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LETTER TO THE EDITOR

Dynamical phase transition in non-symmetric spin glasses

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Abstract. Non-symmetric spin glasses are spin models for which the pair interactions between the spins are random but not symmetric. By studying the time evolution of two configurations in a mean-field model, one finds a transition temperature T_0. For $T > T_0$, two different initial conditions end up by becoming identical after an infinite time. This means that the thermal noise is strong enough to eliminate the memory of the initial conditions. For $T < T_0$, two different initial conditions never become identical.

If one considers the time evolution of a dynamical system in presence of thermal noise, its configuration $\mathscr{C}_t$ at time t depends both on its initial condition $\mathscr{C}_0$ and on the effect of thermal noise between times 0 and t. In the limit $t \to \infty$, there are two possibilities: either $\mathscr{C}_t$ becomes independent of $\mathscr{C}_0$ or not.

Therefore one can expect to observe two possible phases: a high-temperature phase where the thermal noise is strong enough to make the system forget its initial condition $\mathscr{C}_0$ and a low-temperature phase where the system in the limit $t \to \infty$ still depends on $\mathscr{C}_0$. Of course, if one can show that some correlation functions between $\mathscr{C}_0$ and $\mathscr{C}_t$ do not decay to zero in the limit $t \to \infty$, it is clear that the system is in its low-temperature phase. However, it may happen that $\mathscr{C}_t$ depends on $\mathscr{C}_0$ in a very complicated way which cannot be seen in the study of simple order parameters.

In the present letter, a diluted and asymmetric spin glass model is studied which possesses these two phases. By studying the time evolution of the distance between two configurations, one finds that there is a transition temperature T_0 below which the distance between two configurations does not vanish in the long time limit.

Models of non-symmetric spin glasses have already been studied in the mean-field limit (Hertz *et al* 1986, Bausch *et al* 1986, Sompolinsky 1987, Gutfreund *et al* 1987). In the cases considered up to now, the spin glass phase seems to be absent, at least when one looks at the spin-spin correlations at different times. The purpose of this letter is to show that a transition does exist at finite temperature when one considers the distance between two configurations. Whether this dynamical transition can be related to the appearance of a spin glass phase is not clear and will not be discussed in the present work.

The model considered here is a system of N Ising spins ($\sigma_i = \pm 1$) which are connected by random interactions J_{ij}. The interactions J_{ij} and J_{ji} are independent and randomly distributed according to

$$\rho(J_{ij}) = \left(1 - \frac{c}{N}\right)\delta(J_{ij}) + \frac{c}{N}\rho_0(J_{ij}) \qquad (1)$$

where c is a finite number and

$$\rho_0(J_{ij}) = \frac{1}{\sqrt{2\pi}J} \exp\left(-\frac{J_{ij}^2}{2J^2}\right). \tag{2}$$

From (1) it is easy to see that the probability that site i has K non-zero interactions J_{ij} is $c^K \, e^{-c}/K!$. The choice of a Gaussian (2) for ρ_0 makes some of the calculations which follow simpler but everything can be generalised to any other distribution ρ_0.

The dynamics of the model are defined by the following rule. At time t, the fields $h_i(t)$ are computed:

$$h_i(t) = \sum_j J_{ij}\sigma_j(t) + h \tag{3}$$

where h is the external field. Then all the spins σ_i are updated:

$$\begin{aligned} \sigma_i(t+1) &= +1 \qquad \text{with probability} \quad [1+\exp(-2h_i(t)/T)]^{-1} \\ \sigma_i(t+1) &= -1 \qquad \text{with probability} \quad [1+\exp(2h_i(t)/T)]^{-1}. \end{aligned} \tag{4}$$

The parameter T in (4) defines the temperature of the system.

Other probabilistic algorithms could be chosen to define the dynamics at finite temperature. For other dynamics than (4) it may happen that the exact solution which follows could not be generalised.

The dynamics (4) are parallel dynamics since all the spins are updated at the same time. It is easy to show (Derrida *et al* 1987) that random sequential dynamics, where at each time step a randomly chosen spin is updated, lead to the same results in the limit $t \to \infty$.

Let us consider two spin configurations $\{\sigma_i(t)\}$ and $\{\sigma_i'(t)\}$ which evolve according to exactly the same dynamics. This means that, at each time step, the two fields $h_i(t)$ and $h_i'(t)$ are calculated by the formula (3) with the same set of J_{ij} and also that the same random number (Stauffer 1987) is used to decide whether $\sigma_i(t+1)$ and $\sigma_i'(t+1) = +1$ or -1 according to (4). Therefore if $h_i(t) = h_i'(t)$, this implies that $\sigma_i(t+1) = \sigma_i'(t+1)$. Choosing the same random number to update the σ_i and σ_i' means that the two configurations are subjected to the same thermal noise.

Let us define the distance $D(t)$ between the two configurations by

$$D(t) = \frac{1}{4N} \sum_{i=1}^{N} (\sigma_i(t) - \sigma_i'(t))^2 \tag{5}$$

where $D(t)$ is just the fraction of sites i such that $\sigma_i(t) = -\sigma_i'(t)$.

The time evolution of $D(t)$ is given in the thermodynamic limit ($N \to \infty$) by

$$\begin{aligned} D(t+1) = \sum_{K=1}^{\infty} \frac{c^K \, e^{-c}}{K!} \sum_{p=1}^{K} (D(t))^p (1-D(t))^{K-p} \frac{K!}{p!(K-p)!} \\ \times \frac{1}{2\pi J^2 \sqrt{p(K-p)}} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \exp\left(-\frac{x^2}{2(K-p)J^2} - \frac{y^2}{2pJ^2}\right) \\ \times (\{1+\exp[-2(x+|y|+h)/T]\}^{-1} - \{1+\exp[-2(x-|y|+h)/T]\}^{-1}). \end{aligned} \tag{6}$$

Iterating this formula gives the distance $D(t)$ at any finite t if one knows its initial value $D(0)$.

The derivation of (6) is a straightforward generalisation to the problem studied here of a method which was already used for a diluted neural network model (Derrida *et al* 1987) and for random networks of automata (Derrida and Weisbuch 1986). Let us just briefly see how it is derived.

First, as long as c is small (Derrida *et al* 1987),

$$c \ll \log N \tag{7}$$

the quenched model for which the J_{ij} are randomly chosen at time $t=0$ and remain fixed at later times, and the annealed model for which the J_{ij} are changed at each time step both have $D(t)$ given by the same expression (6). The reason for this was explained in detail in Derrida and Weisbuch (1986) and Derrida *et al* (1987) and is a consequence of the fact that the input sites j of almost all sites i are not correlated at any finite time. So when condition (7) is valid, the calculation can be done for the annealed model.

If one looks at formula (6), the term $c^K \mathrm{e}^{-c}/K!$ is the density of sites having K interactions $J_{ij} \neq 0$. Among these K input sites j of a given site, the probability that p of them are different $(\sigma_j(t) = -\sigma_j'(t))$ and $K-p$ are identical $(\sigma_j(t) = \sigma_j'(t))$ is

$$D(t)^p (1 - D(t))^{K-p} \frac{K!}{p!(K-p)!}.$$

The values of the fields $h_i(t)$ and $h_i'(t)$ produced on site i by the spins $\sigma_j(t)$ and $\sigma_j'(t)$ can be written as

$$\begin{aligned} h_i(t) &= \sum_j J_{ij}\sigma_j(t) + h = x + y + h \\ h_i'(t) &= \sum_j J_{ij}\sigma_j'(t) + h = x - y + h \end{aligned} \tag{8}$$

where x is the sum over the spins j which are identical in configurations σ and σ' and y is the sum over the different spins. Clearly x and y are random Gaussian variables of width $(K-p)J^2$ and pJ^2, respectively, since they are the sum of $K-p$ and p random Gaussian variables distributed according to ρ_0 (2). The last factor in (6) tells us the probability that a random number between 0 and 1 is such that $\sigma_i(t+1)$ and $\sigma_i'(t+1)$ have opposite signs (see (4)).

Formula (6) allows the calculation of $D(t)$ at any temperature and at any finite time t (in the thermodynamic limit).

One can then study the long time behaviour. Clearly, everything is described by the mapping $D(t) \to D(t+1)$ given by equation (6). The phase diagram in terms of c, T and h depends on the structure and on the stability of the fixed points (or attractors) of this map.

The easiest thing to study is the fixed point $D=0$. If it is attractive $(\mathrm{d}D(t+1)/\mathrm{d}D(t) < 1)$, this means that two different configurations which are close end up by becoming identical. If it is repulsive, this means that two different configurations (which differ by a finite fraction of spins) never become identical. The surface $T_0(c, h)$ where this fixed point $D=0$ changes its stability is given by

$$\begin{aligned} 1 = \sum_{K=1}^{\infty} \frac{c^K \mathrm{e}^{-c}}{(K-1)!} \frac{1}{2\pi J^2 (K-1)^{1/2}} \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y \exp\left(-\frac{x^2}{2(K-1)J^2} - \frac{y^2}{2J^2}\right) \\ \times (\{\exp[-2(x+|y|+h)/T_0]+1\}^{-1} - \{\exp[-2(x-|y|+h)/T_0]+1\}^{-1}). \end{aligned} \tag{9}$$

Since the expansion of $D(t+1)$ contains linear and quadratic terms in $D(t)$, one can also see that $D(\infty)$ vanishes linearly as $T_0 - T$:

$$D(\infty) \sim T_0 - T. \tag{10}$$

In some special situations, formulae (6) and (9) can be simplified.

For example, in zero field ($h=0$) and in the low-temperature limit ($T\to 0$), the mapping (6) becomes

$$D(t+1)=\sum_{K=1}^{\infty}\frac{c^K e^{-c}}{K!}\sum_{p=1}^{K}(D(t))^p(1-D(t))^{K-p}\frac{K!}{p!(K-p)!}\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{K-p}{p}\right)^{1/2}\right] \tag{11}$$

and the fixed point $D=0$ is only stable if $c<c_0$ with

$$c=2.206\,15\ldots. \tag{12}$$

Therefore for $c<c_0$, at all temperatures including $T=0$, the system is in its high-temperature phase whereas for $c>c_0$, the low-temperature phase exists up to a finite temperature $T_0(c)$ which vanishes as $c\to c_0$. It is interesting to notice that c_0 is a concentration higher than the percolation threshold $c=1$ where some sites start to be connected to an infinite cluster (i.e. to have an infinite number of ancestors).

Another simple situation is the limit where c becomes large. Then one finds that the transition temperature T_0 becomes

$$T_0\sim c\left(\frac{2}{\pi}\right)^{1/2}\left[\cosh\left(\frac{h}{T_0}\right)\right]^{-2}. \tag{13}$$

This result shows that the transition exists even in non-zero field and that $T_0(h)$ decreases when h increases. This is somewhat reminiscent of the de Almeida and Thouless (1978) transition in the mean-field spin glass although the shape of $T_0(h)$ at $h=0$ is different.

In this letter we have seen that a model of a non-symmetric spin glass exhibits a dynamical phase transition at finite temperature. This result could easily be generalised to similar models with non-symmetric distributions of bonds or multispin interactions. Complicated situations could then occur with several attractive fixed points of the mapping (6).

It would be interesting to see whether the phase transition described in this letter is still present when one uses other probabilistic algorithms to define the finite-temperature dynamics.

For symmetric interactions, the analytic method used here does not work. (The annealed and the quenched models become different at the second time step.) However, one could try to study the same quantity $D(t)$ in order to see whether the spin glass transition (of the system at equilibrium) can be seen in the long time behaviour of $D(t)$. Derrida and Weisbuch (1987) have studied the $\pm J$ model in three dimensions and we intend to present our results in a forthcoming work.

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