moments $\langle M_2 \rangle$ which are less sensitive to big clusters. However, its disadvantage compared with the previous method is that it requires the comparison of 3 different sizes.

Lastly since the values of the critical exponents of percolation are known (or at least conjectured) in two dimensions, one can use this information to get better estimates of p_c : one knows [11, 12] the value of $2^{(\gamma/\nu)+1}$ and therefore one can estimate p_c by looking at the value of p where $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$ crosses this value.

It is easy to generalize each of these methods to the case of strip or bar geometries. The only difference is that the ratio $\langle M_4 \rangle / \langle M_2 \rangle^2$ has an extra factor N and becomes $m_4/[(m_2)^2 N]$.

3. The 2D and 3D Ising model.

In this section we present the results obtained by using the first method in the case of strips and bars for the usual Ising model.

We consider an Ising model on a square (2D) or a cubic (3D) lattice in a uniform field h whose Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \Sigma \sigma_i. \tag{11}$$

The free energy f per site on a strip ($d = 2$) or on a bar ($d = 3$) of width N is given by

$$f = -\frac{T}{N^{d-1}} \log(\lambda(J, h)) \tag{12}$$

where λ is the largest eigenvalue of the transfer matrix. Let us call $K = J/T$.

From the knowledge of λ, it is easy to calculate m_2 and m_4 :

$$m_2 = \frac{\mathrm{d}^2}{\mathrm{d}h^2} \log(\lambda(J, h)) \Big|_{h=0} \tag{13a}$$

$$m_4 = \frac{\mathrm{d}^4}{\mathrm{d}h^4} \log(\lambda(J, h)) \Big|_{h=0} \tag{13b}$$

where m_2 and m_4 are related to the total magnetization M of a strip of length L by

$$m_2 = \lim_{L\to\infty} \langle M_2 \rangle / L \tag{14a}$$

$$m_4 = \lim_{L\to\infty} \frac{\langle M_4 \rangle - 3\langle M_2 \rangle^2}{L}. \tag{14b}$$

Technically, the largest eigenvalue λ of the transfer matrix $m(K, h)$ can easily be obtained by constructing the sequence of vector $\mathbf{V}_n(K, h)$ defined by

$$\mathbf{V}_n(K, h) = m\mathbf{V}_{n-1}(K, h) \tag{15}$$

where $\mathbf{V}_0$ is chosen arbitrarily and by using the fact that

$$\lambda = \lim_{n\to\infty} \frac{\| \mathbf{V}_n \|}{\| \mathbf{V}_{n-1} \|}. \tag{16}$$

In (13) and (14) one needs derivatives of λ with respect to h. Numerically it is always a problem to obtain derivatives with accuracy. To avoid this difficulty, we did a perturbation theory. We wrote :

$$\mathbf{V}_n(K, h) = \\ = \mathbf{W}_n^{(0)}(K) + h\mathbf{W}_n^{(1)}(K) + \cdots + h^4 \mathbf{W}_n^{(4)}(K) \tag{17}$$

and

$$m(K, h) = \\ = \mathcal{M}^{(0)}(K) + h\mathcal{M}^{(1)}(K) + \cdots + h^4 \mathcal{M}^{(4)}(K) \tag{18}$$

and we wrote the recursion formula (15) order by order in h. This technique, which was also used to calculate susceptibilities in another problem [13], gives the derivatives of λ with the same accuracy as λ itself.

Knowing the values of m_2 and m_4 as a function of K for each N, we could calculate the ratios

$$R_N(K) = m_4/m_2^2 \quad (19)$$

and we could get estimates of K_c by solving the following equation

$$R_N(K)/N = R_{N-1}(K)/(N - 1) . \quad (20)$$

In principle one can compare two arbitrary sizes N and N' but in order to have the best convergence when N and N' increase, the best choice is $N' = N - 1$ (see Ref. [8]).

The values of $K_c^{(N)}$ obtained by solving equation (20) are given in tables I and II for the 2d and 3d Ising model.

Once $K_c^{(N)}$ was calculated, we could estimate ν and γ/ν using the following formula

$$1 + \frac{1}{\nu} = \log\left(\frac{\frac{dR_N}{dK}(K_c^{(N)})}{\frac{dR_{N-1}}{dK}(K_c^{(N)})}\right) \Big/ \log\left(\frac{N}{N-1}\right) \quad (21)$$

and

$$d - 1 + \frac{\gamma}{\nu} = \frac{\log\left(\frac{m_2^{(N)}(K_c^{(N)})}{m_2^{(N-1)}(K_c^{(N)})}\right)}{\log\left(\frac{N}{N-1}\right)} . \quad (22)$$

The results are given in tables I and II.

One sees that in 2 dimensions the results converge quickly to the exact known values and these calculations using the ratio m_4/m_2^2 are as good as the calculations using the correlation length [14].

In 3 dimensions, the results are not very far from those expected but the sizes ($N \leqslant 4$) are too small to allow any extrapolation.

4. The site percolation on squares.

In this part, we report the results of a standard Monte Carlo calculation done for the site percolation problem on squares of linear size N with periodic boundary conditions.

Squares were generated with a pseudo random number generator used by Ogielsky [15, 16] and we used a known algorithm to calculate the sizes of all the clusters of each sample [17]. For several sizes N, we give the values of $\langle M_2 \rangle$ and $\langle M_4 \rangle$ and of the ratio $\langle M_4 \rangle / \langle M_2 \rangle^2$ as a function of p in tables IA and IB of the Appendix.

These quantities were averaged over 10^5 samples for $N \leqslant 16$ and 10^4 for $N \geqslant 24$.

To estimate the error bars on $\langle M_2 \rangle$ and $\langle M_4 \rangle$ we divided our samples into typically 10 subsets. We calculated the averages for each subset and then estimated the variance assuming that the distribution

Table I. — *Estimates of $K_c^{(N)}$, $\frac{\gamma}{\nu}$ and $\frac{1}{\nu}$ obtained using formulae (20), (21) and (22) for various sizes N for the 2d Ising model. When N increases, the results converge well to the exact known values.*

N	$N - 1$	$K_c^{(n)}$	$\frac{\gamma}{\nu}$	$\frac{1}{\nu}$	$\frac{1}{6N}\left(\frac{m_4}{m_2^2}\right)$
2	1	0.445988	1.83469	1.108	− 1.1517
3	2	0.425934	1.76240	1.120	− 1.0450
4	3	0.432417	1.74908	1.082	− 1.0983
5	4	0.436715	1.74940	1.051	− 1.1488
6	5	0.438689	1.75066	1.033	− 1.1793
7	6	0.439579	1.75124	1.023	− 1.1963
8	7	0.440012	1.75138	1.017	− 1.2061
9	8	0.440244	1.75135	1.013	− 1.2121
10	9	0.440380	1.75126	1.010	− 1.2161
11	10	0.440466	1.75116	1.008	− 1.2189
12	11	0.440522	1.75106	1.005	− 1.2210
exact	values	0.440687	1.75	1.0	

Table II. — *The same as table* I *in dimension* 3. *The expected values are those given in reference* [10] *or* [24].

N	$N - 1$	$K_c^{(n)}$	$\frac{\gamma}{\nu}$	$\frac{1}{\nu}$	$\frac{1}{6N}\left(\frac{m_4}{m_2^2}\right)$
2	1	0.231958	2.66960	1.654	− 0.69034
3	2	0.185454	2.21046	1.933	− 0.46618
4	3	0.192301	1.99987	1.769	− 0.53127
Expected values	[10]	0.2216	1.98 ± 0.02	$[0.63]^{-1} = 1.59$	
Renormalization group prediction	[24]		1.97 ± 0.004	1.59 ± 0.005	

of these subset averages were Gaussian. To estimate $\langle M_4 \rangle / \langle M_2 \rangle^2$, the strictly correct procedure is to combine the final averages of M_2 and M_4. This gave, within the error bars, the same result as averaging these ratios over the 10 subsets. This procedure is a better way of estimating the error bars on $\langle M_4 \rangle / \langle M_2 \rangle^2$ than by simply combining the error bars on $\langle M_4 \rangle$ and $\langle M_2 \rangle$ because the fluctuations of these two quantities are highly correlated.

All these error bars are also given in table IA, IB of the Appendix. (As they are only crude estimates we have just given one value for all values of p.)

As discussed in section 2, we can now use several approaches to estimate p_c, γ and ν from these data.

The first one consists in drawing the curves $\langle M_4 \rangle / \langle M_2 \rangle^2$ *versus* p for each size N. The finite size scaling tells us that these curves for different N should intersect at p_c. In figures 1A and 1B, we have

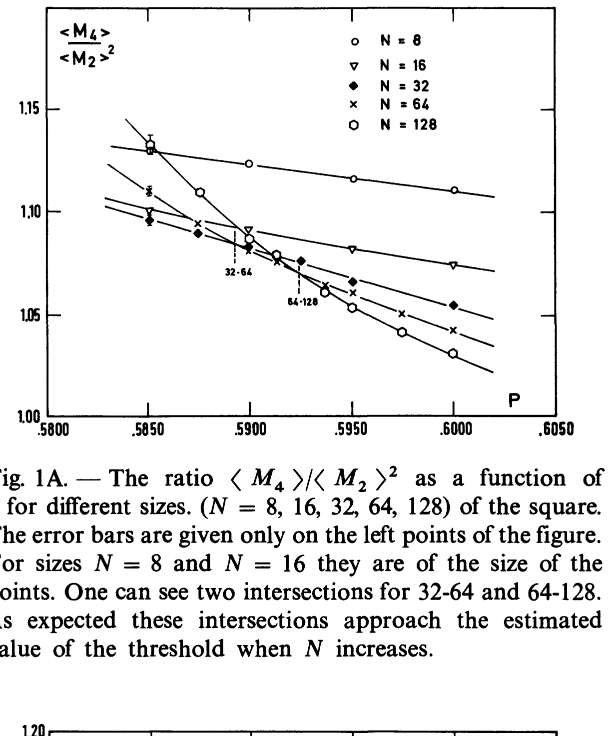

Fig. 1A. — The ratio $\langle M_4 \rangle / \langle M_2 \rangle^2$ as a function of p for different sizes. (N = 8, 16, 32, 64, 128) of the square. The error bars are given only on the left points of the figure. For sizes $N = 8$ and $N = 16$ they are of the size of the points. One can see two intersections for 32-64 and 64-128. As expected these intersections approach the estimated value of the threshold when N increases.

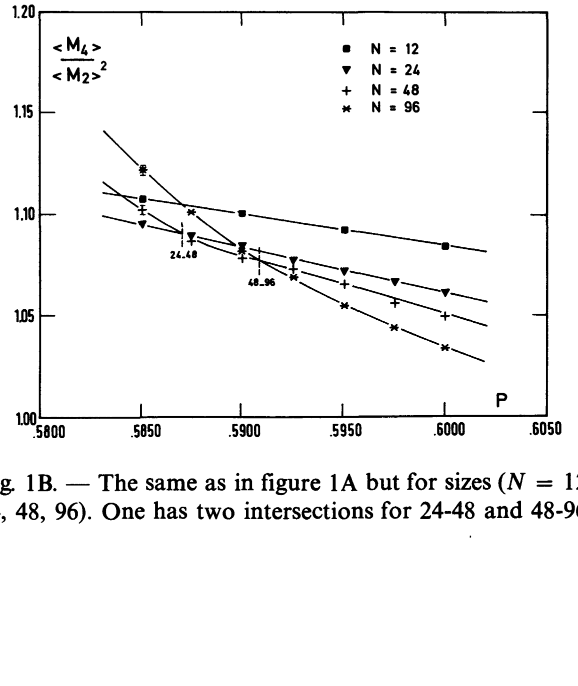

Fig. 1B. — The same as in figure 1A but for sizes (N = 12, 24, 48, 96). One has two intersections for 24-48 and 48-96.

drawn these curves. We see that for small sizes the curves do not intersect, or intersect rather far from the expected value $p_c = 0.59277 \pm 0.00005$ [18] but clearly when one increases the size the results become better and better, the finite size scaling becoming asymptotically valid. The intersection $p_c(N, N')$ of the curves for N and N' gives an estimate of p_c.

In part 3, we compared two neighbouring sizes i.e. $N' = N - 1$. This is expected to give the best convergence of the results when N increases. In the present case, since all the quantities are known within error bars, one cannot determine the intersections accurately when N and N' are too close. Therefore, we have chosen the following compromise $N' = 2N$ which seems empirically reasonable.

The values $p_c(N, 2N)$ estimated from our data are given in table III. We have extrapolated these values, assuming a power law convergence N^{-x}. The best choice for x to extrapolate the data of table III was $x = 2$ and leads to

$$p_c = 0.5928 \pm 0.0006 . \tag{23}$$

However the result (23) depends weakly on the choice of x ($1 \leqslant x \leqslant 3$). The error bar given in (23) contains a contribution due to the extrapolation procedure and also a statistical part.

Once p_c was estimated, we could measure the slopes of the curves at p_c. As explained in section 2, these slopes are proportional to $N^{1/\nu}$.

As before, comparing the values of these slopes for N and $2N$, gives estimates for $y(N, 2N) = \frac{1}{\nu}(N, 2N)$ reported in table IV. As before, these values become better and better when N increases and the same extrapolation procedure gives

$$y = 0.78 \pm 0.05 \tag{24}$$

$$\nu = 1.28 \pm 0.08 . \tag{25}$$

Table III. — *Successive estimates of p_c and $y = 1/\nu$ deduced from the intersection of the curves $\langle M_4 \rangle / \langle M_2 \rangle^2$ for N and 2 N, and from the comparison of the slopes at p_c. The extrapolation was done assuming a power law convergence N^{-x} with $x = 2$ in each case. The final error bars contain a statistical part, and a subjective contribution due to the extrapolation procedure.*

N	$p_c(N, 2N)$	$y(N, 2N)$
24	0.5869 ± 0.0004	0.58 ± 0.02
32	0.5883 ± 0.0006	0.63 ± 0.03
48	0.5911 ± 0.0006	0.78 ± 0.06
64	0.5925 ± 0.0008	0.76 ± 0.06
Extrapolated values	$p_c = 0.5928 \pm 0.0006$	$y = 0.78 \pm 0.05$ $\nu = 1.28 \pm 0.08$

Table IV. — *Value of $\langle M_4 \rangle / \langle M_2 \rangle^2$ at the p_c. The extrapolation procedure is the same as for table* III.

N	$\langle M_4 \rangle / \langle M_2 \rangle^2 (N, p_c)$
9	1.118 ± 0.0004
12	1.095 ± 0.0003
16	1.086 ± 0.0002
24	1.077 ± 0.0005
32	1.074 ± 0.001
48	1.071 ± 0.0008
64	1.068 ± 0.001
96	1.067 ± 0.001
128	1.067 ± 0.002
Extrapolation	1.066 ± 0.001

We have used the same value for $x (x = 2)$. However, the final result does not depend much on the value of x around $x = 2$. This means that our results do not allow us to measure x.

Our calculation gives also an estimate for the value $\lim_{N \to \infty} \frac{\langle M_4 \rangle}{\langle M_2 \rangle^2} (p_c)$. We have given our successive estimates of this ratio at p_c as a function of N in table IV. Our final extrapolated estimate is

$$\lim_{N \to \infty} \frac{\langle M_4 \rangle}{\langle M_2 \rangle^2} (p_c) = 1.066 \pm 0.001 . \qquad (26)$$

The second method consists in comparing the curves $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$ (Figs. 2A, 2B). The intersections of these curves should also converge to p_c when N increases. We give in table V the values of the intersections of these curves for N and $2N$, i.e. the intersections of $\langle M_2 \rangle_{4N} / \langle M_2 \rangle_{2N}$ and of $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$. This leads, after extrapolation to

$$p_c = 0.593 \pm 0.001 . \qquad (27)$$

Table V. — *Successive estimates of p_c deduced from the intersection of the curves $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$ and $\langle M_2 \rangle_{4N} / \langle M_2 \rangle_{2N}$. The extrapolation procedure is the same as in table* III.

N	$p_c(N, 2N)$
8	0.5964 ± 0.0004
6	0.5937 ± 0.0006
8	0.5940 ± 0.0008
12	0.5937 ± 0.0008
16	0.5927 ± 0.0008
24	0.5925 ± 0.001
32	0.5931 ± 0.001
Extrapolated value	0.593 ± 0.001

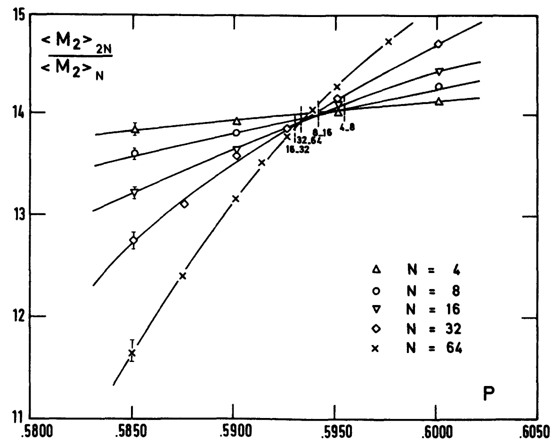

Fig. 2A. — Plot of $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$ for $N = 2^p.4$ in the problem of squares. The error bars are indicated only for the left points. One can see the intersections 4-8, 8-16, 16-32 and 32-64.

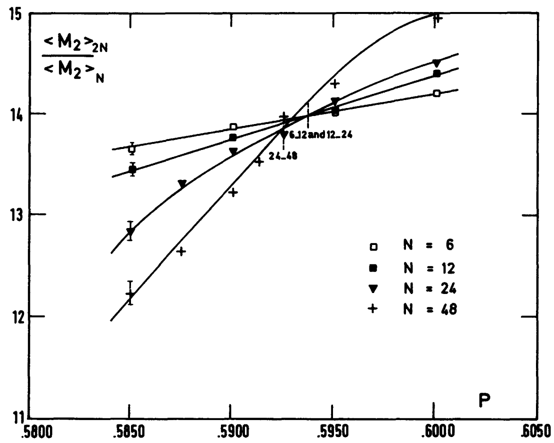

Fig. 2B. — The same as in figure 2A but for sizes $N = 2^p.6$. One sees the intersections of 6-12, 12-24 and 24-48.

The values of these ratios at p_c are related to γ/ν as explained in section 2. They are given as a function of N in table VI leading to the estimate

$$\frac{\gamma}{\nu} = 1.79 \pm 0.01 . \qquad (28)$$

Our last approach is based on the exact knowledge of the exponents [11, 12]. In principle since $\nu = 4/3$ and $\gamma = 43/18$, the value of $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$ at p_c should be

$$\frac{\langle M_2 \rangle_{2N}}{\langle M_2 \rangle_N} = 2^{d + \frac{\gamma}{\nu}} \simeq 13.84 . \qquad (29)$$

We can obtain estimates $p_c(N)$ of the values of p_c for which $\langle M_2 \rangle_{2N} / \langle M_2 \rangle_N$ is equal to 13.84. The $p_c(N)$ are reported in table VI. After a power law

Table VI. — a) *Successive estimates of p_c knowing the presumed exact exponents i.e. the value of $\langle M_2 \rangle_{2N}/\langle M_2 \rangle_N$ at p_c when $N \to \infty$.* b) *Conversely, successive estimates of $\langle M_2 \rangle_{2N}/\langle M_2 \rangle_N$ at our estimated value of p_c. The extrapolations are done as before.*

N	a) $p_c(N)$	b) $\frac{\langle M_2 \rangle_{2N}}{\langle M_2 \rangle_N}$ at p_c
4	0.5850 ± 0.003	14 ± 0.02
6	0.5895 ± 0.003	13.95 ± 0.02
8	0.5910 ± 0.0004	13.94 ± 0.02
12	0.5913 ± 0.0004	13.92 ± 0.02
16	0.5923 ± 0.0003	13.88 ± 0.02
24	0.5923 ± 0.0003	13.87 ± 0.03
32	0.5925 ± 0.0003	13.88 ± 0.04
48	0.5924 ± 0.0006	13.88 ± 0.06
64	0.5927 ± 0.0005	13.84 ± 0.06
Extrapolated values	0.5928 ± 0.0004	13.86 ± 0.08 $\frac{\gamma}{\nu}^{\bullet} = 1.79 \pm 0.01$

extrapolation one gets

$$p_c = 0.5928 \pm 0.0004 \tag{30}$$

which is our third estimate of p_c.

One sees that our three estimates (23), (27) and (30) of p_c are compatible together and with the other values in the literature ($p_c = 0.59277 \pm 0.00005$ [18], $p_c = 0.5927 \pm 0.0002$ [8], $p_c = 0.5923 \pm 0.0007$ [19], $p_c = 0.5927 \pm 0.0001$ [26], $p_c = 0.5925 \pm 0.0003$ [22], $p_c = 0.5929 \pm 0.0003$ [25]). Our best value is given by the third method, which can unfortunately work only if the values of the exponents are known exactly (i.e. in dimension 2).

Our estimates (25) and (28) of the exponent ν and of γ/ν are not very accurate but in agreement with the conjectured values $\nu = 1.3333$ and $\gamma/\nu = 1.79166$.

We believe that the method we have used in this section has two advantages in comparison with the method [17, 18, 20] which consists in calculating the probability that there is a cluster connecting the two opposite sides of a square. The first advantage is that we work with periodic conditions and therefore the finite size effects are not affected by surface effects. The second advantage is that our results show that, with relatively small sizes, one can determine rather accurately p_c and the exponents. We think that our results could be improved by better statistics more than by increasing the sizes of our samples.

5. The site percolation on strips (2d).

Let us first describe the algorithm we have used to study the site percolation problem on strips. This algorithm enables us to calculate the moments m_2 and m_4 for strips or bars of finite width but of arbitrary length.

Suppose that we have constructed a strip of finite length L and width N with periodic boundary conditions. We then add to it a new column (the column $L + 1$) and each site on this column is either occupied with probability p or empty with probability $1 - p$. What information [8] should be kept to calculate the properties of the strip of length $L + 1$ knowing those of the strip of length L ?

One can easily see that the following properties are sufficient. If we know how all the sites of column L are connected together by the strip of length L and the size of the clusters they belong to, in the strip of length L, it is easy to compute these properties for the strip of length $L + 1$. At the same time we can find the sizes of all the clusters which end at column L. So at column L, there is a finite amount of information to be kept : the connections between the sites of column L and the sizes of the clusters which have at least one site belonging to column L.

At each column L, one has a contribution coming from the clusters which end at this column L. By adding all these contributions, one gets m_2 and m_4.

We have calculated estimates of m_2 and m_4 by constructing long strips (typically $L \sim 10^7$). Our results are reported in table II of the Appendix.

Because our strips have finite lengths, the values of m_2 and m_4 have error bars that we estimated in the following way. We have cut a strip of length L into 10 parts (of length $L/10$). We calculated m_2 and m_4 for each part and we estimated the error bars assuming that the distribution of m_2 and m_4 were Gaussian. Here, as in section 3, the fluctuations of the ratio $\frac{1}{N} m_4/m_2^2$ were calculated with the distribution of the subset estimates rather than combining the error bars of m_4 and m_2. To eliminate the boundary effect due to the beginning of the calculation, we always constructed a strip of length $L + L_0$ and we started to count the clusters only after the L_0^{th} column (we choose typically $L_0 = L/10$).

The results given in table II of the Appendix can be analysed in different ways as we did in part 4 for the problem of squares.

We have plotted the ratio m_4/Nm_2^2 *versus* p for different sizes N in figure 3. These curves intersect at values of p which converge quickly to p_c. If we want to compare as in part 4 the results for N and $2N$ we have just two values $p_c(N, 2N)$. This makes the extrapolation impossible. We have adopted the following non rigorous procedure. We have first measured the slopes in the range $0.5920 \leqslant p \leqslant 0.5940$, which gives 3 values of $y(N, 2N)$ which are reported in table VII. These values are extrapolated assuming a power law

$$\nu = 1.25 \pm 0.25 . \tag{31}$$

Here the results seemed to converge like N^{-4} although

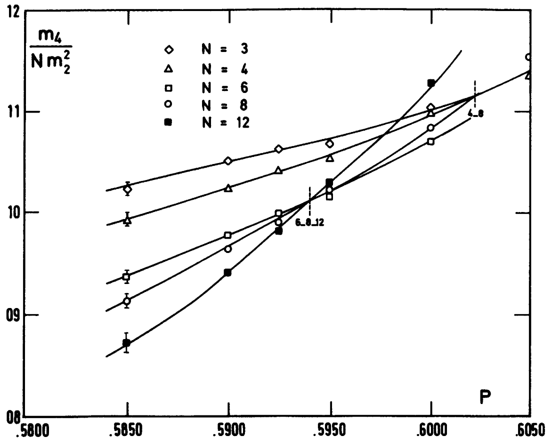

Fig. 3. — Plot of $m\,4/Nm_2^2$ for different sizes *versus* p in the site percolation problem on strips. Here the moments are given per unit length. The error bars are indicated at the left points of the figure. One can see two intersections.

Table VII. — *Successive estimates from* p_c *and* y *deduced from the curves* m_4/Nm_2^2 *for* N *and* $2\,N$. *The extrapolation is done* versus n^{-x} *with* $x = 4$.

N	$p_c(N, 2\,N)$	$y(N, 2\,N)$
3		1.2 ± 0.06
4	0.6025 ± 0.001	0.94 ± 0.08
6	0.5940 ± 0.001	0.83 ± 0.1
Extrapolated values	0.592 ± 0.0015	0.81 ± 0.1 $\nu = 1.24 \pm 0.20$

the value of $x = 4$ cannot be taken seriously. We then have plotted our two values of $p_c(N, 2\,N)$ *versus* N^{-4} which gave

$$p_c = 0.592 \pm 0.0015 . \quad (32)$$

We have finally estimated the limiting value of m_4/Nm_2^2 at p_c and found

$$\lim_{N \to \infty} \frac{m_4}{Nm_2^2} = 9.90 \pm 0.06 . \quad (33)$$

In this strip calculation, it is difficult to estimate error bars. We see that curves for different sizes (6, 8, 12) intersect at the same point which is not p_c. If we expect a good convergence of $p_c(N, N')$ toward p_c when N and N' increase, this seems incompatible with our error bars. We think that the reason for that is the following : for this problem of site percolation on strips, the sizes of the clusters can be arbitrarily large. The distribution of the number of clusters is such that very big and rare clusters contribute to the moments. Therefore the strips have to be long enough in order to have a sufficient number of these clusters.

Table VIII. — *Successive estimates of the ratio* m_4/Nm_2^2 *at* p_c *for the problem on strips. The extrapolation is done* versus n^{-x} *with* $x = 4$.

N	$\frac{m_4}{Nm_2^2}(p_c)$
3	10.6 ± 0.06
4	10.4 ± 0.06
6	10 ± 0.06
8	9.94 ± 0.06
12	9.88 ± 0.08
Extrapolated value	9.90 ± 0.06

For a width $\geqslant 8$ we think that an $L \sim 10^8$ would be a good value. Note however that, because of the correlations in the moments, this effect has a smaller influence on the ratio m_4/Nm_2^2 than on m_4, and also probably than on m_2.

From the curves $m_2(2\,N)/m_2(N)$ *versus* p, our results were rather bad and we could not deduce estimates of critical properties from these ratios.

We see that, from a statistical point of view, these calculations on strips (2D) are more difficult than the calculations on squares but that the results converge more rapidly to their asymptotic values when N increases. The statistical problem is less serious in the 3D case that we shall discuss now.

6. The site percolation problem on bars (3D).

To study the 3D problem, we have used again the algorithm described in 5. The only difference is that we now study bars, which have square sections of linear size N i.e. section of N^2 sites. Our results for m_2 and m_4 are given in table III of the Appendix, with the error bars calculated exactly with the same procedure as in part 3.

We have first plotted the curves m_4/Nm_2^2 *versus* p for different sizes N in figure 4. We see here that the intersections of the curves converge monotonically and rather rapidly to the expected value of $p_c = 0.3117 \pm 0.0003$ [21]. We give the successive estimates $p_c(N, 2\,N)$ in table IX and extrapolated these values assuming a convergence N^{-x} ($x = 2$ seemed the best possible choice). Our estimate of p_c follows

$$p_c = 0.3118 \pm 0.0004 . \quad (34)$$

We have then measured the slopes of the curves at p_c. Our estimated values $y(N, 2\,N)$ are given in table IX and, when extrapolated *versus* n^{-x} with $x = 1$, give the following estimate

$$\nu = 0.90 \pm 0.02 . \quad (35)$$

Note we have taken here a different value of x as for $p_c(N, 2\,N)$. (The extrapolation seemed to be more

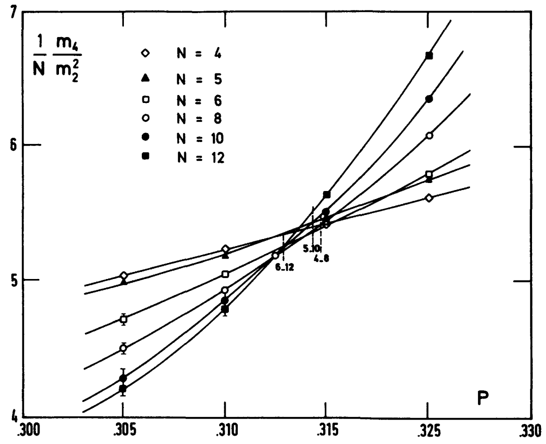

Fig. 4. — Same thing as in figure 3 but for the site percolation problem on bars. The error bars are indicated for the points at the left, but for the sizes 10 and 12 we have made more iterations for the points 0.310, 0.3125 and 0.315. The error bars are indicated for 0.310. One can see 3 intersections 4-8, 5-10, 6-12.

Table IX. — a) *Successive estimates of* p_c *deduced from the intersections of the curves* m_4/Nm_2^2 *for N and 2 N. The extrapolation is made* versus n^{-x} *with* $x=2$. b) *Successive estimates of y deduced from the comparison of the slopes of the curves* $m_4/(Nm_2^2)$ *for N and 2 N at* p_c. *Here* $x = 1$.

N	a) $p_c(N, 2N)$	b) $y(N, 2N)$
4	0.3147 ± 0.0003	1.30 ± 0.02
5	0.3142 ± 0.0004	1.27 ± 0.04
6	0.3127 ± 0.0005	1.23 ± 0.06
Extrapolated values	p_c = 0.3118 ± 0.0004	y = 1.10 ± 0.04 ν = 0.90 ± 0.02

sensitive to x in this 3D case than in 2D on squares or strips.) Finally the successive estimates of $\frac{m_4}{Nm_2^2}$ at p_c are given in table X and extrapolated with $x = 1$ to give

$$\lim_{N\to\infty} \frac{m_4}{Nm_2^2} = 5.0 \pm 0.02 . \quad (36)$$

We can now turn to the ratio $m_2(2N)/m_2(N)$. The curves showing this ratio *versus p* are given in figure 5. We see that we have an intersection at a value $p_c(5,6) = 0.3145 \pm 0.0004$. The values of the ratio at p_c are given in table XI and extrapolated with $x=1$ to give the estimate of

$$\gamma = 1.71 \pm 0.06 . \quad (37)$$

Table X. — *Successive values of* m_4/Nm_2^2 *at the estimated value of* p_c. *The extrapolation is made* versus n^{-x} *with* $x = 1$.

N	m_4/Nm_2^2 at p_c
4	5.30 ± 0.01
5	5.29 ± 0.01
6	5.18 ± 0.01
8	5.11 ± 0.015
10	5.11 ± 0.02
12	5.10 ± 0.02
Extrapolated values	5.0 ± 0.02

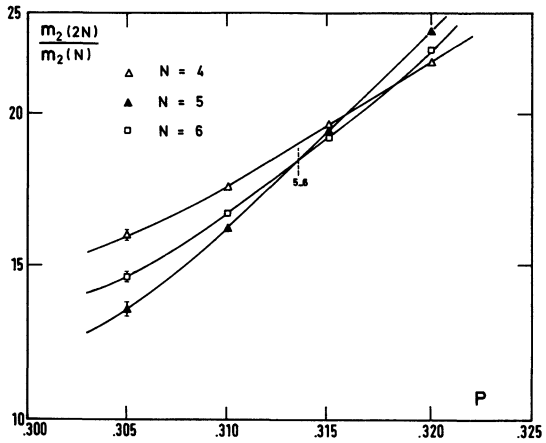

Fig. 5. — Plot of $m_2(2N)/m_2(N)$ *versus p* in the site percolation problem on bars. One can see the intersection (5-6).

Table XI. — *Successive values of the height of* $m_2(2N)/m_2(N)$ *at our estimate of* p_c. *The values are extrapolated* versus n^{-x} *with* $x = 1$.

N	$m_2(2N)/m_2(N)$ at p_c
4	18.5 ± 0.1
5	17.6 ± 0.15
6	17.3 ± 0.3
Extrapolated values	14.9 ± 0.2 $\frac{\gamma}{\nu}$ = 1.90 ± 0.02

Our determination (34) of p_c is in good agreement with the work of Heermann and Stauffer [21] $p_c = 0.3117 \pm 0.0002$ (see also Ref. [22]). Concerning the exponents we find $\nu = 0.90 \pm 0.02$ in agreement with $\nu = 0.89 \pm 0.01$ [21] and our value (37) agrees also rather well with the series estimate $\gamma = 1.74$ [23].

7. Conclusion.

In this paper we have shown that one can use the finite size scaling properties of the moments of the sizes of clusters instead of the correlation length to calculate critical properties. We have calculated these moments by standard methods for the two dimensional site percolation problem on squares and developed an algorithm combining transfer matrices and Monte Carlo sampling to calculate the properties of strip (2D) and bars (3D).

Our calculations on squares gave results compatible with those already known in the literature. Our results for strips were difficult to interpret probably because for strips, there are large but rare clusters which contribute to the moments in a significant way. It is then necessary to construct strips long enough to have a representative number of these clusters. We think that $L = 4 \times 10^7$ was slightly too short and that strips 10 times longer would improve our results a lot. However one should notice that the results for strips converge more rapidly with the size of the system.

For example the curves m_4/Nm_2^2 for $N = 6$ and 12 intersect at $p \sim 0.594$ for strips whereas for the squares the first intersection of the corresponding ratio is at $p \sim 0.587$ for the sizes 24 and 48. In the 3D problem, this is a very important advantage. The problem of large clusters seems less serious for the 3D problem than for the 2D problem. This can be seen in particular in the values of the ratios m_4/Nm_2^2 given in equations (33) and (36). Since the ratio is smaller in $D = 3$ than in $D = 2$, this means that in $D = 3$ the contribution of large clusters is less important. With bars of length $L = 10^7$ and sizes $N < 12$ we were able to obtain results perfectly compatible with those of Heermann and Stauffer, confirming in particular a higher value of the exponent ($\nu \sim 0.90$) than was found in previous calculations ($\nu \sim 0.80$ [17]). Note that our method is just limited by the available computer time and not for example by technical problems of memory size.

We think it would be interesting to carry out a similar calculation in the case of the Ising model. Note however that the MC sampling would be more difficult than for the site percolation problem, the configurations having now to be constructed with their Boltzmann weight.

This method seems however to be a good way of studying some 3D models by the finite size scaling so successfully used in two dimensions and to overcome the difficulties related to the size of the transfer matrices involved in exact calculations.

Acknowledgments.

B. Derrida would like to thank D. Mukamel and Y. Shnidman for discussions which motivated this work.

Table AIa. — *Results for squares of size $N \times N$. In* a) *are given the moments $\langle M_2 \rangle$ and their error bars, in* b) *the moments $\langle M_4 \rangle$ and in* c) *the ratios $\langle M_4 \rangle / \langle M_2 \rangle^2$. The results are obtained with* 10^4 *samples for the dotted points and* 10^5 *samples for the others.*

a) P	$N = 8$	$N = 16$	$N = 32^+$	$N = 64^+$	$N = 128^+$
0.5850	$0.1196.10^4$	$0.1625.10^5$	$0.215.10^6$	$0.274.10^7$	$0.319.10^8$
0.5875			0.224	0.295	0.365
0.5900	0.1230	0.1701	0.232	0.315	0.414
0.59125				0.324	0.438
0.5925			0.240	0.333	0.459
0.59375				0.346	0.484
0.5950	0.1263	0.1775	0.250	0.353	0.504
0.5975				0.374	0.551
0.6000	0.1297	0.1850	0.267	0.393	0.592
error bars	$\pm 0.00020.10^4$	$\pm 0.00020.10^5$	$\pm 0.001.10^6$	$\pm 0.0020.10^7$	$\pm 0.002.10^8$

b) P	$N = 8$	$N = 16$	$N = 32^+$	$N = 64^+$	$N = 128^+$
0.5850	$0.1615.10^7$	$0.2908.10^9$	$0.508.10^{11}$	$0.830.10^{13}$	$0.115.10^{16}$
0.5875			0.547	0.946	0.148
0.5900	0.1701	0.3156	0.581	$0.107.10^{14}$	0.186
0.59125				0.113	0.207
0.5925			0.623	0.119	0.225
0.59375				0.126	0.249
0.5950	0.1781	0.3410	0.668	0.132	0.268
0.5975				0.147	0.315
0.6000	0.1866	0.3676	0.753	0.161	0.361
error bars	$\pm 0.0004.10^7$	$\pm 0.0005.10^9$	$\pm 0.004.10^{11}$	$\pm 0.001.10^{14}$	$\pm 0.002.10^{16}$

c) P	$N = 8$	$N = 16$	$N = 32^+$	$N = 64^+$	$N = 128^+$
0.5850	$0.1130.10^1$	$0.1101.10^1$	$0.1096.10^1$	$0.1110.10^1$	$0.1133.10^1$
0.5875			0.1089	0.1094	0.1110
0.5900	0.1123	0.1091	0.1083	0.1081	0.1087
0.59125				0.1076	0.1079
0.5925			0.1075	0.1070	0.1070
0.59375				0.1064	0.1060
0.5950	0.1116	0.1182	0.1065	0.1060	0.1053
0.5975				0.1050	0.1040
0.6000	0.1110	0.1074	0.1054	0.1042	0.1030
error bars	$\pm 0.00070.10^4$	$\pm 0.00004.10^1$	$\pm 0.0002.10^1$	$\pm 0.0002.10^1$	$\pm 0.0004.10^1$

Table AIb. — *The same as in table* AIa *but for other sizes.*

a) P	$N = 6$	$N = 12$	$N = 24^+$	$N = 48^+$	$N = 96^+$
0.5850	$0.4031.10^3$	$0.5520.10^4$	$0.7423.10^5$	$0.9529.10^6$	$0.1166.10^8$
0.5875			0.7664	$0.1021.10^7$	0.1291
0.5900	0.4125	0.5726	0.7893	0.1076	0.1422
0.59125				0.1095	0.1482
0.5925			0.8128	0.1120	0.1566
0.59375				0.1156	0.1614
0.5950	0.4225	0.5932	0.8319	0.1174	0.1679
0.5975			0.8539	0.1226	0.1801
0.6000	0.4314	0.6140	0.8842	0.1278	0.1914
error bars	$\pm$ $0.0005.10^3$	$\pm$ $0.0008.10^4$	$\pm$ $0.0003.10^5$	$\pm$ $0.0004.10^7$	$\pm$ $0.0006.10^8$

b) P	$N = 6$	$N = 12$	$N = 24^+$	$N = 48^+$	$N = 96^+$
0.5850	$0.188.10^6$	$0.3378.10^8$	$0.603.10^{10}$	$0.100.10^{13}$	$0.151.10^{15}$
0.5875			0.640	0.113	0.183
0.5900	0.196	0.3607	0.675	0.125	0.219
0.59125				0.129	0.237
0.5925			0.712	0.135	0.259
0.59375				0.143	0.276
0.5959	0.204	0.3841	0.742	0.147	0.297
0.5975			0.778	0.159	0.339
0.6000	0.212	0.4087	0.830	0.171	0.379
error bars	$\pm$ $0.0004.10^6$	$\pm$ $0.0008.10^8$	$\pm$ $0.003.10^{10}$	$\pm$ $0.001.10^{13}$	$\pm$ $0.002.10^5$

c) P	$N = 6$	$N = 12$	$N = 24^+$	$N = 48^+$	$N = 96^+$
0.5850	$0.1155.10^1$	$0.1108.10^1$	$0.1095.10^1$	$0.1102.10^1$	$0.1122.10^1$
0.5875			0.1089	0.1088	0.1101
0.5900	0.1150	0.1100	0.1084	0.1078	0.1082
0.59125				0.1077	0.1078
0.5925			0.1078	0.1073	0.1069
0.59375				0.1067	0.1061
0.5950	0.1143	0.1092	0.1072	0.1066	0.1055
0.5975			0.1067	0.1056	0.1044
0.6000	0.1138	1.084	0.1081	0.1049	0.1034
error bars	$\pm$ $0.0001.10^1$	$\pm$ $0.0001.10^1$	$\pm$ $0.0001.10^1$	$\pm$ $0.0002.10^1$	$\pm$ $0.0002.10^1$

Table AII. — *Results for strips of transversal size N. In* a) *are given the second moments m_2 per unit length and their error bars; in* b) *the fourth moments and in* c) *the ratios m_4/m_2^2.*
The strips are 10^7 *unit long for the dotted* (+) *points, and* 4 × 10^7 *long for the others.*

a) P	3	4	6	8	12
0.5850⁺	$0.3112.10^2$	$0.6845.10^2$	$0.1060.10^3$	$0.452.10^3$	$0.135.10^4$
0.5900	0.3249	0.7201	0.1000	0.4909	0.1506
0.5925	0.3319	0.7388	0.2275	0.5125	0.1592
0.5950	0.3391	0.7579	0.2353	0.5331	0.1684
0.6000⁺	0.3534	0.7988	0.2523	0.579	0.188
(+)	$\pm 0.0004.10^2$	$\pm 0.0006.10^2$	$\pm 0.0004.10^3$	$\pm 0.0005.10^3$	$\pm 0.0003.10^4$
	± 0.0002	± 0.0003	± 0.0002	± 0.0002	± 0.0002

b) P	3	4	6	8	12
0.5850⁺	$0.296.10^5$	$0.186.10^6$	$0.239.10^7$	$0.149.10^8$	$0.191.10^9$
0.5900	0.331	0.212	0.283	0.185	0.257
0.5925	0.350	0.227	0.310	0.208	0.300
0.5950	0.368	0.242	0.337	0.232	0.348
0.6000⁺	0.409	0.281	0.408	0.290	0.476
(+)	$\pm 0.002.10^5$	$\pm 0.002.10^6$	$\pm 0.002.10^7$	$\pm 0.001.10^8$	$\pm 0.008.10^9$
	± 0.001	± 0.001	± 0.001	± 0.0005	± 0.004

c) P	3	4	6	8	12
0.5850⁺	$0.306.10^2$	$0.398.10^2$	$0.562.10^2$	$0.731.10^2$	$0.105.10^3$
0.5900	0.314	0.409	0.585	0.770	0.113
0.5925	0.318	0.416	0.598	0.793	0.118
0.5950	0.3205	0.421	0.610	0.816	0.123
0.6000⁺	0.327	0.440	0.641	0.866	0.135
(+)	$\pm 0.002.10^2$	$\pm 0.002.10^2$	$\pm 0.004.10^2$	$\pm 0.004.10^2$	$\pm 0.002.10^3$
	± 0.001	± 0.001	± 0.002	± 0.002	± 0.001

Table AIII. — *Results for bars of transversal size $N \times N$. In* a) *are given the second moments m_2 per unit length and their errors bars; in* b) *the fourth moments and in* c) *the ratios m_4/m_2^2. The strip are 5×10^6 unit long for the dotted* (+) *points and 10^7 long for the others.*

P	$N = 4$	$N = 5$	$N = 6$	$N = 8$	$N = 10$	$N = 12$
0.305	$0.1034.10^3$	$0.2625.10^3$	$0.5404.10^3$	$0.1651.10^4$	$0.384.10^{4+}$	$0.748.10^{4+}$
0.310	0.1107	0.2930	0.6188	0.1993	0.489	$0.1008.10^5$
0.3125				0.2190	0.553	0.1172
0.315	0.1230	0.3272	0.7103	0.2411	0.626	0.1367
0.320	0.1343	0.3657	0.8160	0.2934	0.810^+	0.1874^+
(+)					$0.002.10^4$	$0.0002.10^5$
	$\pm\ 0.0002.10^3$	$\pm\ 0.0004.10^3$	$\pm\ 0.0008.10^3$	$\pm\ 0.0003.10^4$	$\pm\ 0.001.10^4$	$\pm\ 0.0001.10^5$

P	$N = 4$	$N = 5$	$N = 6$	$N = 8$	$N = 10$	$N = 12$
0.305	$0.215.10^6$	$0.171.10^7$	$0.826.10^7$	$0.981.10^8$	$0.629.10^{9+}$	$0.277.10^{10+}$
0.310	0.265	0.223	$0.116.10^8$	$0.157.10^9$	$0.116.10^{10}$	0.569
0.3125				0.199	0.159	0.856
0.315	0.330	0.294	0.164	0.253	0.217	$0.1268.10^{11}$
0.320	0.406	0.385	0.232	0.419	0.416^+	0.2814^+
(+)					$0.002.10^{10}$	$0.0004.10^{11}$
	$\pm\ 0.001.10^6$	$\pm\ 0.002.10^7$	$\pm\ 0.001.10^8$	$\pm\ 0.001.10^9$	$\pm\ 0.001.10^{10}$	$\pm\ 0.0002.10^{11}$

P	$N = 4$	$N = 5$	$N = 6$	$N = 8$	$N = 10$	$N = 12$
0.305	$0.2010.10^2$	$0.250.10^2$	$0.283.10^2$	$0.360.10^2$	$0.428.10^{2+}$	$0.494.10^{2+}$
0.310	0.2089	0.260	0.303	0.395	0.485	0.576
0.3125				0.414	0.521	0.623
0.315	0.2175	0.274	0.325	0.435	0.553	0.677
0.320	0.2255	0.288	0.349	0.486	0.635^+	0.801^+
(+)					$0.004.10^2$	$0.004.10^2$
	$\pm\ 0.0005.10^2$	$\pm\ 0.001.10^2$	$\pm\ 0.002.10^2$	$\pm\ 0.002.10^2$	$\pm\ 0.002.10^2$	$\pm\ 0.002.10^2$

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