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Magnetic properties and the function $q(x)$ of the generalised random-energy model

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Abstract. We give a general expression for the free energy of the generalised random-energy model (GREM) in terms of the average partition function $\langle Z \rangle$ and the average squared partition function $\langle Z^2 \rangle$. Then, using the notion of the partial partition function, we show how one can introduce a magnetic field in the GREM. The de Almeida–Thouless line and the magnetisation in the spin glass phase are computed. The moments $\langle Z^p \rangle$ of the partition function are calculated and we show that a replica calculation with a full breaking of replica symmetry leads to the correct free energy. The function $q(x)$ is then computed.

1. Introduction

At present, there are few spin glass models that can be solved exactly and show the whole structure (breaking of replica symmetry, ultrametricity, transition in a magnetic field) that resulted from the approach developed by Parisi (1980) in order to solve the problem posed by Sherrington and Kirkpatrick (1975, 1978).

The random-energy model (Derrida 1980, 1981) and some extensions of it (Mottishaw 1986) are exactly soluble and do have this structure although the spin glass phase is particularly simple because the system is completely frozen at its ground-state energy.

The generalised random-energy model (GREM) (Derrida 1985) remains an exactly soluble model but has much more structure (Derrida and Gardner 1986). In our previous work the general solution of the GREM was derived and we discussed how one could associate any spin glass model with a GREM that had the same pair correlations between energies.

In the present work, we will present several recent results on the GREM. First, in § 2 a simple expression for the free energy in terms of $\langle Z \rangle$ and $\langle Z^2 \rangle$ is given. In §§ 3 and 4 we show how one can introduce a magnetic field in the GREM and we compute its magnetic properties. In § 5 the integer moments $\langle Z^p \rangle$ are computed. Lastly in § 6, we show that the replica approach with a full breaking of replica symmetry gives the correct free energy of the GREM and the function $q(x)$ is computed.

The reader who has read our previous work is encouraged to start after equation (12).

2. Simple expression for the free energy of the GREM

The GREM, as we shall see, depends on a lot of parameters. In this section we shall show that the free energy of the GREM always has a very simple expression in terms of the average partition function $\langle Z\rangle$ and of the average squared partition function $\langle Z^2\rangle$.

Let us start by recalling the definition of the GREM of size N (N should be considered as the number of spins) (Derrida 1985, Derrida and Gardner 1986).

One can represent the energies E_ν of the configurations ν ($1 \leqslant \nu \leqslant 2^N$) as the end points of a tree of n levels (see figure 1). To each level i ($1 \leqslant i \leqslant n$) of the tree, one associates three quantities α_i, a_i and q_i. q_i must be an increasing function of i with $q_1 = 0$ and $q_{n+1} = 1$.

At each level i, one branch divides into α_i^N branches. Therefore at level i there are $(\alpha_1\alpha_2 \ldots \alpha_i)^N$ branches. On each bond of the tree at level i, one chooses a random variable $\varepsilon_i^{(\nu)}$ according to a distribution $\rho_i(\varepsilon_i^{(\nu)})$ whose width is a_i:

$$\rho_i(\varepsilon_i^{(\nu)}) = (N\pi a_i J^2)^{-1/2} \exp[-(\varepsilon_i^{(\nu)})^2/NJ^2 a_i]. \tag{1}$$

The energy E_ν of the configuration ν is given by definition by

$$E_\nu = \sum_{i=1}^{n} \varepsilon_i^{(\nu)} \tag{2}$$

where the $\varepsilon_i^{(\nu)}$ in (2) are the n bonds that connect the configuration ν to the top of the tree (see figure 1).

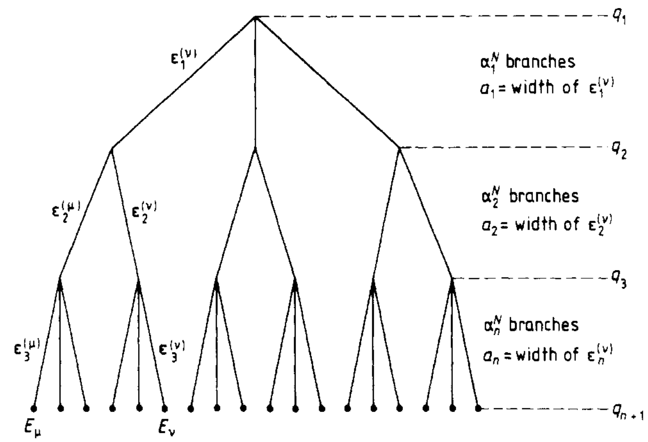

Figure 1. The configurations ν ($1 \leqslant \nu \leqslant 2^N$) of the GREM are the end points of a tree of n levels. The space of configurations is ultrametric.

By definition, two configurations μ and ν have an overlap $q^{\mu\nu} = q_i$ if $\varepsilon_j^{(\mu)} = \varepsilon_j^{(\nu)}$ for $j \leqslant i-1$ and $\varepsilon_j^{(\mu)} \neq \varepsilon_j^{(\nu)}$ for $j \geqslant i$. So the overlap q_i is given by the level on the tree where the branches coming from μ and ν join.

If we choose the following normalisation:

$$\sum_{i=1}^{n} a_i = 1 \tag{3a}$$

$$\sum_{i=1}^{n} \log \alpha_i = \log 2 \tag{3b}$$

then we can show that there are 2^N energies E_ν distributed according to

$$P_\nu(E_\nu) = (N\pi J^2)^{-1/2} \exp[-(E_\nu)^2/NJ^2]. \tag{4}$$

We can also compute the number e^{Nu} of configurations μ that have an overlap $q^{\mu\nu} = q_i$ with a given configuration ν:

$$u = \sum_{j=i}^{n} \log \alpha_j \tag{5}$$

whereas the correlation between two configurations μ and ν that have an overlap $q^{\mu\nu} = q_i$ is given by

$$P_{\mu,\nu}(E_\mu, E_\nu) \sim \exp\left(-\frac{(E_\mu + E_\nu)^2}{2N(1+v)J^2} - \frac{(E_\mu - E_\nu)^2}{2N(1-v)J^2} \right) \tag{6}$$

where

$$v = \sum_{j=1}^{i-1} a_j. \tag{7}$$

In our previous work (Derrida and Gardner 1986), we derived the general solution of the GREM in the following two cases: case A where the sequence $a_i/\log \alpha_i$ is a decreasing function of i; and case B where the sequence $a_i/\log \alpha_i$ has a single maximum. If the temperatures T_i are defined by

$$T_i = (J/2)(a_i/\log \alpha_i)^{1/2} \tag{8}$$

then in case A, all temperatures T_i are transition temperatures and the free energy is given by

$$(1/N)\langle \log Z \rangle = \log 2 + J^2/4T^2 \qquad \text{if } T > T_1$$

$$\frac{1}{N}\langle \log Z \rangle = \sum_{j=i+1}^{n} \log \alpha_j + a_j \frac{J^2}{4T^2} + \frac{J}{T}\sum_{j=1}^{i} (a_j \log \alpha_j)^{1/2} \qquad \text{if } T_{i+1} < T < T_i$$

$$\frac{1}{N}\langle \log Z \rangle = \frac{J}{T}\sum_{j=1}^{n} (a_j \log \alpha_j)^{1/2} \qquad \text{if } T < T_n. \tag{9}$$

In case B, the solution is slightly more complicated. One has first to find the level i_0 such that

$$\frac{a_{i_0}}{\log \alpha_{i_0}} > \sum_{j=1}^{i_0} a_j \Big/ \sum_{j=1}^{i_0} \log \alpha_j > \frac{a_{i_0+1}}{\log \alpha_{i_0+1}}. \tag{10}$$

Then the highest critical temperature is given by

$$\theta = \frac{J}{2}\left(\sum_{j=1}^{i_0} a_j \Big/ \sum_{j=1}^{i_0} \log \alpha_j \right)^{1/2} \tag{11}$$

and the free energy is given by

$$(1/N)\langle \log Z\rangle = \log 2 + J^2/4T^2 \qquad \text{if } T > \theta$$

$$\frac{1}{N}\langle \log Z\rangle = \sum_{i_0+1}^{n}\left(\log \alpha_j + a_j \frac{J^2}{4T^2}\right) + \frac{J}{T}\left(\sum_1^{i_0} a_j\right)^{1/2} \times \left(\sum_1^{i_0} \log \alpha_j\right)^{1/2} \qquad \text{if } \theta > T > T_{i_0+1}$$

$$\frac{1}{N}\langle \log Z\rangle = \sum_{i+1}^{n}\left(\log \alpha_j + a_j \frac{J^2}{4T^2}\right) + \frac{J}{T}\sum_{i_0+1}^{i} (a_j \log \alpha_j)^{1/2} + \frac{J}{T}\left(\sum_1^{i_0} a_j\right)^{1/2}\left(\sum_1^{i0} \log \alpha_j\right)^{1/2} \qquad \text{if } T_i > T > T_{i+1} \quad \text{and } i \geqslant i_0 + 1$$

$$\frac{1}{N}\langle \log Z\rangle = \frac{J}{T}\left[\left(\sum_1^{i_0} a_j\right)^{1/2}\left(\sum_1^{i_0} \log \alpha_j\right)^{1/2} + \sum_{i_0+1}^{n} (a_j \log \alpha_j)^{1/2}\right] \qquad \text{if } T_n > T \qquad (T_n < \theta). \tag{12}$$

We will show here that for any choice of the a_i and $\log \alpha_j$, the free energy always has a simple expression in terms of $\langle Z(T)\rangle$ and $\langle Z^2(T)\rangle$. Let us compute these averages:

$$\langle Z(T)\rangle = \sum_{\nu=1}^{2^N} \int P_\nu(E_\nu) \exp\left(-\frac{E_\nu}{T}\right) \mathrm{d}E_\nu. \tag{13}$$

Therefore

$$\frac{1}{N}\log\langle Z(T)\rangle = \sum_{j=1}^{n} \log \alpha_j + a_j \frac{J^2}{4T^2} = \log 2 + \frac{J^2}{4T^2}. \tag{14}$$

Similarly $\langle Z^2(T)\rangle$ is given by

$$\langle Z^2(T)\rangle = \sum_{\mu=1}^{2^N}\sum_{\nu=1}^{2^N} \iint p_{\mu,\nu}(E_\mu, E_\nu) \exp\left(-\frac{E_\mu + E_\nu}{T}\right) \mathrm{d}E_\mu \, \mathrm{d}E_\nu. \tag{15}$$

In the thermodynamic limit ($N \to \infty$), the sum (15) is dominated by a saddle point corresponding to an optimal overlap $q^{\mu\nu} = q_i$ between configurations μ and ν:

$$\frac{1}{N}\log\langle Z^2(T)\rangle = \max_{1\leqslant i\leqslant n+1}\left(\sum_{j=1}^{i-1} \log \alpha_j + \frac{J^2}{T^2} a_j + \sum_{j=1}^{n} 2\log \alpha_j + \frac{J^2}{2T^2} a_j\right. \tag{16}$$

When one looks for the i that maximises (16), one finds that in case A

$$(1/N)\log\langle Z^2(T)\rangle = 2\log 2 + J^2/2T^2 \qquad \text{if } T/2^{1/2} > T_1$$

$$\frac{1}{N}\log\langle Z^2(T)\rangle = \sum_{j=1}^{i} \log \alpha_j + \frac{J^2}{T^2} a_j + \sum_{j=i+1}^{n} 2\log \alpha_j + \frac{J^2}{2T^2} a_j \qquad \text{if } T_i > T/2^{1/2} > T_{i+1}$$

$$(1/N)\log\langle Z^2(T)\rangle = \log 2 + J^2/T^2 \qquad \text{if } T_n > T/2^{1/2} \tag{17}$$

and in case B

$$(1/N)\log\langle Z^2(T)\rangle = 2\log 2 + J^2/2T^2 \qquad \text{if } T/2^{1/2} > \theta$$

$$\frac{1}{N}\log\langle Z^2(T)\rangle = \sum_{j=1}^{i_0} \log\alpha_j + \frac{J^2}{T^2}a_j + \sum_{j=i_0+1}^{n} 2\log\alpha_j$$

$$+\frac{J^2}{2T^2}a_j \qquad \text{if } \theta > T/2^{1/2} > T_{i_0+1}$$

$$\frac{1}{N}\log\langle Z^2(T)\rangle = \sum_{j=1}^{i} \log\alpha_j + \frac{J^2}{T^2}a_j + \sum_{j=i+1}^{n} 2\log\alpha_j$$

$$+\frac{J^2}{2T^2}a_j \qquad \text{if } T_i > T/2^{1/2} > T_{i+1} \quad \text{and } i > i_0$$

$$(1/N)\log\langle Z^2(T)\rangle = \log 2 + J^2/T^2 \qquad \text{if } T_n > T/2^{1/2}. \tag{18}$$

In both cases, one can see by looking at (9), (12), (14), (17) and (18) that

$$\frac{\mathrm{d}}{\mathrm{d}T}(T\langle\log Z(T)\rangle) = \frac{\mathrm{d}}{\mathrm{d}T}(T\log\langle Z(T)\rangle) + \log\left(\frac{\langle Z^2(2^{1/2}T)\rangle}{\langle Z(2^{1/2}T)\rangle^2}\right) \tag{19}$$

and from (19) one can deduce a formula valid in all cases:

$$\langle\log Z(T)\rangle = \log\langle Z(T)\rangle - \frac{1}{T}\int_T^\infty \log\left(\frac{\langle Z^2(2^{1/2}T_1)\rangle}{\langle Z(2^{1/2}T_1)\rangle^2}\right)\mathrm{d}T_1. \tag{20}$$

This formula gives the average free energy $\langle\log Z(T)\rangle$ in terms of $\langle Z(T)\rangle$ and $\langle Z^2(T)\rangle$ in closed form. It is interesting to notice that to compute the free energy at temperature T, one needs to know $\langle Z(T)\rangle$ and $\langle Z^2(T)\rangle$ at other temperatures.

Let us mention here that Capocaccia *et al* (1986) were recently able to write the general solution of the GREM in another very simple and compact form.

It is not surprising that such a formula exists. In our previous work, we saw that to each spin glass model (finite- or infinite-dimensional) on a lattice

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j \tag{21}$$

with the interactions J_{ij} between nearest neighbours distributed according to

$$\rho(J_{ij}) \sim \exp[-(J_{ij}^2/2J^2)z] \tag{22}$$

(z is the coordination number), one could associate a GREM that has the same pair correlations between energies and therefore has the same $\langle Z(T)\rangle$, $\langle Z^2(T)\rangle$ and $\langle Z(T)Z(T')\rangle$ for any temperatures T and T'. We had shown (see equation (73) of Derrida and Gardner 1986) that the free energy could be expressed as a function of the specific heat of the ferromagnetic model on the same lattice. It is easy to see that for a spin glass model defined by (21) and (22) on an arbitrary lattice, one can relate $\langle Z^2(T)\rangle$ to the partition function $Z_{\text{ferro}}(\tau)$ of the ferromagnetic model on the same lattice

$$\langle Z^2(T)\rangle = \left\langle \sum_{\sigma_i,\tau_i}\prod_{ij}\exp\frac{J_{ij}}{T}(\sigma_i\sigma_j + \tau_i\tau_j)\right\rangle = 2^N\exp\left(\frac{NJ^2}{2T^2}\right)Z_{\text{ferro}}\left(\frac{T^2 z}{J^2}\right). \tag{23}$$

So if one wants to know the solution of the GREM associated with a given spin glass model, one needs to compute $\langle Z(T)\rangle$ and $\langle Z^2(T)\rangle$ of this spin glass model and then the free energy of the GREM is given by (20).

One should also notice that a GREM is always defined by the pair correlations betwen the energies. In principle a full knowledge of these pair correlations is equivalent to a full knowledge of the average $\langle Z(T)Z(T')\rangle$ for all T and T'. It is remarkable that it is only the information contained in $\langle Z^2(T)\rangle$ that is relevant to the computation of the average free energy. This is due to the fact that for all the spin glass models defined by (21) and (22) one has

$$\begin{aligned}\langle Z(T)Z(T')\rangle &= 2^N \exp\left[N\frac{J^2}{4}\left(\frac{1}{T^2}+\frac{1}{T'^2}\right)\right] Z_{\text{ferro}}\left(\frac{TT'z}{J^2}\right) \\ &= \exp\left[N\frac{J^2}{4}\left(\frac{1}{T}-\frac{1}{T'}\right)^2\right]\langle Z^2((TT')^{1/2})\rangle \end{aligned} \tag{24}$$

and so all the information in $\langle Z(T)Z(T')\rangle$ is already contained in $\langle Z^2(T)\rangle$.

For a given spin glass model, the α_i, a_i and q_i that define the GREM must be chosen in order to give the same pair correlations. For the p spin glass model (the Sherrington–Kirkpatrick (SK) model being that with $p=2$), we had seen in our previous work (Derrida and Gardner 1986) that one must choose

$$\log \alpha_i = \log[t/(1-t)]\,\mathrm{d}t \qquad a_i = 2p(2t-1)^{p-1}\,\mathrm{d}t \qquad q_i = 2t-1 \tag{25}$$

for $\frac{1}{2} \leqslant t \leqslant 1$. That is

$$\log \alpha_i = \log \alpha(q)\,\mathrm{d}q = \tfrac{1}{2}\log\left(\frac{1+q}{1-q}\right)\mathrm{d}q \qquad a_i = a(q)\,\mathrm{d}q = pq^{p-1}\,\mathrm{d}q. \tag{26}$$

In the following, we shall replace $\log \alpha_i$ and a_i by $\log \alpha(q)\,\mathrm{d}q$ and $a(q)\,\mathrm{d}q$ in order to have q as a continuous variable. To obtain the free energy in this case, one only needs to replace sums in (9), (10), (11) and (12) by integrals.

3. The partial partition function and magnetic properties of spin glass models

In this section we shall show that there exists a very general relation between the partial partition function of a spin glass model in zero field and its magnetic properties. The content of this section is *a priori* true for any spin glass model. In § 4 this relation will be used to discuss the magnetic properties of the GREM.

Consider a spin glass model on a finite- or an infinite-dimensional lattice whose Hamiltonian is given by (21) and (22). If one chooses a spin configuration $\mathscr{C}_0$ and an overlap q, one can define a partial partition function $z(T, q, \mathscr{C}_0)$ by

$$z(T,q,\mathscr{C}_0) = \sum_{\mathscr{C},\,q(\mathscr{C},\mathscr{C}_0)=q} \exp\left(-\frac{E(\mathscr{C})}{T}\right) \tag{27}$$

where $E(\mathscr{C})$ is the energy of the spin configuration $\mathscr{C}$. So instead of summing over all spin configurations as usual for the partition function, one only sums over those configurations that have a given overlap q with a reference configuration $\mathscr{C}_0$.

As usual with disordered systems, one can try to compute the average $(1/N)$ $\langle \log z(T, q, \mathscr{C}_0)\rangle$. Clearly because the distribution $\rho(J_{ij})$ chosen in (22) is symmetric, the result will be gauge-invariant and so will not depend on $\mathscr{C}_0$. Therefore one can choose for $\mathscr{C}_0$ the ferromagnetic configuration, i.e. the configuration for which all spins $\sigma_i = +1$. It is clear that $z(T, q, \mathscr{C}_0)$ is then the partition function restricted to all the configurations that have a given magnetisation $m = q$. Se we see that given the free energy of a spin glass in a field, one can deduce $(1/N)\langle \log z(T, q, \mathscr{C}_0)\rangle$ immediately.

In the SK model, one knows that along the de Almeida and Thouless (1978) line, there is an instability. Let us call the magnetisation of the SK model along this line $m_c(T)$. At a given temperature, if $m > m_c(T)$ the system is in the paramagnetic phase and if $m < m_c(T)$ it is in its spin glass phase. Then because of the relation discussed above, one sees that at each temperature $T < T_c$, there is a critical value $q_c(T) = m_c(T)$ for the partial partition function $\langle \log z(T, q, \mathscr{C}_0)\rangle$. For $q > q_v(T)$, the system is in its high-temperature phase whereas for $q < q_c(T)$ the system is a spin glass phase.

In other words, if one chooses an arbitrary configuration $\mathscr{C}_0$, at short distances $(q > q_c(T))$ the system looks like it does in its high-temperature phase, i.e. there is only one valley whereas at larger distances $(q < q_c(T))$, the system is in a spin glass phase, i.e. there are several valleys.

In the next section, the relationship discussed here will be used to obtain the magnetic properties of the GREM.

4. The GREM in a magnetic field

In the GREM, it is easy to compute $\langle \log z(T, q, \mathscr{C}_0)\rangle$. Instead of considering the whole tree described in figure 1, one takes only the branch of all the configurations that have an overlap of at least q with a reference configuration $\mathscr{C}_0$. Then we can use the results described in § 2 since this branch is itself a GREM. (In the GREM one finds the same answer in the thermodynamic limit if one sums over all the configurations that have an overlap q with $\mathscr{C}_0$ or over all the configurations that have an overlap of at least q.)

Let us first discuss the case A, i.e. the case where $a(q)/\log \alpha(q)$ is a decreasing function of q. Then one finds that

$$\frac{1}{N}\langle \log z(T, q, \mathscr{C}_0)\rangle = \int_q^1 \mathrm{d}q' \left(\log \alpha(q') + a(q')\frac{J^2}{4T^2}\right) \qquad \text{if } T > \frac{J}{2}\left(\frac{a(q)}{\log \alpha(q)}\right)^{1/2} \tag{28}$$

and

$$\frac{1}{N}\langle \log z(T, q, \mathscr{C}_0)\rangle = \frac{J}{T}\int_q^Q \mathrm{d}q' (a(q')\log \alpha(q'))^{1/2} + \int_Q^1 \mathrm{d}q' \log \alpha(q') + a(q')\frac{J^2}{4T^2} \qquad \text{if } T < \frac{J}{2}\left(\frac{a(q)}{\log \alpha(q)}\right)^{1/2} \tag{29}$$

where Q is given by

$$T = (J/2)(a(Q)/\log \alpha(Q))^{1/2}. \tag{30}$$

Using the fact that $\langle \log z(T, q, \mathscr{C}_0)\rangle$ is simply the free energy at fixed magnetisation, one can easily deduce the free energy $\langle \log Z(T, h)\rangle$ of the GREM in a magnetic field h:

$$\frac{1}{N}\langle \log Z(T, h)\rangle = \max_m \left(\frac{1}{N}\langle \log z(T, m, \mathscr{C}_0)\rangle + \frac{mh}{T}\right). \tag{31}$$

This gives the equation of state

$$h = T \log \alpha(m) + a(m)J^2/4T \qquad \text{for } T > T_c(h) \tag{32}$$

$$h = J(a(n) \log \alpha(m))^{1/2} \qquad \text{for } T < T_c(h) \tag{33}$$

where the critical line $T_c(h)$ is given by

$$T_c(h) = (J/2)(a(m)/\log \alpha(m))^{1/2} \tag{34}$$

where m is given by

$$h = J(a(m) \log \alpha(m))^{1/2}. \tag{35}$$

So we find a de Almeida–Thouless line whose equation is given by (34) and (35). One should notice that since $a(q)/\log \alpha(q)$ is a decreasing function of q, the critical temperature, $T_c(h)$ decreases as the magnetisation (or the field) increases (see (34)). In the random-energy model (Derrida 1981), $T_c(h)$ was an increasing function of the fields. So we see that the effect of the correlations between energies that are taken into account in the GREM is to change this tendency and to give a $T_c(h)$ that decreases with h. (In order to ensure that the magnetisation m is an increasing function of the field h, see (33), which is of course the only situation that is physically meaningful, we need to consider only a GREM for which $a(q) \log \alpha(q)$ is an increasing function of q. This condition is satisfied for the choice (26) which corresponds to the SK model, i.e. $p = 2$.)

Another important result is that magnetisation is always independent of temperature in the whole spin glass phase (see (33)). A direct consequence is that the magnetic susceptibility is constant in the spin glass phase. One should notice that the assumption that magnetisation depends only on h and not on T in the spin glass phase was proposed a few years ago (Parisi and Toulouse 1980, Vannimenus *et al* 1981) and gives a very good approximation for the SK model. Here we have shown that for any choice of the $\alpha(q)$ and $a(q)$, the magnetisation m is always a function of the field h only in the spin glass phase.

To illustrate this first case, let us take the $a(q)$ and $\log \alpha(q)$ that correspond to the pair correlations of the SK model

$$a(q) = 2q \qquad \log \alpha(q) = \tfrac{1}{2} \log[(1+q)/(1-q)]. \tag{36}$$

The de Almeida–Thouless line (see Figure 2) is then given by

$$h^2 = J^2 m \log\left(\frac{1+m}{1-m}\right) \qquad T^2 = J^2 m/\log\left(\frac{1+m}{1-m}\right). \tag{37}$$

From (32) and (33), one finds that in zero field there is a transition at $T_0 = J/2^{1/2}$ and that

$$m = h/T(1 + T_0^2/T^2) - h^3/3T^3(1 + T_0^2/T^2)^4 + O(h^5) \qquad \text{if } T > T_0 \tag{38}$$

$$m = h/2T_0 - h^3/48T_0^3 + O(h^5) \qquad \text{if } T < T_0. \tag{39}$$

Thus one finds a cusp in the susceptibility (see figure 3) as well as in the non-linear susceptibility. One should notice that the de Almeida–Thouless line has the following shape for h small:

$$(T_c(0) - T_c(h))/T_c(0) \sim h^2/6J^2. \tag{40}$$

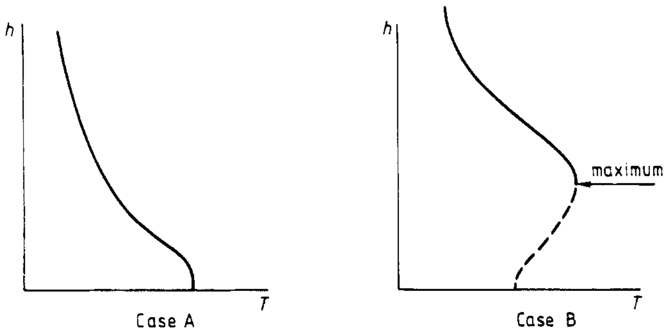

Figure 2. The shape of the de Almeida–Thouless line in the GREM. In case A, $T_c(h)$ decreases as h increases. In case B, $T_c(h)$ has a maximum. This maximum is a kind of tri-critical point. The function $q(x)$ is continuous when one crosses the upper part of the line (full curve) and has a jump when one crosses the lower part of the line (broken curve).

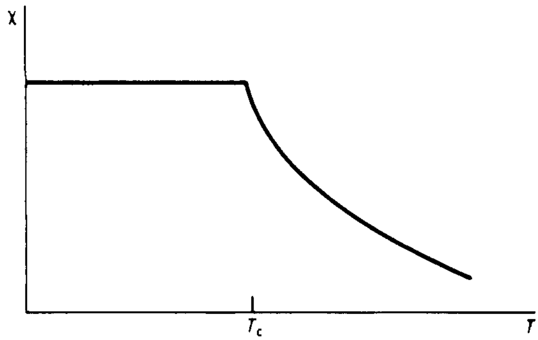

Figure 3. The zero-field magnetic susceptibility of the GREM (case A and case B).

So we see that the de Almeida–Thouless line does not have the shape $T_c(0) - T_c(h) \sim h^{2/3}$ and that the non-linear susceptibility does not diverge as expected in the SK model. The reason is that the SK model and the GREM defined by (36) have the same pair correlations between energies but are not identical models.

Let us now discuss briefly the case B, i.e. the case where the ratio $a(q)/\log \alpha(q)$ increases, has a single maximum for $q = Q_{\max}$ and then decreases. A typical example of this case B is

$$a(q) = pq^{p-1} \qquad \log \alpha(q) = \tfrac{1}{2}\log[(1+q)/(1-q)] \qquad \text{for } p > 2 \tag{41}$$

If $q < Q_{\max}$, then let us call $Q_0(q)$ the solution of

$$\int_q^{Q_0(q)} a(q')\,\mathrm{d}q' \Big/ \int_q^{Q_0(q)} \log \alpha(q')\,\mathrm{d}q' = \frac{a(Q_0(q))}{\log \alpha(Q_0(q))}. \tag{42}$$

(If there is no solution of (42), then $Q_0(q) = q$ if the left-hand side is larger than the right-hand side and $Q_0(q) = 1$ otherwise.) If $q > Q_{\max}$, $Q_0(q)$ is defined by

$$Q_0(q) = q. \tag{43}$$

Then one finds that there is a de Almeida–Thouless line given by

$$\frac{h}{J}=\frac{1}{2}\left[a(m)\left(\frac{\log \alpha(Q_0(m))}{a(Q_0(m))}\right)^{1/2}+\log \alpha(m)\left(\frac{a(Q_0(m))}{\log \alpha(Q_0(m))}\right)^{1/2}\right]$$

$$T_c(h)=(J/2)[a(Q_0(m))/\log \alpha(Q_0(m))]^{1/2}. \tag{44}$$

For $m > Q_{\max}$, the sequence $a(q)/\log \alpha(q)$ is decreasing and since $Q_0(m) = m$, formula (44) is equivalent to (34) and (35). For $m < Q_{\max}$, i.e. for small field, formula (44) is more complicated. So we see that there are two parts along the de Almeida–Thouless line: the part where $m < Q_{\max}$ and the part where $m > Q_{\max}$. It is easy to see that $Q_0(m)$ is a continuous function of m which is always larger than $Q_{\max}$ and such that $Q_0(Q_{\max}) = Q_{\max}$. Therefore the minimal value of $Q_0(m)$ is $Q_{\max}$ and is reached for $m = Q_{\max}$. The result on the shape of the de Almeida–Thouless line in case B is the following. When h increases, $T_c(h)$ increase and magnetisation increases. When m reaches $Q_{\max}$, $T_c(h)$ has a maximum and then when h increases further, $T_c(h)$ decreases. In § 6 it will be seen that the maximum of $T_c(h)$ is like a tri-critical point: for $m < Q_{\max}$, the function $q(x)$ has a jump, whereas for $m > Q_{\max}$, $q(x)$ is continuous.

In case B, the equation of state is given by

$$\log \alpha(m)+(J^2/4T^2)a(m)=h/T \qquad \text{if } T>T_c(h)$$

$$\frac{1}{2}\left[a(m)\left(\frac{\log \alpha(Q_0(m))}{a(Q_0(m))}\right)^{1/2}+\log \alpha(m)\left(\frac{a(Q_0(m))}{\log \alpha(Q_0(m))}\right)^{1/2}\right]=\frac{h}{J} \qquad \text{if } T<T_c(h). \tag{45}$$

As in case A, we find that magnetisation is independent of T in the spin glass phase. A consequence, here again, is that the magnetic susceptibility is constant in the spin glass phase.

5. The moments of the partition function

We have seen in § 2 that the only information about the correlations between energies that was necessary to obtain the average free energy of the GREM was the information contained in $\langle Z^2(T)\rangle$. In this section we are going to show that all integer moments $\langle Z^p(T)\rangle$ of the partition function can be expressed in terms of $\langle Z(T)\rangle$ and $\langle Z^2(T)\rangle$. By definition $\langle Z^p(T)\rangle$ is given by

$$\langle Z^p(T)\rangle=\sum_{\nu_1=1}^{2^N}\cdots\sum_{\nu_p=1}^{2^N}\int\cdots\int p_{\nu_1,\ldots,\nu_p}(E_{\nu_1},\ldots,E_{\nu_p})$$

$$\times \exp\left(-\frac{E_{\nu_1}+\ldots+E_{\nu_p}}{T}\right)\mathrm{d}E_{\nu_1}\ldots \mathrm{d}E_{\nu_p}. \tag{46}$$

As in the calculation for $\langle Z^2(T)\rangle$ in § 2, this sum is dominated by a saddle point which corresponds to certain overlaps between the configurations. If we assume that for any integer p, there is no breaking of replica symmetry, i.e. all the overlaps are the same and equal to Q, then

$$\frac{1}{N}\log\langle Z^p(T)\rangle = \max_Q\left(\int_0^Q \mathrm{d}q \log\alpha(q) + p^2 a(q)\frac{J^2}{4T^2} + \int_Q^1 \mathrm{d}q\, p\log\alpha(q) + pa(q)\frac{J^2}{4T^2}\right). \tag{47}$$

The value of Q that maximises this sum is given by the solution of

$$a(Q)/\log\alpha(Q) = 4T^2/J^2p. \tag{48}$$

If (48) has no solution, then one should choose $Q = 1$ or $Q = 0$ depending on which maximises (47). If (48) has a solution, this solution is unique in case A because $a(q)/\log\alpha(q)$ is monotonic. In case B, the function $a(q)/\log\alpha(q)$ has a single maximum and it is always the solution of (48) which is larger than $Q_{\max}$, which should be considered because it is the local maximum. (The other solution, which is in the increasing part of $a(q)/\log\alpha(q)$, is always a local minimum.) However, in case B one has to make sure that $Q = 0$ does not give a larger value to (47). In case B one expects a first-order transition for $\langle Z^p(T)\rangle$ where Q jumps from 0 to a finite value larger than $Q_{\max}$.

From (47) and (48), one can show that

$$\frac{1}{N}[\log\langle Z^p(T)\rangle - \log\langle Z(T)\rangle^p] = (p-1)\int_0^Q \mathrm{d}q\left(pa(q)\frac{J^2}{4T^2} - \log\alpha(q)\right) \tag{49}$$

where Q is the solution of (48). One can easily see that the right-hand side of (49) is a function of T^2/p except for the factor $p - 1$. Therefore one has the following formula for the integer moments $\langle Z^p(T)\rangle$:

$$(1/N)[\log\langle Z^p(p^{1/2}T)\rangle - \log\langle Z(p^{1/2}T)\rangle^p] = (p-1)(1/N) \times [\log\langle Z^2(2^{1/2}T)\rangle - \log\langle Z(2^{1/2}T)\rangle^2]. \tag{50}$$

So once $\langle Z^2(T)\rangle$ is known, all the other integer moments can be computed using (50).

Let us remark that the result (50) relies on the assumption that the saddle point that gives $\langle Z^p(T)\rangle$ is replica-symmetric. This is not *a priori* obvious. Let us just mention that at least in case A we can prove it. We will not give the proof here because it is rather complicated and requires new notation.

We see that if one tries to make the continuation $p \to 0$, there is no way that (50) can give (20). Therefore the replica calculation without breaking is unable to lead to the solution of the GREM.

6. The replica calculation and the function $q(x)$

For the random-energy model, it was shown that a calculation with a single breaking of replica symmetry can give the correct free energy (Derrida 1981, Gross and Mezard 1984). In this section we shall show that a breaking of symmetry in the manner described by Parisi leads to the solution of the GREM with a non-trivial function $q(x)$.

Let us start with the expression (46) of $\langle Z^p(T)\rangle$. There are p configurations. Consider that these p configurations belong to the same branch of the tree (of figure 1) for overlaps $0 < q < Q_1$. At Q_1 they divide themselves into p_1 groups of p/p_1 configurations each up to the overlap Q_2. At Q_2 each branch of p/p_1 configurations each bifurcates into p_2/p_1 branches of p/p_2 configurations. Therefore for $Q_2 < q < Q_3$, there are altogether p_2

groups of p/p_2 configurations each. One can repeat this breaking K times. For such a breaking $\langle Z^p(T)\rangle$ is given by

$$\frac{1}{N}\log\langle Z^p(T)\rangle = \underset{\substack{p_1,p_2\cdots p_K\\ Q_1,Q_2\cdots Q_K}}{\text{extremum}}\bigg(\int_0^{Q_1} \mathrm{d}q \log \alpha(q) + p^2 a(q)\frac{J^2}{4T^2} + \int_{Q_1}^{Q_2} \mathrm{d}q\, p_1 \log \alpha(q)$$
$$+\frac{p^2}{p_1}a(q)\frac{J^2}{4T^2} + \ldots + \int_{Q_K}^{1} \mathrm{d}q\, p_K \log \alpha(q) + \frac{p^2}{p_K}a(q)\frac{J^2}{4T^2}\bigg). \quad (51)$$

Clearly one must have

$$Q_1 < Q_2 < \ldots < Q_K \quad (52)$$

and

$$1 < p_1 < p_2 \ldots < p_K < p. \quad (53)$$

In the limit $p \to 0$, according to the usual replica calculation (Parisi 1980a,b,c), one should choose the $Q_1 \ldots Q_K$ and $p_1 \ldots p_K$ that minimise (51) with the constraint (52) but where (53) is replaced by

$$1 > p_1 > p_2 \ldots < p_K > p \to 0. \quad (54)$$

If one takes an infinite number of breakings, then for each value of Q_i there is a number p_i. In the continuum limit, let us define $x(q)$ by

$$p_i = p/x(Q_i) \quad (55)$$

then one finds from (51) that

$$\frac{1}{N}\langle\log Z\rangle = \lim_{p\to 0}\frac{1}{N}\frac{\log\langle Z^p\rangle}{p} = \min_{q(x)}\bigg(\int_0^1 \frac{1}{x(q)}\log\alpha(q)\,\mathrm{d}q + \frac{J^2}{4T^2}\int_0^1 x(q)a(q)\,\mathrm{d}q\bigg). \quad (56)$$

Because of (52), (54) and (55), $q(x)$ must be an increasing function of x. (In (56) it is $x(q)$ that appears; this is obviously defined as the inverse function of $q(x)$.)

The function $q(x)$ or its inverse $x(q)$ plays exactly the same role here as in the Parisi solution. One can see that by considering the following quantity:

$$\sum_{\alpha<\beta}(Q_{\alpha\beta})^2 = \frac{p}{2}\bigg[Q_1^2\Big(p-\frac{p}{p_1}\Big) + \sum_{i=2}^{K} Q_i^2\Big(\frac{p}{p_{i-1}} - \frac{p}{p_i}\Big)\bigg]. \quad (57)$$

In the limit $p \to 0$, one finds

$$\sum_{\alpha<\beta}(Q_{\alpha\beta}^2) = \frac{p}{2}\bigg(Q_1^2(p - x(Q_1)) + \sum_{i=2}^{K} Q_i^2(x(Q_{i-1}) - x(Q_i))\bigg) = -\frac{p}{2}\int_0^1 q^2(x)\,\mathrm{d}x. \quad (58)$$

So the replica approach to the GREM reduces to expression (56) and to finding the best function $q(x)$ or equivalently $x(q)$.

In case A, the result is

$$x(q) = (2T/J)(\log\alpha(q)/a(q))^{1/2} \qquad \text{for } 0 < q < q_{\max}$$
$$x(q) = 1 \qquad \text{for } q_{\max} < q < 1. \quad (59)$$

Since $a(q)/\log\alpha(q)$ is a decreasing function of q, then $x(q)$ given by (59) is an increasing function of q. Since $x(q)$ has to fulfil the condition $0 < x < 1$, at each temperature $T < T_c$, there is a maximal value $q_{\max}$ of q given by

$$1 = (2T/J)(\log \alpha(q_{\max})/a(q_{\max}))^{1/2}. \tag{60}$$

When $x(q)$ is replaced by expression (59), one finds the free energy given by (9). Therefore the Parisi *ansatz* leads to the correct free energy in this case A.

It is of course easy to include the presence of a magnetic field. As discussed in §§ 3 and 4, the overlaps must then be larger than or equal to the magnetisation m. Therefore in (56) the integrals must go from m to 1. Therefore one finds a flat part of $q(x)$ at $q_{\min} = m$ (figure 4).

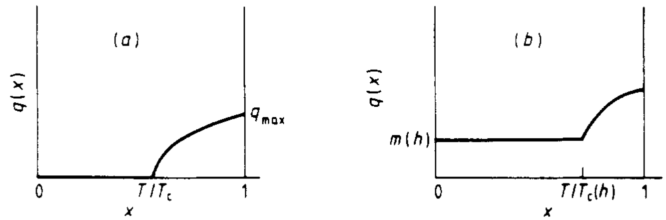

Figure 4. The function $q(x)$ of the GREM in the spin glass phase in case A for (a) $h = 0$ and (b) $h \neq 0$. One should notice that there is no plateau at $q_{\max}$ and also that $q_{\min}$ is equal to the magnetisation.

In case B, one again has to find the function $q(x)$ that gives an extremum of (56). Since the ratio $a(q)/\log \alpha(q)$ has a maximum, one finds

$$x(q) = (2T/J)(\log \alpha(q)/a(q))^{1/2} \qquad \text{for } Q < q < q_{\max} \tag{61}$$

where $q_{\max}$ is again given by (60) and where Q is given by

$$\int_0^Q \log \alpha(q)\,\mathrm{d}q \Big/ \int_0^Q a(q)\,\mathrm{d}q = \frac{\log \alpha(Q)}{a(Q)}. \tag{62}$$

Again one can check that by replacing $x(q)$ by expression (61), one finds the correct solution of the GREM given by (12) in case B.

To obtain the expression for $q(x)$ in presence of a field one has only to replace the lower bound 0 in the integrals (56) by $m(h)$. So if the field is small enough and therefore if m is small enough, the function $a(q)/\log \alpha(q)$ for $m < q < 1$ still has a maximum and therefore the function $q(x)$ has a jump: this situation corresponds to the part of the de Almeida–Thouless line where $T_c(h)$ is an increasing function of h (broken curve in figure 2(b)). When h is strong enough, i.e. when $T_c(h)$ reaches its maximal value, the function $a(q)/\log \alpha(q)$ for $m < q < 1$ starts to be monotonic and the jump disappears (full curve in figure 2(b)).

Discontinuous functions $q(x)$ have already been found for other spin glass models: the Potts glass (Gross *et al* 1985) and the p spin glass model (Gardner 1985).

All the diagrams in figures 4 and 5 correspond to $a(q)$ and $\log \alpha(q)$ given by (26) with $p = 2$ in case A and $p > 2$ in case B. To end this section, we will make a few remarks.

Firstly, Parisi's *ansatz* gives the exact free energy of the GREM. This was not obvious *a priori* even if the GREM already had an ultrametric structure in itself because the breaking could have been more complicated. (In Parisi's scheme, at a given overlap, all

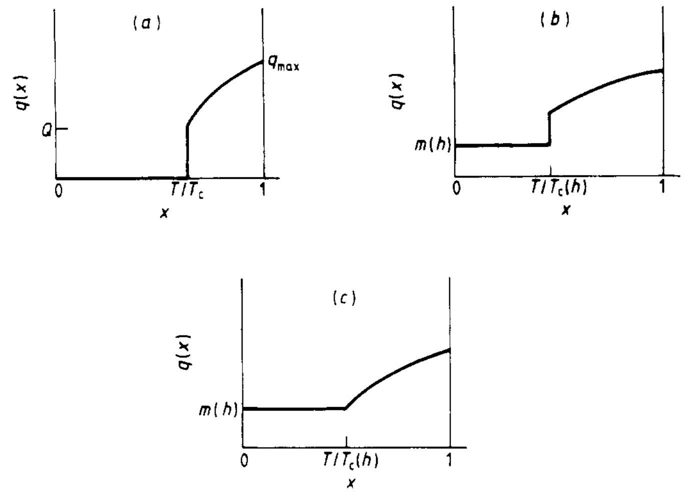

Figure 5. The function $q(x)$ in case B ((a) is for $h = 0$). There is a jump in $q(x)$ for (b) small field (broken curve in figure 2) but this jump disappears if the field is (c) strong enough (full curve in figure 2).

the groups have the same size. One could imagine breakings for which not all these sizes would be equal.)

Secondly, the function $q(x)$ has roughly the shape expected in spin glass models. One big difference however from Parisi's $q(x)$ for the SK model is that for the GREM the plateau at q_{max} is missing. The length Y of the plateau at q_{max} has been interpreted as

$$Y = \sum_{\alpha} P_{\alpha}^{2} \tag{63}$$

(Mezard *et al* 1984) where P_α is the probability that the system is in the pure state α. In the SK model as well as in the GREM there are an infinite number of pure states in the spin glass phase. However, $Y = 0$ in the GREM, i.e. there is no pure state that has a finite weight; whereas in the SK model near T_c, $Y \sim 1$, i.e. there is one pure state that carries almost all the whole weight.

Thirdly, in expression (59), one can see that $q(x)$ is a function of x/T only. This was also observed by Vannimenus *et al* (1981) when they made the approximation that magnetisation depends only on the field in the spin glass phase.

Fourthly, we have introduced here the breaking of replica symmetry in an abstract way by analogy with Parisi's *ansatz* (Parisi 1980a,b,c). It is interesting to notice that the function $q(x)$ found here is the same as the one derived by De Dominicis and Hilhorst (1985, 1986) who computed the function $q(x)$ using the physical interpretation (Parisi 1983, Mezard *et al* 1984a,b), i.e. computing the probability distribution of overlaps.

Lastly, one can see that in the presence of the magnetic field, there is a plateau at the value q_{min} which is always equal to the magnetisation m. This is not surprising because one always has $q_{min} \geqslant m$. This is due to the ultrametric structure of the GREM. If one takes two configurations having a magnetisation m, this means that they have an overlap m with the ferromagnetic configuration. Therefore they must have a mutual overlap that is at least m.

7. Conclusion

In this paper, we have given a general expression (20) for the free energy of the GREM. We have seen how one could introduce a magnetic field and have computed the de Almeida–Thouless line $T_c(h)$. In case A, $T_c(h)$ decreases with h, whereas in case B, $T_c(h)$ has a maximum. Lastly we have described in figures 4 and 5 the shapes of the function $q(x)$ and have shown that Parisi's *ansatz* gives the correct free energy of the GREM.

We have discussed the analogies with and the differences from the Parisi solution of the SK model:

(i) the de Almeida–Thouless line;
(ii) the flat susceptibility in the spin glass phase;
(iii) the function $q(x)$; but
(iv) no plateau in $q(x)$ at $q_{\max}$;
(v) the de Almeida–Thouless does not have the same shape in the GREM and in the SK model.

We have seen that the magnetisation depends only on the field in the spin glass phase and that the varying part of $q(x)$ is only a function of x/T. This is very reminiscent of the approximation proposed a few years ago to calculate the properties of the SK model (Parisi and Toulouse 1980, Vannimenus *et al* 1981).

There remain several questions that we would like to mention at the end of this paper.

The main problem, of course, would be to generalise the GREM in order to treat triple and higher correlations between energies. This means that one could try to generalise formula (20) to cases where the moments $\langle Z^p(T)\rangle$ are not given by (50). It seems reasonable to think that (20) is only the special case (when (50) is valid) of a more complicated formula involving $\langle Z^3(T)\rangle$, $\langle Z^4(T)\rangle$ etc. It would be very interesting to get such a formula because it would give a systematic expansion for studying spin glasses in any dimension and more generally disordered spin systems (dilute magnets, random field models etc).

The GREM by itself gives a whole class of spin glass models that can be solved exactly. To our knowledge it is the first spin glass model for which one can show that Parisi's *ansatz* gives the correct free energy. It would be interesting to study the replica treatment in more detail, and in particular to look at the stability of the replica solution and to see whether it is fully or only marginally (De Dominicis and Kondor 1983) stable.

In our previous work (Derrida and Gardner 1986), we have seen that one could associate any spin glass model in finite or infinite dimension with a GREM that has the same correlations between energies. In infinite dimension, the correlations between the energies of two configurations depend on their overlap. However in finite dimension, these correlations do not depend on the overlap but on the number of bonds that are changed (see § 6 of Derrida and Gardner 1986). Therefore our way of introducing the magnetic field in the GREM is only legitimate if it is a GREM associated with an infinite-ranged spin glass model. For a GREM associated with a finite-dimensional spin glass model, one can still talk of partial partition functions but this would correspond to considering a GREM associated with a spin glass model whose distribution $\rho(J_{ij})$ of bonds is gaussian but not symmetric

$$\rho(J_{ij}) \sim \exp[-(J_{ij} - J_0)^2/2J^2]z \qquad \text{with } J_0 \neq 0.$$

Until now, we have always associated a given spin glass model with the GREM that has the same pair correlations between energies. This constraint was sufficient to fix the functions $a(q)$ and $\log \alpha(q)$. At the end of the present work, one sees that one could associate a GREM in a different way. For example one could choose the functions $a(q)$ and $\log \alpha(q)$ such that the GREM has exactly the same de Almeida–Thouless line (see equations (32) and (33)) as a given spin glass model. In the case of the SK model, the properties of this GREM would be those described by Vannimenus *et al* (1981) since this GREM would have a magnetisation m depending only on the field in the spin glass phase.

Lastly we have seen that the de Almeida–Thouless line $T_c(h)$ has a maximum at h_0 when h varies from 0 to ∞ for the GREM corresponding to the p spin glass model ($p > 2$). For $h > h_0$, the function $q(x)$ has a discontinuity, whereas the function $q(x)$ is continuous for $h > h_0$. So the point h_0, $T_c(h_0)$ is a kind of tri-critical point. It would be interesting to know whether such a tri-critical point does exist along the de Almeida–Thouless line of the true p spin glass model.

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